2011 50th IEEE Conference on Decision and Control and
European Control Conference (CDC-ECC)
Orlando, FL, USA, December 12-15, 2011

Heavy-tailed weighted networks from local attachment strategies

Pablo Moriano and Jorge Finke
Department of Manufacturing Engineering
Pontificia Universidad Javeriana
pamoriano@javerianacali.edu.co, finke@ieee.org

Abstract— **Large networks arise by the gradual addition of nodes attaching to an existing and evolving network component. There are a wide class of attachment strategies which lead to distinct structural features in growing networks. This paper introduces a mechanism for constructing, through a process of distributed decision-making, substrates for the study of collective dynamics on power-law weighted networks with both a desired scaling exponent and a desired clustering coefficient. The analytical results show that the connectivity distribution converges to the scaling behavior often found in social and engineering systems. To illustrate the approach of the proposed framework we generate network substrates that resemble the empirical citation distributions of (i) patents granted by the U.S. Patent and Trademark Office from 1975 to 1999; and (ii) opinions written by the Supreme Court and the cases they cite from 1754 to 2002.**

I. Introduction

Understanding the formation of structure lies at the very heart of the study of complex networks. A network encompasses a large number of interconnected elements (units or agents) whose interaction with each other and with the surroundings leads to emerging properties that can only be attributed to the network as a whole [1]. Often, networks gradually develop distinct structural patterns without centralized information or coordination schemes. Studying the emergence of these patterns promises to enhance our understanding of collective human dynamics [2], corrupt behavior [3], [4], and economic development [5]-[7].

Random graph models fail to capture key features of real-world networks (e.g., clustering coefficients and degree correlations). Recent efforts to understand network structure have focused on connectivity distributions underlying a number of social and engineering systems which, rather than following the bell-shape of random networks (bounded by Chebyshev's inequality), have heavy tails [8], [9]. Heavy-tailed distributions in empirical data suggests the existence of causal mechanisms that shape the structure and function of complex networks [10]. In the era of "big data," the development of formal frameworks that quantify patterns of interaction of real-world networks has set the research agendas across various disciplines, but only recently across the data-driven computational social sciences.

Power-laws, a particular type of heavy-tailed distributions, have received significant attention in recent years. For a network with a power-law connectivity distribution, the probability that a node connects to x other nodes is proportional to $x^{-\alpha}$ for some positive constant α (the probability cumulative function $P[X > x]$ is a straight line on a log-log plot). The distribution satisfies that for any positive constant λ, $\lim_{x\to\infty} e^{\lambda x} P[X > x] = \infty$, characterizing the fact that its tail has no exponential bound. As a result, the connectivity of the nodes of the network comprises different orders of magnitude, with a few nodes being highly connected (e.g., see [11], [12]).

Key to the above computational models is the existence of *hubs* (highly interconnected nodes). Identifying hubs allows us to measure key structural properties and plausibly predict the behavior of complex networks [13]. In the context of the spread of disease, measuring time-varying patterns in regions that are more vulnerable to infection allows us to design strategies that respond more effectively to the potential spread of large-scale epidemics [14]. The structuralist approach to these strategies embraces how interconnected regions influence one another (as a result of the evolution of social systems) to quantify collective human behavior.

In order to capture some of the different types of relationships between the elements of a network, e.g., duration, emotional intensity, or intimacy, mathematical models define *weights* as an intrinsic property between nodes [15]-[18]. Previous models of weighted networks have focused on attachment strategies in which nodes are added according to probability distributions on the existing weights across the entire network [18], [19]. The model introduced in [19] captures the evolution of weights driven by preferential strength attachment, a mechanism in which newly added nodes are more likely to connect to nodes associated with larger weights. Lacking a competitive advantage of (possibly) newly added nodes (node fitness), the resulting network exhibits a power-law distribution where hubs correspond to the nodes that have been part of the network the longest.

This paper introduces a wide class of attachment strategies which promote the formation of hubs based on both the longevity and fitness of surrounding neighbors. Because the connectivity dynamics of the nodes depend on their attractiveness to compete for weights (as in [20]), older nodes are not necessarily more successful in acquiring weights. To our knowledge the proposed mechanisms is novel in that it generates weighted directed networks with power-law strength distributions (i) in a distributed fashion (decision-making strategies are based on local information; we do not assume any type of global information to generate the desired network structure); (ii) for an arbitrary scaling exponent $\alpha > 2$ and varying clustering coefficients $c \in (0, 1)$; (iii)

for values greater than a particular threshold $\omega_{min} > 0$; we assume the power-law is obeyed at the tail of a distribution; and (iv) at a rate of emergence captured by a smoothness parameter $\delta_i > 0$.

The remaining sections are organized as follows: Section 2 introduces a model that captures the connectivity and growth dynamics of the gradual addition of nodes to an existing network component and proposes attachment strategies for local rearrangement of weights between pairs of nodes. In Section 3, Theorem 1 shows that for any connected network there exists a distribution of the total weight from neighboring nodes (node strength) that is asymptotically stable (i.e., the proposed strategies lead to an asymptotically stable strength distribution). Theorem 2 proves that consecutive achievements of this network state leads to weighted networks with power-law strength distribution. In Section 4, we present simulations that capture the effect of node fitness and present an application of the proposed model to generate various citation networks. Section 5 draws some conclusions and future research directions.

II. A MODEL OF NETWORK CONNECTIVITY AND GROWTH

Let $\mathcal{H}_1 = \{1, \ldots, n_1\}$ be a finite set of nodes at generation $k = 1$. Nodes represent elements (acting units) that establish connections to other nodes. We represent the relationship between nodes using a weighted matrix $\mathcal{W}_1 = [w_{ij}]_{n_1 \times n_1}$, where $w_{ij} \in \mathbb{R}_+ = (0, \infty)$ quantifies the relationship between node i and j. If $w_{ij} > 0$, then there exists some kind of action from i to j with weight w_{ij}. It may capture, for instance, the extent to which node i influences node j. Let $\mathcal{G}_k = (\mathcal{H}_k, \mathcal{W}_k)$ represent the network at generation k. For a fixed generation, let $p(i) = \{j : w_{ji} > 0\}$ represent all nodes which influence node i (incoming neighbors). Similarly, let $q(i) = \{j : w_{ij} > 0\}$ represent all nodes influenced by node i (outgoing neighbors). A gain function $g_i(s_i)$ is associated to each node $i \in \mathcal{H}_k$ and characterizes the marginal benefit that results from its current set of influences, where $s_i = \sum_{j \in p(i)} w_{ji}$, $s_i \in \mathbb{R}_+$. Note that s_i is a scalar that represents the *incoming strength* of node i (e.g., the extent to which neighbors $p(i)$ influence node i). The following network assumptions are needed:

- A1 *Finite network strength:* The total weight of the initial network $P_1 = \sum_{i=1}^{n_1} s_i$, $P_1 \in \mathbb{R}_+$, is finite. In other words, the extent to which any node in the network can be influenced by other nodes is bounded.
- A2 *Connectedness:* Every node is influenced to some extent by another node. At each generation k, $s_i \geq \epsilon > 0$, $\forall i \in \mathcal{H}_k$.
- A3 *Bounded marginal gains:* The gain function $g_i(s_i) > 0$ associated to node $i \in \mathcal{H}_k$ satisfies

$$-a_i \leq \frac{g_i(y_i) - g_i(z_i)}{y_i - z_i} \leq -b_i \tag{1}$$

for any $y_i, z_i \in \mathbb{R}_+$, $y_i \neq z_i$ and some constants $a_i \geq b_i > 0$. In other words, the marginal gain associated with each node decreases with increasing strength. Equation (1) eliminates the possibility that a very small difference in node strength may result in an unbounded change in gain.

Assumption A3 captures an inverse relationship between the gain level of a node and its incoming strength (e.g., the attention of a node often degrades at some cost as other nodes attach to it). If for example, $g_i(s_i) = \frac{1}{s_i}(n_i/\delta_i)^{-\beta_i}$ for $\delta_i > 0$, $n_i > 0$, and $\beta_i \in (0, 1)$ for all $i \in \mathcal{H}_k$, the upper and lower bound in (1) are satisfied with $b_i = (n_i/\delta_i)^{-\beta_i} P_k^{-2}$ and $a_i = (n_i/\delta_i)^{-\beta_i} \epsilon^{-2}$, respectively.

Next, we use $t \geq 0$ to specify the time index of events. Let $t = \tau_k$ be the time instant when a new node is added to the network $\mathcal{G}_k$ (i.e., the start of generation k). Let τ_k^+ be the instant right before the new node is added to generation k. When $t = \tau_{k+1}$, $\mathcal{G}_k$ evolves into generation $k + 1$. For a network generation k let the set of states

$$\mathcal{S}_k = \left\{ s \in \mathbb{R}_+^{n_k} : \sum_{i=1}^{n_k} s_i = P_k \right\}$$

be the simplex over which the s_i dynamics evolve. Constraints on our model below will ensure that for all nodes $i \in \mathcal{H}_k$, $s_i(t) \in \mathcal{S}_k$ for all $\tau_k \leq t < \tau_{k+1}$. We assume that as $t \to \tau_k$, $t \to \infty$ for generation k and as $t \to \tau_{k+1}$, $t \to \infty$ for generation $k+1$. Let $s(t) = [s_1(t), \ldots, s_{n_k}(t)]^\top \in \mathcal{S}_k$ be the state vector for generation k at time t (i.e., the incoming distribution of strength of the entire network).

A. Connectivity dynamics

We first focus on the dynamics of $s(t)$ for $\tau_k \leq t < \tau_{k+1}$ (i.e., within a fixed generation). In particular, we want to define a set of states, such that any strength distribution that belongs to this set

$$\mathcal{S}_k^* = \{ s \in \mathcal{S}_k : \text{for all } i, j \in \mathcal{H}_k,\ g_i(s_i) = g_j(s_j) \} \tag{2}$$

represents a distribution where all nodes in $\mathcal{H}_k$ have equal gain levels. To capture the connectivity dynamics that lead to $\mathcal{S}_k^*$, let $e_{\mu(i)}^{\sigma(i)}$ represents the event when node i weakens its relation from some nodes j in $p(i)$ while strengthening its relation to other nodes in $\mathcal{H}_k$ (it may strengthen a relation by either increasing the value of w_{ij} for $j \in q(i)$ or establishing a weight $w_{ij} > 0$ to some new outgoing neighbor $j \in \mathcal{H}_k - q(i)$). Let the list $\sigma(i) = (\sigma_j(i), \sigma_{j'}(i), \ldots, \sigma_{j''}(i))$ such that $j < j' < \cdots < j''$ and $j, j', \ldots, j'' \in \mathcal{H}_k$ be composed of elements $\sigma_r(i)$ that denote the weight to be added to node $r \in \mathcal{H}_k$. For convenience, we will denote this list by $\sigma(i) = (\alpha_j(i) : j \in \mathcal{H}_k)$. Similarly, let the list $(\mu_j(i) : j \in p(i))$ be composed of elements $\mu_r(i)$ that denote the weight to be subtracted from node $r \in p(i)$.

Let $\{e_{\mu(i)}^{\sigma(i)}\}$ denote the set of *all* possible combinations of how node i can weaken or strengthen its relations to other nodes. Let the set of events be described by $\mathcal{E}_1 = \mathcal{P}\left(\{e_{\mu(i)}^{\sigma(i)}\}\right) - \{\emptyset\}$ ($\mathcal{P}(\cdot)$ denotes the power set). We call $e_1(t)$ an event of type 1; they drive the connectivity dynamics within a network generation. Notice that each event $e_1(t) \in \mathcal{E}_1$ is defined as a *set*, with each element of $e_1(t)$ representing the potential rearrangement of multiple weights between nodes. Multiple elements in $e_1(t)$ represent the simultaneous rearrangements among multiple nodes.

An event $e_1(t)$ may only occur if it belongs to the set defined by an enable function $g_{e_1} : \mathcal{S}_k \longrightarrow \mathcal{P}(\mathcal{E}_1) - \{\emptyset\}$. We specify g_{e_1} as follows:

- If for node $i \in \mathcal{H}_k$, $g_i(s_i) \geq g_j(s_j)$ for all $j \in q(i)$, then $e_{\mu(i)}^{\sigma(i)} \in e_1(t)$ such that $\sigma(i) = (0, \ldots, 0)$ and $\mu(i) = (0, \ldots, 0)$ is the only enabled event. Hence, node i does not modify its relationships to others nodes (i.e., the strength of node i does not change).
- If for node $i \in \mathcal{H}_k$, $g_i(s_i) < g_j(s_j)$ for some $j \in q(i)$, then the only $e_{\mu(i)}^{\sigma(i)} \in e_1(t)$ are ones with $\sigma(i) = (\sigma_j(i) : j \in \mathcal{H}_k)$ and $\mu(i) = (\mu_j(i) : j \in p(i))$ such that:

C1 $$\sum_{j \in \mathcal{H}_k} \sigma_j(i) = \sum_{j \in p(i)} \mu_j(i)$$

C2 $$\sigma_{j^*}(i) \geq \frac{1}{a_i} \gamma_{ij^*} \left(g_{j^*}(s_{j^*}) - g_i(s_i) \right)$$

C3 $$\sum_{r \in p(i)} \mu_r(i) \leq \frac{1}{b_i} \left(g_{j^*}(s_{j^*}) - g_i(s_i) \right) - \sigma_{j^*}(i)$$

for some $j^* \in \{j : \ g_j(s_j) \geq g_r(s_r)$, for all $r \in q(i)\}$ and γ_{ij}. The parameter $\gamma_{ij} \in (0, 1)$ regulates the speed at which weights are rearranged and allow us to alter the amount of transitivity between the elements of the network (i.e., if a node j is connected to node j' and node j' to node j'', the probability that node j is also connected to node j''). Low values of γ_{ij} lead to slower convergence processes which increase the probability of forming transitive triples and lead to high clustering coefficients [21].

Condition C1 implies that a node can only establish or strengthen its relations to other nodes by weakening incoming weights (the sum of incoming weights must equal that of outgoing weights). It is required so that C2 and C3 are well defined at all times. To interpret C2 and C3 it is useful to remember that reducing (increasing) the strength of a node always increases (decreases, respectively) the gain at that node. Both conditions constrain how nodes can modify their weights in terms of outgoing neighboring node gains. Condition C2 implies that if the gain of node i differs from any of its outgoing neighbors, then the relation to the neighbor with the highest gain must be strengthened by some amount. Condition C3 implies that when node i weakens incoming weights, node i cannot exceed the highest gain of at least one outgoing neighbor. Together they guarantee that the highest gain node is strictly monotone decreasing over time (as we prove in Theorem 1).

Next, state transitions are defined by the operator $f_{e_1} : \mathcal{S}_k \longrightarrow \mathcal{S}_k$ where $e_1(t) \in \mathcal{E}_1$. For a fixed generation k, if $e_1(t) \in g_{e_1}(s(t))$, $e_{\mu(i)}^{\sigma(i)} \in e_1(t)$, then $s(t+1) = f_{e_1(t)}(s(t))$, where

$$s_i(t+1) = s_i(t) + \sum_{\{j \in \mathcal{H}_k,\, e_{\mu(i)}^{\sigma(i)} \in e_1(t)\}} \sigma_i(j) - \sum_{\{j \in p(i),\, e_{\mu(i)}^{\sigma(i)} \in e_1(t)\}} \mu_j(i) \tag{3}$$

Equation (3) means that the strength at node i at time $t+1$ equals the strength of node i at time t, plus the total weight added by the nodes that strengthened their relationship to node i, minus the total weight reduced by nodes that weakened their relation to node i at time t. Note that (3) implies conservation of network strength for a network generation k so that $P_k = \sum_{i=1}^{n_k} s_i(t)$ is constant. Therefore, if $s(\tau_k) \in \mathcal{S}_k$, then $s(t) \in \mathcal{S}_k$ for $\tau_k \leq t < \tau_{k+1}$.

Let E_1 denote the set of all infinite sequence of events $\mathcal{E}_1$. Let E_t^1 denote the sequence of events $e_1(0), \ldots, e_1(t-1)$ and let the value of the function $S(s(0), E_t^1, t)$ denote the state reached at time t from the initial state $s(0)$ by the application of the sequence E_t^1 of events of type 1. We assume that each event of type 1 occurs infinitely often on each event trajectory $E_t^1 E^1$ within each generation. Note that this assumption is met if nodes persistently try to rearrange weights. The enable function g_{e_1} together with state transition operator f_{e_1} define the evolution of the connectivity dynamics of the network.

B. Growth dynamics

We now turn our attention to the evolution of the network as it grows. To capture a nodes's advantage of longevity let k_i be the generation when node i becomes part of the network and define $n_i = \frac{k_i}{k}$ as the fraction of generations node i has *not* been part of the network component. Moreover, to capture a node's competitive advantage in acquiring weights we associate to every node a fitness β_i, where $\beta_i \in (0, 1)$ is chosen from a random distribution. Let the gain function associated to node $i \in \mathcal{H}_k$ during generation k be defined as

$$g_i(s_i) = \frac{1}{s_i} \left(\frac{\delta_i}{n_i} \right)^{\beta_i} \tag{4}$$

Higher values of β_i characterize nodes that are more attractive in the sense that they can carry more weight without greatly reducing their gain. Both, high values of n_i (representing the fact that node i has been part of the growing network for only a few generations) and low values of β_i (representing the fact that the node has a low competitive advantage for acquiring weights) have a negative effect on the gain of node i. Below we will see how β_i allows us to define the scaling exponent of power-law strength distributions. In particular, Section 3 shows that if $\beta_i = \beta \ \forall i \in \mathcal{H}_k$ then $\alpha = 1/\beta + 1$ represents the linear growth constant in the case of networks that follow preferential strength attachment [19]. Finally, the smoothness parameter $\delta_i > 0$ allows us to quantify the curvature at the start of the power-law distribution (δ_i is chosen from a random distribution as specified below). Associating the parameter δ_i to the nodes allows us to smooth out the curve that establishes the emergence of power-laws.

Let $e^{\sigma(i)}$ represents the attachment of a new node i to the network at the beginning of generation k (when $t = \tau_k$). Let $m = \sum_r \sigma_r(i)$ be the total (constant) weight of a newly added node. A node attaches to the network by (i) randomly distributing $\sigma(i)$ (its weight) across some of the established nodes; and (ii) establishing a non-empty set of incoming neighbors (i.e., some node must attach to it). We call this occurrence an event of type 2. Let $\mathcal{E}_2 = \{e^{\sigma(i)}\}$ denote all possible combinations of how node i can attach to the network component. An event $e_2(k) \in \mathcal{E}_2$ may occur if it is defined by an enable function $g_{e_2} : \mathcal{S}_k \longrightarrow e^{\sigma(i)}$. We specify g_{e_2} as follows:

- Node i attaches to the network only if the associated gain function $g_i(s_i)$ follows the general form of (4).
- Node i has smoothness, longevity, and fitness parameters that satisfy:

C4 $\quad \delta_i \sim \begin{cases} \frac{1}{\Delta\sqrt{\pi}} e^{-(\delta-a)^2/\Delta^2} \text{ as } \Delta \to 0 & \text{, if } a = 1 \\ \frac{1}{\Delta} \text{ for } \delta \in [0, \Delta] \text{ (0 otherwise)} & \text{, if } a = 0 \end{cases}$

C5 $\quad n_i = 1$

C6 $\quad \beta_i = \beta \; \forall i \in \mathcal{H}_k$

where $a \in \{0, 1\}$ captures the emergence of the scaling behavior in the growth of the network. When $a = 1$ the process of emergence of power-law distributions features a sharp knee in the probability cumulative function of the network strength $P[s_i > \omega]$. When $a = 0$ the process leads to a more gradual emergence of power-law distributions. The parameter Δ defines the support of the uniform distribution (here, Δ=1). Condition C5 follows from letting $k_i = k$ for the newly added node i (at generation k node i has been part of network for one generation). Finally, condition C6 specifies an equal fitness for every node (as is the case for networks with linear growth and preferential attachment [11]).

The transition $e_2(k) \in \mathcal{E}_2$ is defined by the operator $f_{e_2} : \mathcal{S}^*_k \longrightarrow \mathcal{S}_{k+1}$. If $e_2(k) \in g_{e_2}(s(\tau_k))$, then $s(\tau_{k+1}) = f_{e_2(k)}(s(\tau^+_k))$ where $s_i(\tau_{k+1}) = m$ only if node i is the newly added node. Let E_2 denote the set of all infinite sequence of events $\mathcal{E}_2$. Let E^2_k denote the sequence of events of type 2, $e_2(1), \dots, e_2(k)$. We assume that each event of type 2 occurs infinitely often on each event trajectory $E^2_k E^2$. The assumption is met if nodes persistently attach to the existing network component. The enable function g_{e_2} together with the transition operator f_{e_2} define the growth dynamics of the network.

III. Analysis

Here, we present stability properties of the invariant set $\mathcal{S}^*_k$ for every generation k. Second, we deduce bounds on the average gain level of the network. We then prove that the strength distribution converges a scaling behavior for values greater than the threshold ω_{min}.

Theorem 1. *(Asymptotic stability)*
Suppose A1-3 *and* C1-3 *hold. Then $\mathcal{S}^*_k$ is an invariant set and has region of asymptotic stability equal to $\mathcal{S}_k$.*

Because $\mathcal{S}^*_k$ is globally asymptotically stable, there is only one desired strength distribution for each group of nodes at every generation k (i.e., $|\mathcal{S}^*_k| = 1$). Thus, for any initial strength distribution Theorem 1 guarantees that $\mathcal{S}^*_k$ will be reached. In particular, for any generation k, initial network state $s(0)$, by applying the event sequence E^1_t, $S(s(0), E^1_t, t) \to \mathcal{S}^*_k$ as $t \to \infty$ for generation k. Let

$$C_k = \frac{1}{n_1 + k} \sum_{i \in \mathcal{H}_k} g_i(s_i(\tau^+_k)) \tag{5}$$

be the average gain of the network an instant before the start of generation $k + 1$.

Lemma 1. *(Equilibrium value)*
Suppose A1-3 *and* C1-6 *hold. Moreover, $\forall k$ let $s(\tau^+_k) \in \mathcal{S}^*_k$. Then the average gain level $C_k \to 1/m(1-\beta)$ when $a = 1$ and $k \to \infty$. The expected average gain level is bounded by $E[C_k] < \left(\frac{1+\beta^2}{1-\beta^2}\right)\frac{1}{m}$ when $a = 0$ and $k \to \infty$.*

Broadly speaking, Lemma 1 implies that if $\delta_i \sim \delta(1)$ when $a = 1$, then at the desired strength distribution $\mathcal{S}^*_k$ every node $i \in \mathcal{H}_k$ has the gain level $C_k \to 1/m(1-\beta)$ as $k \to \infty$. Moreover, Lemma 1 implies that if $\delta_i \sim U(0, 1)$ when $a = 0$, then as $k \to \infty$ every node $i \in \mathcal{H}_k$ has a expected average gain level $E[C_k] < \left(\frac{1+\beta^2}{1-\beta^2}\right)\frac{1}{m}$ at the desired strength distribution $\mathcal{S}^*_k$.

Theorem 2. *(Power-law distribution)*
Suppose A1-3 *and* C1-6 *hold. Moreover, $\forall k$ let $s(\tau^+_k) \in \mathcal{S}^*_k$. Then the strength distribution $P[s_i > \omega]$ of the network $\mathcal{G}_k(\mathcal{H}_k, \mathcal{W}_k)$ follows a power-law with scaling exponent $\alpha = 1/\beta + 1$ as $k \to \infty$. In particular, the scaling behavior holds for values greater than*

i. $\omega_{min} = m(1-\beta)$ *for* $P[s_i > \omega]$ *for distinct emergent distributions* $(a = 1)$*; and*
ii. $E[\omega_{min}] = \frac{1-\beta^2}{1+\beta^2}\frac{m}{2^\beta}$ *for* $P[s_i > \omega]$ *for a gradual emergent distribution* $(a = 0)$.

Theorem 2 implies that as the network grows, it develops a power-law structure, driven by the marginal benefit of the allocation of weights across nodes. It quantifies the value ω_{min} above which the scaling behavior emerges.

IV. Simulations

To gain insight into the connectivity dynamics of the network let $\beta = \frac{1}{2}$, $m = 1$, $n_1 = 2$, and $a = 1$. We let the network grow for 1000 generations. The theoretical prediction for the power-law exponent is $\alpha = 1/\beta + 1 = 3$. Fig. 1 illustrates the clustering coefficient c for various values of $\gamma_{ij} = \gamma$ as the network evolves. Note that the clustering properties remain constant as the network increases in size.

Fig. 2 shows the effect of varying node fitness, where β_i is chosen from a uniform distribution with support $(0, 1)$. The left plot in Fig. 2 shows the time evolution of the node's strength for different values of β_i illustrating that $s_i(\tau^+_k)$ follows a power-law for different values of $\beta_i \sim U(0, 1)$. Because of the node's relative fitness, there are nodes with higher strength s_i but lower longevity advantage n_i. It is possible for a node to join the network at a more recent generation and become more attractive than other nodes that have been part of the network for longer. Here, the node added at generation $k = 105$ with $\beta_{105} = 0.9$ overcomes older nodes with $\beta_{55} = 0.6$, and $\beta_5 = 0.3$. The right plot shows the strength cumulative probability distribution for the

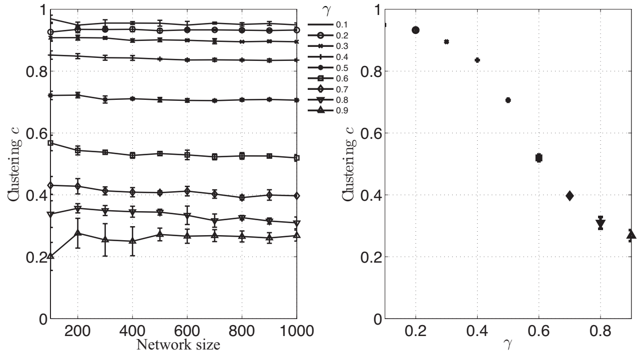

Fig. 1. The left plot shows the clustering coefficient as a function of network size n_k at various values of γ. The right plot shows the clustering coefficient as a function of γ.

entire network, suggesting a power-law with a logarithmic corrective term (similar to the theoretical prediction in [20] where $p_\omega \sim \frac{1}{\log(\omega)}\omega^{-(1+C^*)}$ with $C^* = 1.255$).

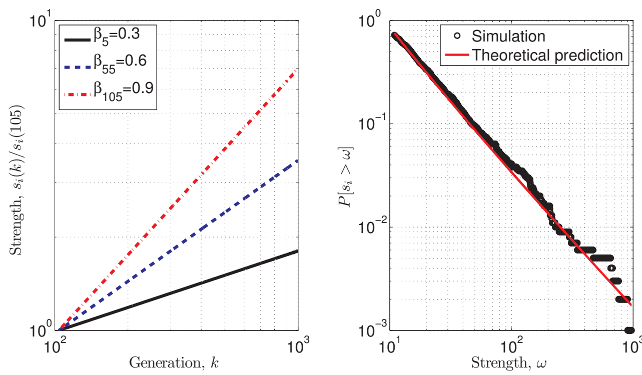

Fig. 2. The left plot shows the evolution on the strength of three nodes added to the network $\mathcal{G}_{1000}$ using fitness $\beta_5 = 0.3$, $\beta_{55} = 0.6$, and $\beta_{105} = 0.9$ from $\beta_i \sim U(0,1)$ with $m = 30$ when $a = 1$. The right plot shows $P[s_i > \omega] \sim Ei(-C^* \log(\omega))$ or $p_\omega \sim \frac{1}{\log(\omega)}\omega^{-(1+C^*)}$ where $Ei(x)$ is the exponential integral function (i.e., a power-law with an inverse logarithmic correction term emerges).

Figs. 3 shows the effect of varying ω_{min} and δ_i on the power-law distribution range. Few real-world networks follow power-laws over their entire range of the distribution [22]. As we increase m we shift the value above which the network obeys a power-law. Fig. 4 shows how the model can capture a slow rate of emergence of power-laws by letting $a = 0$.

Finally, Fig. 5 shows empirical data on the citation distribution of patents granted by the U.S. Patents and Trade Office, and the opinions written by the U.S. Supreme Court and the cases they cited. The left plot in Fig. 5 represents citations on the main subnetwork of U.S. patents granted between January 1963 and December 1999 and references made to these patents between 1975 and 1999 [23]. Finally, the right plot in Fig. 5 shows the majority opinions written by the U.S. Supreme Court and the cases they cite from 1754 to 2002 [24]. For both examples we estimate α^*, ω^*_{min}, and c^* from actual data. We summarize the main attributes in Table I.

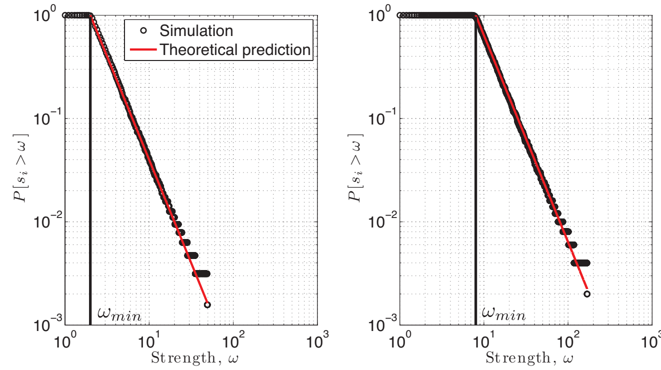

Fig. 3. Cumulative probability distribution with a fixed smoothness parameter for $m = 4$ and $\omega_{min} = 2$ (left); and for $m = 16$ and $\omega_{min} = 8$ (right).

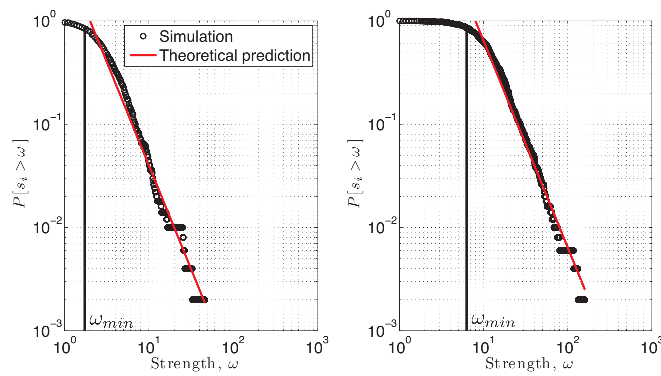

Fig. 4. Cumulative probability distribution with a varying smoothness parameter for $m = 4$ and $\omega_{min} = 1.69$ (left); and for $m = 16$ and $\omega_{min} = 6.79$ (right).

TABLE I

PROPERTIES FOR TWO CITATION NETWORKS

	Actual			Model		
	α^*	ω^*_{min}	c^*	α	ω_{min}	c
Patents	4.68	19 ± 2	0.037	4.63	20	0.044
Court opinions	4.29	55 ± 20	0.107	4.25	60	0.112

V. Discussion

The proposed model generates power-law distributions from consecutive achievements of stable strength distributions $\mathcal{S}^*_k$ and may be of interest in the following context. First, it can be shown that the state $\mathcal{S}^*_k$ is a Nash, which implies that when a network reaches the equilibrium there is not any node that can gain by unilaterally rearranging weights to neighboring nodes (there are no incentives to change or establish new relationships). By focusing on the dynamics that drive the network to $\mathcal{S}^*_k$ we capture the coupling between different nodes, characterizing how relationships between any pair of nodes affects other nodes in the network. Second, the proposed strategies allow us to control the connectivity

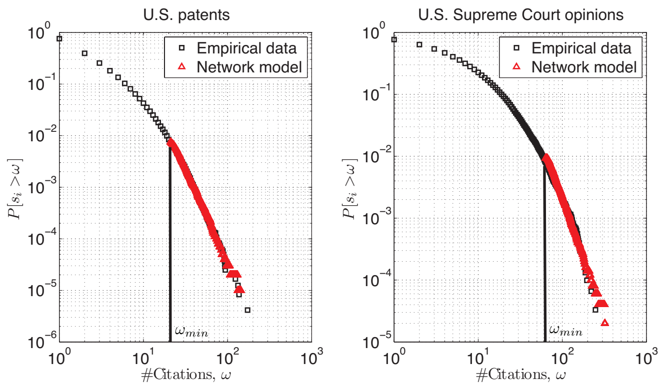

Fig. 5. The left plot shows $P[s_i > \omega]$ for for U.S. patents data found in [23]. The model uses $m = 28$, $\beta = 0.28$, and $\gamma = 0.98$. The right plot shows $P[s_i > \omega]$ for U.S. Supreme Court opinions found in [24]. The model uses $m = 87$, $\beta = 0.31$, and $\gamma = 0.93$.

dynamics of nodes based on local attachment strategies (C1-6), allowing us to generate large network substrates through distributed decision-making. Finally, the ability to control the rate at which attachment strategies lead to the scaling behavior allows us to modify transitivity properties of the network.

We focused on two types of network incentives: (i) Longevity rewards nodes that have been part of the network for a long time (they have the ability to acquire more weight compared to recently added ones); (ii) Fitness rewards nodes that are highly competent (they are more suitable to compete and maintain weights). Modeling nodes with varying fitness allows "latecomers" to overcome nodes that have been in the network for more generations.

Finally, the proposed framework can be extended to generate exponential strength distributions following similar ideas as in Theorems 1 and 2. In particular, if we consider the gain function of the general from $g_i(s_i) = \frac{1}{s_i} \ln \left| \frac{\delta_i}{n_i} + \kappa \right|$ where $\kappa > 0$, the proposed strategies lead to weighted networks with $P[s_i > \omega] \sim e^{-\omega}$. A mathematical framework that allows us to generate various strength distributions for different intervals over an entire distribution range provides an important direction for future research.

VI. Appendix

The proofs of Theorem 1 and 2 can be found in the supplement to the paper available at the authors' websites (http://escher.puj.edu.co/∼pamoriano or http://www.jfinke.org).

References

[1] L. A. N. Amaral and J. M. Ottino, "Complex networks: Augmenting the framework for the study of complex systems," *The European Physical Journal B*, vol. 38, no. 2, pp. 147–162, 2004.
[2] A. Cho, "Ourselves and our interactions: The ultimate physics problem," *Science*, vol. 325, no. 5939, pp. 406–408, 2009.
[3] J. Bohannon, "Counterterrorism's new tool: Metanetwork analysis," *Science*, vol. 325, no. 5939, pp. 409–411, 2009.
[4] H. M. Blalock, *Understanding Social Inequality: Modeling Allocation Processes*. Sage Publications, Newbury Park, 1991.
[5] F. Schweitzer, G. Fagiolo, D. Sornette, F. Vega-Redondo, A. Vespignani, and D. R. White, "Economic networks: The new challenges," *Science*, vol. 325, no. 5939, pp. 422–425, 2009.
[6] E. Ostrom, "A general framework for analyzing sustainability of social-ecological systems," *Science*, vol. 325, no. 5939, pp. 419–422, 2009.
[7] J. D. Farmer and D. Foley, "The economy needs agent-based modelling," *Nature*, vol. 460, no. 7256, pp. 685–686, 2009.
[8] J. Finke, N. Quijano, and K. M. Passino, "Emergence of scale-free networks from ideal free distributions," *Europhysics Letters*, vol. 82, no. 2, p. 28004(6), 2008.
[9] L. Li, D. Alderson, J. C. Doyle, and W. Willinger, "Towards a theory of scale-free graphs: Definition, properties, and implications," *Internet Mathematics*, vol. 2, no. 4, pp. 431–523, 2005.
[10] A. Clauset, C. R. Shalizi, and M. E. J. Newman, "Power-law distributions in empirical data," *SIAM Review*, vol. 51, no. 4, pp. 661–703, 2009.
[11] R. Albert and A.-L. Barabási, "Statistical mechanics of complex networks," *Reviews of Modern Physics*, vol. 74, no. 1, pp. 47–97, 2002.
[12] M. E. J. Newman, "The structure and function of complex networks," *SIAM Review*, vol. 45, no. 2, pp. 167–256, 2003.
[13] L. A. Meyers, M. E. J. Newman, M. Martin, and S. Schrag, "Applying network theory to epidemics: Control measures for mycoplasma pneumoniae outbreaks," *Emerging Infectious Diseases*, vol. 9, no. 2, pp. 204–210, 2003.
[14] N. A. Christakis and J. H. Fowler, "Social network sensors for early detection of contagious outbreaks," *PLoS ONE*, vol. 5, no. 9, p. e12948(8), 2010.
[15] M. Granovetter, "The strength of weak ties," *American Journal of Sociology*, vol. 78, no. 6, pp. 1360–1380, 1973.
[16] Q. Ou, Y.-D. Jin, T. Zhou, B.-H. Wang, and B.-Q. Yin, "Power-law strength-degree correlation from resource-allocation dynamics on weighted networks," *Physical Review E*, vol. 75, no. 2, p. 21102(5), 2007.
[17] A. Barrat, M. Barthelemy, R. Pastor-Satorras, and A. Vespignani, "The architecture of complex weighted networks," *PNAS: Proceedings of the National Academy of Sciences of the United States of America*, vol. 101, no. 11, pp. 3747–3752, 2004.
[18] S. H. Yook, H. Jeong, A.-L. Barabási, and Y. Tu, "Weighted evolving networks," *Physical Review Letters*, vol. 86, no. 24, pp. 5835–5838, 2001.
[19] A. Barratt, M. Barthelemy, and A. Vespignani, "Weighted evolving networks: Coupling topology and weights dynamics," *Physical Review Letters*, vol. 92, no. 22, p. 228701(4), 2004.
[20] G. Bianconi and A.-L. Barabási, "Competition and multiscaling in evolving networks," *Europhysics Letters*, vol. 54, no. 4, pp. 436–442, 2001.
[21] T. Opsahl and P. Panzarasa, "Clustering in weighted networks," *Social Networks*, vol. 31, no. 2, pp. 155–163, 2009.
[22] M. E. J. Newman, "Power laws, Pareto distributions and Zipf's law," *Contemporary Physics*, vol. 46, no. 5, pp. 323–351, 2005.
[23] G. H. Hall, A. B. Jaffe, and M. Trajtenberg, "The NBER patent citation data file: Lessons, insights and methodological tools." NBER Working Papers 8498, National Bureau of Economic Research, October 2001.
[24] J. H. Fowler and S. Jeon, "The authority of Supreme Court precedent," *Social Networks*, vol. 30, no. 1, pp. 16–30, 2008.