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Non-Exponential Stabilization of Linear Time-Invariant Systems by Linear Time-Varying Controllers

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Abstract— **This paper proposes non-exponential stabilization of linear time-invariant systems by linear time-varying controllers. We consider state feedback and dynamic output feedback to make the states of the closed-loop systems decay non-exponentially. We first introduce a non-exponential stability concept that the state of a zero-input time-varying system converges to the origin with an upper bound provided by a specified function. Then, we give non-exponential stabilizability conditions and time-varying controllers to achieve the behavior defined by a desired upper bound function for the closed-loop system. By the proposed methods, we can realize various non-exponential behaviors, which may improve control performance.**

I. INTRODUCTION

It is well known that all the behaviors of linear time-invariant (LTI) systems can be represented by exponential functions [1]. This implies that stabilization of LTI systems by LTI feedback can realize only exponential decays as the behaviors of the stable closed-loop systems. However, if we apply linear time-varying (LTV) feedback, the behaviors of the closed-loop systems are not restricted in the class of exponential functions. This paper proposes non-exponential stabilization of LTI systems by LTV controllers.

The idea of LTV control for LTI systems has been utilized to achieve fast positioning and vibration suppression in mechanical systems [2], [3], where time-varying optimal regulators were extensively used. However, non-exponential behaviors were not recognized, or non-exponential stabilization was not intended explicitly. There is another research considering non-exponential stabilizing LTV controllers [4]. It considered LTV systems and showed that complete controllability is necessary and sufficient for the existence of a state feedback controller which makes the state of the closed-loop system decay non-exponentially with a bound provided by any specified function.

This paper deals with LTI systems and considers non-exponential stabilization by LTV state feedback and dynamic output feedback. We provide methods to obtain controllers to achieve desired closed-loop behaviors. For this aim, we present a concept of stability to represent non-exponential decaying behaviors of LTV systems properly with an upper bound provided by a specified function. This stability concept is in the class of asymptotic stability (AS) [5], but different from the traditional one in the sense that the concept explicitly contain a concrete decaying function, while the traditional AS requires only the existence of a decaying function.

Based on the new concept we deal with non-exponential behaviors of LTV systems. We first derive a stability condition of a Liapunov differential equation type. We show also that under a certain additional condition, the Liapunov equation gives a lower bound of the non-exponential behavior. Then, we consider a state feedback controller and a dynamic output feedback controller for non-exponential stabilization. We derive stabilizability conditions with proposal of design methods of controllers to achieve the desired behaviors of the closed-loop systems. The conditions are given in terms of Riccati differential equations (RDEs). Although it is in general not possible to solve analytically the differential equations with time-varying coefficients, the equations can be solved numerically by utilizing recent digital computers. Time-varying controllers also can be realized and implemented to actual systems by computers without much difficulty. Non-exponential stabilization by the proposed methods is illustrated by numerical examples.

We use the following notation. I_n denotes the $n \times n$ identity matrix. A symmetric matrix $X(t)$ is said to be *strictly positive definite* (*positive semidefinite*) if $X(t) \geq \alpha I$ for a positive α ($X(t) \geq O$) holds for all t. $\|\cdot\|$ denotes the Euclidean norm for a vector and the corresponding induced norm for a matrix. $\mathrm{eig_M}(\cdot)$ denotes the maximum eigenvalue of a symmetric matrix.

II. MOTIVATING EXAMPLES

We present motivating examples in this section. We consider an LTI system and introduce feedback control laws which render solutions of the closed-loop systems behaving non-exponentially.

Example 1. We consider the LTI scalar system

$$\dot{x} = x + u \tag{1}$$

and state feedback $u = k(t)x$ with the time-varying gain $k(t) = -2(t+1)$, $t \geq 0$. Then, the LTV closed-loop system is $\dot{x} = -(2t+1)x$. The solution $x(t)$ of this system for the initial time $t = 0$ and any initial state x_0 is described by

$$x(t) = e^{-t^2-t}x_0, \quad t \geq 0. \tag{2}$$

The behavior of (2) is not exponential, and converges to zero faster than any exponential function.

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Example 2. We consider the LTI scalar system (1) and state feedback $u = k(t)x$ with $k(t) = -1 - (1+t)^{-1}$, $t \geq 0$. Then, the solution $x(t)$ of the LTV closed-loop system $\dot{x} = \{k(t)+1\}x$ for the initial time $t = 0$ and any initial state x_0 is described by

$$x(t) = \frac{1}{1+t}x_0, \quad t \geq 0. \tag{3}$$

The behavior of (3) is not exponential, and converges to zero more slowly than any exponential function.

Example 3. We consider the LTI scalar system (1) and state feedback $u = k(t)x$ with

$$k(t) = -\frac{t^2(3-0.01t^4)}{(1+0.01t^4)^2} - 2, \quad t \geq 0. \tag{4}$$

Then, the solution $x(t)$ of the LTV closed-loop system $\dot{x} = \{k(t)+1\}x$ for the initial time $t = 0$ and any initial state x_0 is described by

$$x(t) = \exp\left(-\frac{t^3}{1+0.01t^4} - t\right)x_0, \quad t \geq 0. \tag{5}$$

The behavior of (5) is faster than any exponential function for small t and converges to zero exponentially for sufficiently large t.

In this paper, we present control methods for non-exponential stabilization of LTI systems by LTV controllers so that we realize desired behaviors of the closed-loop systems, which are broader than the class of exponential functions.

III. Non-Exponential Stability

In this section, we consider the LTV system

$$\dot{x} = A(t)x, \tag{6}$$

where $t \geq 0$ is the time, $x \in \mathbb{R}^n$ is the state variable, and $A(t) \in \mathbb{R}^{n\times n}$ is a time-varying matrix with continuous elements. Then, we introduce a definition of stability, which explicitly represent non-exponential behaviors of the system (6). Since we study the behaviors of the closed-loop systems obtained from LTI systems by LTV controllers after the control actions start, we fix the initial time at 0 without loss of generality.

Definition 1. Consider a continuously differentiable function $\lambda(t)$ defined for $t \geq 0$, which satisfies $\lambda(t) > 0$, $\lambda(0) = 1$, and $\lambda(t) \to 0$ as $t \to \infty$. The zero solution of (6) is said to be λ asymptotically stable for the initial time $t = 0$ (λAS) if there exists a positive constant κ such that any solution $x(t)$ satisfies

$$\|x(t)\| \leq \kappa\|x_0\|\lambda(t), \quad \forall t \geq 0, \tag{7}$$

where $x_0 = x(0)$ is an arbitrary initial state.

The concept of λAS looks the same as that of AS, but is not exactly. In [5], AS of the zero solution for the initial time $t = 0$ is defined as the existence of continuous functions ϕ and σ such that the inequality

$$\|x(t)\| \leq \phi(\|x_0\|)\sigma(t, x_0), \quad \forall t \geq 0 \tag{8}$$

holds, where $\phi(r)$ is strictly increasing in $r \geq 0$ and $\sigma(t, x_0)$ is strictly decreasing in $t \geq 0$ satisfying $\phi(0) = 0$ and $\sigma(t, x_0) \to 0$, $t \to \infty$, respectively. For the linear system (6), the inequality (8) is equivalent to (7). However, the concept of AS is concerned only with the existence of $\phi(\cdot)$ and $\sigma(\cdot,\cdot)$. Neither the estimation of the function $\sigma(\cdot,\cdot)$ in analysis nor control system design under a given $\sigma(\cdot,\cdot)$ has been considered. In this paper, we are interested in the concrete shape of the function $\lambda(\cdot)$, and the concept of λAS explicitly contains a specified $\lambda(\cdot)$. Therefore, λAS depends not only on the behavior of the system but also the specified function $\lambda(t)$.

Although we introduced the function $\lambda(t)$ to represent non-exponential upper bounds of behaviors in (7), we employ an exponential-like expression $\|x(t)\| \leq \kappa\|x_0\|e^{-\delta(t)}$ for (7) in the following discussions, where $\delta(t) = -\ln\lambda(t)$ is a continuously differentiable function and satisfies $\delta(0) = 0$ and $\delta(t) \to \infty$, $t \to \infty$.

We present a sufficient condition for λAS.

Theorem 1: Let $\lambda(t) = e^{-\delta(t)}$ is a given function defined for $t \geq 0$. Then, the zero solution of (6) is λAS if for a continuous positive semidefinite matrix $Q(t)$, there exists a continuously differentiable and strictly positive definite matrix $P(t)$ such that the Liapunov differential equation

$$\begin{aligned}\dot{P}(t) + P(t)\{A(t) + \dot{\delta}(t)I_n\} \\ +\{A(t) + \dot{\delta}(t)I_n\}^{\mathrm{T}}P(t) \;=\; -Q(t).\end{aligned} \tag{9}$$

holds for all $t \geq 0$.

This theorem is implied by the idea of [4]. We omit the proof here, which is the same as the first half of the proof for Theorem 2.

We note that we may replace the differential equation (9) by the differential inequality in which the left side of (9) is less than or equal to 0 in Theorem 1. Although differential inequality conditions are more popular recently, a reason why we adopt the equality condition in the theorem is that the differential equation (9) can be solved numerically, while there is no established method for solving the differential inequality.

There is one more reason why we employ the differential equation (9) rather than the differential inequality. As stated in the following theorem, (9) may give more information of the behavior of the system (6) than (7). We can find a lower bound of the behavior under the additional condition to Theorem 1 that $P(t)$ and $Q(t)$ are bounded.

Theorem 2: Let $\lambda(t) = e^{-\delta(t)}$ is a given function defined for $t \geq 0$. Then, the zero solution of (6) is λAS and any solution $x(t)$ satisfies

$$\kappa_1\|x_0\|\lambda(t)e^{-\rho t} \leq \|x(t)\| \leq \kappa_2\|x_0\|\lambda(t), \;\forall t \geq 0, \tag{10}$$

where κ_1, κ_2, ρ are positive constants, if for a bounded, continuous, and positive semidefinite matrix $Q(t)$, there exists a bounded, continuously differentiable, and strictly positive definite matrix $P(t)$ such that the Liapunov differential equation (9) holds for all $t \geq 0$.

The inequality (10) implies that the behavior of the LTV system (6) is restricted in the band described by $\lambda(t)$ and $\lambda(t)e^{-\rho t}$. This is a desirable property of control systems and can be included in the design methods of LTV controllers given in Section IV.

We present the proof of Theorem 2. As remarked above, the first half proves Theorem 1 as well.

Proof of Theorem 2. We first show that the inequality

$$\|x(t)\| \;\leq\; \kappa_2\|x_0\|\lambda(t) \tag{11}$$

holds from the condition that $P(t)$ is strictly positive definite, $Q(t)$ is positive semidefinite, and the equation (9) holds. We consider the following function as a candidate of the Liapunov function

$$V(t,x) = e^{2\delta(t)}x^{\mathrm{T}}P(t)x. \tag{12}$$

Since $P(t)$ is strictly positive definite, that is, $P(t) \geq \alpha_1 I_n$ for a positive constant α_1,

$$V(t,x) \geq \alpha_1 e^{2\delta(t)}x^{\mathrm{T}}x \tag{13}$$

holds. We compute the total derivative of $V(t,x)$ along the solution $x(t)$ of the system (6). Then, utilizing the differential equation (9) and positive semidefiniteness of $Q(t)$, we see

$$\frac{d}{dt}V(t,x(t)) \leq 0. \tag{14}$$

We integrate this inequality from 0 to t to obtain $V(t,x(t))-V(0,x_0) \leq 0$. From (13) and $\delta(0)=0$, the inequality (11) holds by setting a positive constant

$$\kappa_2 = \sqrt{\alpha_1^{-1}\mathrm{eig_M}\{P(0)\}}. \tag{15}$$

We next show that the inequality

$$\kappa_1\|x_0\|\lambda(t)e^{-\rho t} \;\leq\; \|x(t)\| \tag{16}$$

holds from the condition that $P(t)$ is bounded and strictly positive definite, that is, $\alpha_1 I_n \leq P(t) \leq \alpha_2 I_n$ for positive constants α_1 and α_2, $Q(t)$ is positive semidefinite and bounded, that is, $O \leq Q(t) \leq \beta I_n$ for a positive constant β, and the equation (9) holds. We compute the total derivative of $V(t,x)$ of (12) along the solution $x(t)$ of (6) to obtain

$$\begin{aligned}\frac{d}{dt}V(t,x(t)) \;&=\; -x^{\mathrm{T}}(t)Q(t)x(t)\\ &\geq\; -\alpha_1^{-1}\beta V(t,x(t)),\end{aligned} \tag{17}$$

which implies

$$V(0,x(0))e^{-2\rho t} \leq V(t,x(t)), \tag{18}$$

where $\rho = (1/2)\alpha_1^{-1}\beta$. Then, the inequality (16) holds by setting a positive constant $\kappa_1 = (\alpha_1\alpha_2^{-1})^{1/2}$. Thus, we obtained the inequality (10). This completes the proof of Theorem 2. □

IV. Non-Exponential Stabilization

We deal with the LTI system

$$\dot{x} = Ax + Bu, \quad y = Cx, \tag{19}$$

where $x \in \mathbb{R}^n$ is the state variable, $u \in \mathbb{R}^m$ is the control input, $y \in \mathbb{R}^\ell$ is the measured output, and $A \in \mathbb{R}^{n\times n}$, $B \in \mathbb{R}^{n\times m}$, $C \in \mathbb{R}^{\ell\times n}$ are constant matrices. We present non-exponential stabilization methods by state feedback and dynamic output feedback with stabilizability conditions. In this paper, for a specified function $\lambda(t)$, the system (19) is said to be λ stabilizable if there exists a controller such that the zero solution of the closed-loop system is λAS.

A. State Feedback Stabilization

In this subsection, we assume that all the components of the state x in (19) can be measured, that is, we suppose $C = I_n$. We apply the LTV state feedback

$$u = K(t)x \tag{20}$$

to (19), where $K(t) \in \mathbb{R}^{m\times n}$ is a time-varying matrix with continuous elements. Thus, the closed-loop system is represented as

$$\dot{x} = \{A + BK(t)\}x. \tag{21}$$

Using the condition of Theorem 1, we can derive a λ stabilizability condition and a time-varying feedback gain $K(t)$ as follows.

Theorem 3: Let $\lambda(t) = e^{-\delta(t)}$ is a given function defined for $t \geq 0$. Suppose that for continuous positive semidefinite matrices $Q(t)$ and $R(t)$, there exists a continuously differentiable and strictly positive definite matrix $P(t)$ such that the RDE

$$\begin{aligned}\dot{P}(t) + P(t)\{A + \dot{\delta}(t)I_n\} + \{A + \dot{\delta}(t)I_n\}^{\mathrm{T}}P(t)\\ -P(t)BR(t)B^{\mathrm{T}}P(t) = -Q(t)\end{aligned} \tag{22}$$

holds for all $t \geq 0$. Then, the system (19) is λ stabilizable by the state feedback controller (20) with the feedback gain

$$K(t) = -\frac{1}{2}R(t)B^{\mathrm{T}}P(t). \tag{23}$$

Remark 1: Theoretically, Theorem 3 seems valid for any positive semidefinite $Q(t)$, even for $Q(t) = O$. However, an arbitrary positive semidefinite $Q(t)$ may not generate a strictly positive definite solution $P(t)$ of (22). Therefore, we need to choose $Q(t)$ properly. Such a $Q(t)$ is given as

$$\begin{aligned}Q(t) \;=\; Q_0(t) - \varepsilon\{A + A^{\mathrm{T}} + 2\dot{\delta}(t)I_n\}\\ +\varepsilon^2 BR(t)B^{\mathrm{T}},\end{aligned} \tag{24}$$

where ε is a positive constant and $Q_0(t)$ is a positive semidefinite matrix so that $Q(t)$ is positive semidefinite. Using this $Q(t)$, as the proof of Theorem 1 in [4] implied, it can be shown that for any $\lambda(t)$ and any continuous positive definite $R(t)$, controllability of the system (19) is sufficient for the existence of a strict positive definite $P(t) \geq \varepsilon I_n$ (for more details and derivation of (24), see Appendix A) and thus the system is λ stabilizable. We note, however, that when $\lambda(t)$ is specified, controllability of the system may not be necessary for λ stabilizability. λ stabilizability depends on the specified $\lambda(t)$.

The above choice of $Q(t)$ is an example to obtain a strictly positive definite $P(t)$. We do not need to use it. In Section V, we show another choice in an example.

Remark 2: When the derivative of the specified $\delta(t)$ is bounded, the matrix $Q(t)$ of (24) can be chosen as bounded. In this case, for continuous, bounded, and strictly positive definite matrices $Q(t)$ and $R(t)$, we can have a differentiable, bounded, and strictly positive definite $P(t)$ [7] and realize λ stabilization with a lower bound as in Theorem 2.

B. Output Feedback Stabilization

This subsection considers the case where the measured output y is limited as in (19). We present a stabilizability condition and an LTV output feedback controller described by

$$\dot{\xi} = \hat{A}(t)\xi + \hat{B}(t)y, \quad u = \hat{C}(t)\xi, \tag{25}$$

where $\xi \in \mathbb{R}^n$ is the state variable of the controller, $\hat{A}(t) \in \mathbb{R}^{n\times n}$, $\hat{B}(t) \in \mathbb{R}^{n\times \ell}$, $\hat{C}(t) \in \mathbb{R}^{m\times n}$ are time-varying matrices with continuous elements.

The closed-loop system composed of the system (19) and the controller (25) is represented using the combined state variable $x_{cl} = [\, x^{\mathrm{T}} \ \xi^{\mathrm{T}} \,]^{\mathrm{T}}$ as

$$\dot{x}_{cl} = A_{cl}(t)x_{cl}, \quad A_{cl}(t) = \begin{bmatrix} A & B\hat{C}(t) \\ \hat{B}(t)C & \hat{A}(t) \end{bmatrix}. \tag{26}$$

A commonly used stabilization method for LTI systems by LTI controllers is a combination of state feedback and state estimation. It is well known as the Separation Theorem that stabilizing state feedback gains and observers for state estimation can be designed independently (see, e.g., [6]). That is, any combination of a stabilizing state feedback gain and an observer stabilizes the given system in LTI cases. It may seem that extension of this idea to the case of stabilization of LTI systems by LTV controllers is straightforward, but is not true in non-exponential stabilization. A combination of a state feedback gain for which the zero solution of the closed-loop system is λAS and an observer whose zero solution is λAS does not generally guarantee the same stability property in the total closed-loop system. This fact is shown in Appendix B by a counter-example.

This fact motivates us to derive the following theorem which gives an LTV dynamic output feedback controller for non-exponential stabilization of LTI systems.

Theorem 4: Let $\lambda(t) = e^{-\delta(t)}$ is a given function defined for $t \geq 0$. Suppose that for continuous positive semidefinite matrices $Q_1(t)$, $Q_2(t)$, $R_1(t)$, and $R_2(t)$, there exist matrices $X(t)$, $Y(t)$ such that the following conditions hold.

(i) $X(t)$ is continuously differentiable and strictly positive definite, and satisfies the RDE

$$\begin{aligned} &\dot{X}(t) + X(t)\{A + \dot{\delta}(t)I_n\} + \{A + \dot{\delta}(t)I_n\}^{\mathrm{T}}X(t) \\ &\quad -X(t)BR_1(t)B^{\mathrm{T}}X(t) = -Q_1(t) \end{aligned} \tag{27}$$

for all $t \geq 0$.

(ii) $Y(t)$ is bounded, continuously differentiable, and positive definite, and satisfies the RDE

$$\begin{aligned} &-\dot{Y}(t) + \{A + \dot{\delta}(t)I_n\}Y(t) + Y(t)\{A + \dot{\delta}(t)I_n\}^{\mathrm{T}} \\ &\quad -Y(t)\{C^{\mathrm{T}}R_2(t)C - X(t)BR_1(t)B^{\mathrm{T}}X(t)\}Y(t) \\ &= -Q_2(t) \end{aligned} \tag{28}$$

with the above $X(t)$ for all $t \geq 0$.

Then, the system (19) is λ stabilizable by the output feedback controller (25) with coefficient matrices

$$\begin{aligned} \hat{A}(t) &= A + B\hat{C}(t) - \hat{B}(t)C - Y(t)\hat{C}^{\mathrm{T}}(t)B^{\mathrm{T}}X(t) \\ \hat{B}(t) &= \frac{1}{2}Y(t)C^{\mathrm{T}}R_2(t), \ \hat{C}(t) = -\frac{1}{2}R_1(t)B^{\mathrm{T}}X(t). \end{aligned} \tag{29}$$

The proof is omitted due to page limitations.

Remark 3: The design method of Theorem 4 is composed of two steps. First, we compute the solution $X(t)$ of the RDE (27). As stated in Remark 1, if the system (19) is controllable, we can choose suitable $Q_1(t)$ and $R_1(t)$ to obtain a desirable $X(t)$. Then, using the obtained $X(t)$, we compute the solution $Y(t)$ of the RDE (28). Observability of the system (19) seems sufficient for the existence of a desirable $Y(t)$. However, it has not been shown yet.

Although we have considered non-exponential stabilization of LTI systems here, it is obvious that we can simply and directly extend Theorems 3 and 4 to LTV systems.

V. Numerical Example

We present numerical examples of non-exponential stabilization by Theorems 3 and 4. We consider the LTI system

$$\dot{x} = \begin{bmatrix} 0 & 1 \\ -2 & -2 \end{bmatrix} x + \begin{bmatrix} 0 \\ 1 \end{bmatrix} u, \quad y = \begin{bmatrix} 1 & 0 \end{bmatrix} x. \tag{30}$$

Since the poles of the system are $-1 \pm i$, the zero solution is exponentially asymptotically stable. By the proposed methods, we achieve desired fast decay of the measured output $y(t)$ through the fast decay of the norm $\|x(t)\|$ of the state.

A. State Feedback Case

We present an example of state-feedback non-exponential stabilization of (30) by Theorem 3. We consider the non-exponential behaviors represented by $\lambda_1(t) = \exp(-\delta_1(t))$ and $\lambda_2(t) = \exp(-\delta_2(t))$ with

$$\delta_1(t) = t^3 + t, \quad \delta_2(t) = \frac{t^3}{1 + 0.01t^4} + t, \quad t \geq 0. \tag{31}$$

We note that they are almost identical when t is small.

In both cases $\lambda_1(t)$ and $\lambda_2(t)$, we choose the same matrices $Q(t) = \mathrm{diag}\,[\,0.1,\, 0.1\,]$ and $R(t) = 1$ in the RDE (22). Then, we solve it numerically and obtain time-varying gains by (23).

Figs.1$\sim$5 illustrate the time responses of the norm $\|x(t)\|$ of the state for the initial state $x_0 = [\,1 \ 0\,]^{\mathrm{T}}$, the logarithmically scaled $\|x(t)\|$, the measured output $y(t)$, the computed feedback gain $K(t) = [\, k_1(t) \ k_2(t) \,]$, and the control input $u(t)$, respectively. In the figures, the solid lines and the crosses indicate the behaviors for $\lambda_1(t)$ and $\lambda_2(t)$, respectively. The broken lines in Figs. 1$\sim$3 are exponential behaviors of the original system (30), which are given for comparison.

The results show that we can achieve non-exponential behaviors of the norm $\|x(t)\|$ of the state and fast decays of the measured output $y(t)$ by the proposed LTV controllers in both cases of $\lambda_1(t)$ and $\lambda_2(t)$, which are almost identical in Figs. 1 and 3. We note that the decay of the norm $\|x(t)\|$ of the state is not monotonically decreasing. This phenomenon is unavoidable when the dimension of the input is less than that of the state.

Although the LTV gain for $\lambda_1(t)$ is not bounded, the control input is bounded and almost identical to that generated by the bounded gain for $\lambda_2(t)$. Thus, it is suggested that we should adopt $\lambda_2(t)$ rather than $\lambda_1(t)$ for realistic implementation.

B. *Output Feedback Case*

We consider output-feedback non-exponential stabilization of (30) by Theorem 4. In this example, we adopt the same decaying function $\lambda_2(t)$ as (31).

First, we note that the RDE (27) is identical to (22) of the state feedback case if we replace $Q_1(t)$, $R_1(t)$ with $Q(t)$, $R(t)$. Therefore, by setting $Q_1(t) = \mathrm{diag}\,[0.1,\ 0.1]$ and $R_1(t) = 1$, we use the solution $X(t)$ already computed in above example. Using that $X(t)$, we solve the RDE (28) numerically to obtain $Y(t)$. Here, we choose the matrices $Q_2(t) = O$ and $R_2(t) = 1000\,\|X(t)BB^{\mathrm{T}}X(t)\|$. Then, using the matrices $R_1(t)$, $R_2(t)$, $X(t)$, and $Y(t)$, we obtain a stabilizing output feedback controller (29).

For comparison of the output feedback case with the state feedback case, simulation results are given in Figs. 6, 7, where the initial values x_0 and ξ_0 of the states x of the system and ξ of the controller are set as $x_0 = [1\ 0]^{\mathrm{T}}$ and $\xi_0 = [0\ 0]^{\mathrm{T}}$, respectively. The crosses and the solid lines indicate the state feedback case and the output feedback case, respectively. To magnify the difference between the two cases, we focus on only a short time interval here. We see that the behaviors of the norm $\|x(t)\|$ of the state and the measured output $y(t)$ are not so different in the two cases.

VI. Conclusions

In this paper, we considered non-exponential stabilization of LTI systems by LTV controllers. For this purpose we first proposed the concept of λAS, which can specify various non-exponential upper bounds of initial state responses of LTV systems. Then, we derived stabilization methods by state feedback and dynamic output feedback, which realize specified λAS properties in the closed-loop systems. Since the stabilization methods are given in terms of matrix differential equations, not differential inequalities, we can compute the solutions numerically without much difficulty to obtain controllers.

Acknowledgements

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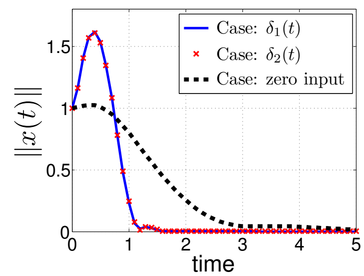

Fig. 1. Norm of the state $x(t)$

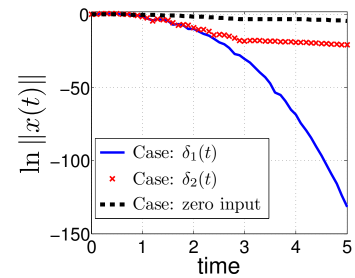

Fig. 2. Norm of the state $x(t)$ (log scale)

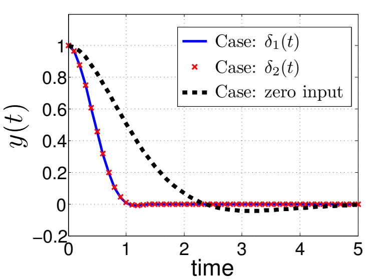

Fig. 3. Measured output $y(t)$

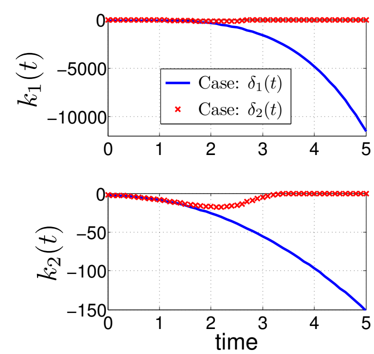

Fig. 4. Time-varying gain $K(t)$

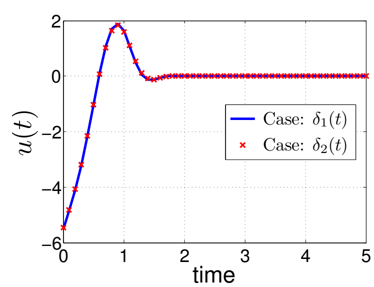

Fig. 5. Control input $u(t)$

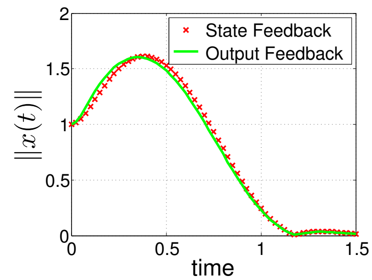

Fig. 6. Norm of the state $x(t)$

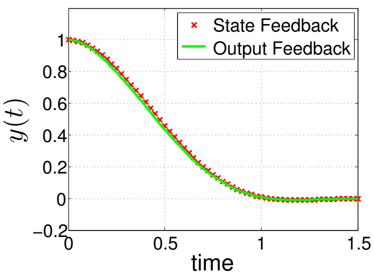

Fig. 7. Measured output $y(t)$

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Appendix

A. Supplement to Remark 1

We show that the choice of $Q(t)$ of (24) and controllability of the system (19) (controllability of the pair $\{A, B\}$) generate a strict positive definite solution $P(t)$ of the RDE (22). We show Remark 1 by the followings (i)∼(iii).

(i) Let $Q_0(t)$ be a positive semidefinite matrix such that

$$Q_0(t) \geq \varepsilon\{A + A^{\mathrm{T}} + 2\dot{\delta}(t)I_n\} - \varepsilon^2 BR(t)B^T \tag{32}$$

holds.

(ii) Solve the RDE

$$\begin{aligned}&\dot{P}_0(t) + P_0(t)\{A + \dot{\delta}(t)I_n - \varepsilon BR(t)B^{\mathrm{T}}\}\\ &\quad +\{A + \dot{\delta}(t)I_n - \varepsilon BR(t)B^{\mathrm{T}}\}^{\mathrm{T}}P_0(t)\\ &\quad -P_0(t)BR(t)B^{\mathrm{T}}P_0(t) = -Q_0(t)\end{aligned} \tag{33}$$

to obtain continuously differentiable and positive semidefinite solution $P_0(t)$. If $\{A, B\}$ is controllable, $\{A+\dot{\delta}(t)I_n - \varepsilon BR(t)B^T, B\}$ is completely controllable [4]. Then, the RDE (33) has a positive semidefinite solution $P_0(t)$ [7].

(iii) Setting $Q(t)$ as

$$\begin{aligned}Q(t) \;=\; & Q_0(t) - \varepsilon\{A + A^{\mathrm{T}} + 2\dot{\delta}(t)I_n\}\\ & +\varepsilon^2 BR(t)B^T,\end{aligned} \tag{34}$$

we see from (33) that

$$\begin{aligned}&\dot{P}_0(t) + \{P_0(t) + \varepsilon I_n\}\{A + \dot{\delta}(t)I_n\}\\ &\quad +\{A + \dot{\delta}(t)I_n\}^{\mathrm{T}}\{P_0(t) + \varepsilon I_n\}\\ &\quad -\{P_0(t) + \varepsilon I_n\}BR(t)B^{\mathrm{T}}\{P_0(t) + \varepsilon I_n\} = -Q(t)\end{aligned} \tag{35}$$

holds. This implies that the RDE (22) has a strictly positive definite solution $P(t) = P_0(t) + \varepsilon I_n$.

B. Counter-Example to Separation Theorem

A combination of a state feedback controller for which the zero solution of the closed-loop system is λAS and an observer whose zero solution is λAS does not generally guarantee the same stability property in the total closed-loop system. We show this fact by the following example.

Counter-Example. We consider non-exponential stabilization specified by

$$\lambda(t) = \frac{1}{1+t}, \quad t \geq 0 \tag{36}$$

for the LTI scalar system

$$\dot{x} = ax + bu, \quad y = cx, \tag{37}$$

where $a = b = c = 1$. We apply an LTV controller composed of the feedback of the estimated state and the observer

$$u = k(t)\xi, \quad \dot{\xi} = \{a - h(t)c\}\xi + bu + h(t)y, \tag{38}$$

where ξ is the estimated state, and $k(t)$ and $h(t)$ are the state feedback gain and the observer gain, respectively. If we set $k(t)$ and $h(t)$ as

$$k(t) = -1 - \frac{1}{1+t}, \quad h(t) = 1 + \frac{1}{1+t}, \quad t \geq 0, \tag{39}$$

then the solutions of the systems $\dot{x}_{c1} = \{a + bk(t)\}x_{c1}$ and $\dot{x}_{c2} = \{a - h(t)c\}x_{c2}$ are described by

$$x_{ci}(t) = \frac{1}{1+t}x_{ci0}, \quad t \geq 0, \quad i = 1, 2 \tag{40}$$

for any initial state $x_{ci0} = x_{ci}(0)$. Therefore, the zero solutions of the state feedback system and the observer are λAS. However, the zero solution of the closed-loop system

$$\begin{bmatrix}\dot{x}\\ \dot{\xi}\end{bmatrix} = \begin{bmatrix}a & bk(t)\\ h(t)c & a + bk(t) - h(t)c\end{bmatrix}\begin{bmatrix}x\\ \xi\end{bmatrix} \tag{41}$$

is neither λAS nor AS. This is seen from the equivalent system

$$\begin{bmatrix}\dot{x}\\ \dot{x} - \dot{\xi}\end{bmatrix} = \begin{bmatrix}-\frac{1}{1+t} & 1 + \frac{1}{1+t}\\ 0 & -\frac{1}{1+t}\end{bmatrix}\begin{bmatrix}x\\ x - \xi\end{bmatrix}. \tag{42}$$

The solution of this system is described by

$$\begin{bmatrix}x(t)\\ x(t) - \xi(t)\end{bmatrix} = \frac{1}{1+t}\begin{bmatrix}1 & t + \ln(1+t)\\ 0 & 1\end{bmatrix}\begin{bmatrix}x_0\\ x_0 - \xi_0\end{bmatrix} \tag{43}$$

for the initial time $t = 0$ and any initial state $[x_0\ x_0 - \xi_0]^{\mathrm{T}} = [x(0)\ x(0) - \xi(0)]^{\mathrm{T}}$. The behavior of (43) does not converge to zero when $t \to \infty$ if $x_0 \neq \xi_0$.