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# Stability conditions for linear continuous time difference systems with discrete and distributed delay

Daniel Melchor-Aguilar

*Abstract*— **In this paper, we consider some classes of continuous time difference systems with dicrete and distributed delay terms. For these infinite-dimensional systems we derive sufficient delay-dependent conditions for the exponential stability by using Lyapunov-Krasovskii functionals.**

**Keywords: continuous time difference systems, exponential stability, Lyapunov-Krasovskii functionals**

## I. INTRODUCTION

Continuous time difference systems play a fundamental role in investigating the stability properties of neutral type delay systems, whereas stability of the difference system is a necessary condition for stability of the corresponding neutral delay system [2], [8]. There are also a number of applications such as in economics, gas dynamics, lossless propagation and models of heredity where the stability of continuous time difference systems is so important [13].

In this context, stability properties of linear continuous time difference systems have been widely studied and several stability conditions based on spectral radius and norm of matrices have been reported [2], [8].

Lyapunov theorems for continuous time difference systems with discrete time delays have been developed in [14], [15] and [16]. As such class of continuous time difference systems can be regarded like *discrete time equations* evolving on an appropriate infinite-dimensional space [2], the results in [14], [15] and [16] propose Lyapunov functions satisfying along solutions a first difference type condition.

However, there are some difficulties in the application of these Lyapunov approaches to more general continuous time difference systems as, for instance, those including both discrete and distributed delay terms. The main reason of this is that the proposed functions are such that their first difference type condition, along solutions of such class of systems, include not only discrete delay terms but also distributed delay terms whose negativity cannot be directly assured.

In this paper, we propose a Lyapunov-Krasovskii approach for investigating the exponential stability of linear continuous time difference systems with discrete and distributed delay terms. We address this problem as a robust stability one. More explicitly, by assuming only stability of the discrete delay part of the system and interpreting the distributed delay term as a perturbation, we present Lyapunov-Krasovskii functionals guaranteeing the exponential stability of the whole system. Our contribution is based on the recent papers [11] and [12], where we have introduced Lyapunov-Krasovskii theorems for special classes of integral delay systems arising in several stability problems of time-delay systems.

The paper is structured as follows: Section II presents the problem formulation. Some preliminary results are provided in section III. Basic facts about solutions are given and Lyapunov-Krasovskii type stability conditions are introduced. The main results are given in section IV. First, we consider a general case for which a simple to check delay-dependent stability condition is derived. Next, a particular case of continuous time difference systems with multiple distributed delay terms and constant system matrices is addressed. In this case, delay-dependent conditions for exponential stability can be expressed in terms of linear matrix inequalities. Several examples illustrating the results are provided in section V. Concluding remarks end the paper.

## II. PROBLEM FORMULATION

Consider the following continuous time difference system

$$x(t) = Ax(t-h) + \int_{-h}^{0} G(\theta)x(t+\theta)d\theta, t \geq 0, \quad (1)$$

where $A \in \mathcal{R}^{n\times n}$ and the matrix function $G(\theta)$ has piecewise continuous bounded elements defined for $\theta \in [-h, 0]$.

Systems of the form of (1) can be found as delay approximations of the partial differential equations for describing the propagation phenomena in excitable media [1], in the stability analysis of additional dynamics introduced by some systems transformations [6], in delay-dependent stability analysis of neutral type systems [4], [9], [10], as well as in the stability analysis of some difference operators in neutral type functional differential equations [2].

For the sake of simplicity of the problem formulation let us consider that the matrix function $G(\theta)$ is a $n\times n$ constant matrix, i.e., $G(\theta) = G, \forall\theta \in [-h, 0]$.

In this case, it is known that (1) is asymptotically stable if the inequality

$$\|A\| + h\|G\| < 1 \quad (2)$$

holds, see for instance [8].

When $G = 0$, the inequality (2) leads to $\|A\| < 1$ which is evidently more restrictive than Schur stability of the matrix $A$ (all eigenvalues of the matrix lie in the open unit disc of the complex plane).

This naturally raises the following question: Is it not possible to obtain less conservative conditions by assuming that matrix $A$ is Schur stable and considering the integral delay term as a perturbation?

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D. Melchor-Aguilar is with División de Matemáticas Aplicadas, IPICYT, 78216 San Luis Potosí, SLP, México dmelchor@ipicyt.edu.mx

Some difficulties occur if we address the problem from the standpoint of existing Lyapunov results for difference systems in continuous time.

First we note that by the change of time $t' = t - h$, the system (1), with $G(\theta) = G, \forall\theta \in [-h, 0]$ , can be written as

$$x(t'+h) = Ax(t') + G\int_{t'}^{t'+h} x(\theta)d\theta, t' \geq h. \quad (3)$$

Assuming that matrix $A$ is Schur stable, there exists a unique positive definite matrix $P$ satisfying the Lyapunov matrix equation

$$A^T PA - P = -Q,$$

where $Q$ is any given positive definite matrix.

Following [14], [15], and [16], when $G = 0$, the function $v(t') = x^T(t')Px(t')$ is a Lyapunov function for the corresponding system. In particular, in this case, we have

$$\begin{aligned}\Delta v(t') &\triangleq v(t'+h) - v(t') \\ &= x^T(t')\left(A^T PA - P\right)x(t') \\ &= -x^T(t')Qx(t').\end{aligned}$$

It is clear that one cannot conclude directly the stability of system (3) by using the function $v(t')$ as a Lyapunov function candidate for the system since its first difference type condition along solutions of (3) includes products of $x(t')$ and $\int_{t'}^{t'+h} x(\theta)d\theta$ which cannot be compensated for negativity of $\Delta v(t')$.

We will introduce below a Lyapunov-Krasovskii approach that will give a positive answer to the above question. The method consists in a combination of the Lyapunov-Krasovskii approach that we recently developed for integral delay systems and stability properties of linear continuous time difference systems with a discrete pure delay.

Throughout this paper, the Euclidean norm for vectors and the induced matrix norm for matrices are used. We denote by $A^T$ the transpose of $A$, $I$ stands for the identity matrix, while $\lambda_{\min}(A)$ and $\lambda_{\max}(A)$ denote the smallest and largest eigenvalues of a symmetric matrix $A$, respectively.

## III. PRELIMINARIES

### A. Solutions and stability concept

In order to determine a particular solution of (1) an initial vector function $\varphi(\theta), \theta \in [-h, 0)$, should be given. We assume that $\varphi$ belongs to the space of continuous vector functions $\mathcal{C}([-h,0), \mathcal{R}^n)$ , equipped with the uniform norm $\|\varphi\|_h = \sup_{\theta\in[-h,0)} \|\varphi(\theta)\|$ .

For a given initial function $\varphi \in \mathcal{C}([-h,0), \mathcal{R}^n)$ , let $x(t,\varphi), t \geq 0$, be the unique solution of (1) satisfying $x(t,\varphi) = \varphi(t), t \in [-h, 0)$ . This solution has a jump discontinuity at $t = 0$ given by

$$\begin{aligned}\Delta x(0,\varphi) &= x(0,\varphi) - x(-0,\varphi) \\ &= A\varphi(-h) + \int_{-h}^{0} G(\theta)\varphi(\theta)d\theta - \varphi(-0).\end{aligned}$$

As a neutral delay system, this discontinuity is propagated along the solution leading to jump discontinuities at time instants multiple of $h$.

Except at the time instants $t = jh, j = 0, 1, 2, ...,$ the solution $x(t, \varphi)$ is a continuous function of $t$. Clearly, if the condition

$$A\varphi(-h) + \int_{-h}^{0} G(\theta)\varphi(\theta)d\theta = \varphi(-0)$$

holds, then the solution $x(t,\varphi)$ is continuous for all $t \geq -h$.

When matrix function $G(\theta)$ is continuously differentiable on the interval $[-h, 0]$, where a right-hand side continuous derivative at $-h$ and a left-hand side continuous derivative at 0 are assumed to exist, the solutions of (1) can be related with particular solutions of some neutral functional differential equations. More precisely, consider the neutral functional differential equation

$$\begin{aligned}\frac{d}{dt}\left[z(t) - Az(t-h)\right] &= G(0)z(t) - G(-h)z(t-h) \\ &\quad - \int_{-h}^{0} \dot{G}(\theta)z(t+\theta)d\theta. \quad (4)\end{aligned}$$

Denote by $z(t,\psi), t \geq 0$, the solution of (4) satisfying $z(t,\psi) = \psi(t), t \in [-h, 0]$, where the initial function $\psi$ belongs to the space of piecewise continuous vector functions $\mathcal{PC}([-h,0], \mathcal{R}^n)$ , see [2].

The following result, which proof is omitted for the sake of brevity, relates the solutions of (1) with some particular solutions of (4).

*Lemma 1:* Assume in (1) that matrix function $G(\theta)$ is continuously differentiable on $[-h, 0]$. For a given initial function $\varphi \in \mathcal{C}([-h,0), \mathcal{R}^n)$ , define the function

$$\psi(\theta) = \begin{cases} \varphi(\theta), \theta \in [-h, 0), \\ A\varphi(-h) + \int_{-h}^{0} G(\theta)\varphi(\theta)d\theta, \theta = 0. \end{cases}$$

Then $x(t,\varphi) = z(t,\psi)$.

*Definition 1:* [2] System (1) is said to be exponentially stable if there exist $\alpha > 0$ and $\mu > 0$ such that any solution of (1) satisfies the inequality

$$\|x(t,\varphi)\| \leq \mu e^{-\alpha t} \|\varphi\|_h, t \geq 0. \quad (5)$$

*Remark 1:* The neutral functional differential equation (4) is not exponentially stable. Indeed, any constant vector is a solution of (4).

The remark implies that, even in the particular case when matrix function $G(\theta)$ is continuously differentiable on $[-h, 0]$, existing stability results for neutral functional differential equations [3] cannot be directly applied to the stability analysis of (1).

It is worth mentioning that similar conclusions can be obtained on the application of the results in [5] for coupled systems described by retarded functional differential equations and functional difference equations to the stability analysis of (1) .

### B. A Lyapunov type theorem

For any $t \geq 0$ we denote the restriction of the solution $x(t,\varphi)$ on the interval $[t-h, t)$ by $x_t(\varphi) = x(t+\theta,\varphi), \theta \in [-h, 0)$ . When the initial function $\varphi$ is irrelevant we simply write $x(t)$ and $x_t$ instead of $x(t,\varphi)$ and $x_t(\varphi)$.

The jump discontinuities of the solutions of (1) imply that $x_t(\varphi)$ belongs to $\mathcal{PC}([-h,0),\mathcal{R}^n)$ for $t \geq 0$. This means that in a Lyapunov-Krasovskii functional setting, the functionals should be defined on the infinite-dimensional space $\mathcal{PC}([-h,0),\mathcal{R}^n)$.

*Theorem 2:* Let system (1) be given and assume that matrix $A$ is Schur stable. System (1) is exponentially stable if there exists a continuous functional $v : \mathcal{PC}([-h,0),\mathcal{R}^n) \to \mathcal{R}$ such that $t \to v(x_t(\varphi))$ is differentiable for all $t \geq 0$ and satisfies the following conditions:

1) $\alpha_1 \int_{-h}^{0} \|\varphi(\theta)\|^2 d\theta \leq v(\varphi) \leq \alpha_2 \int_{-h}^{0} \|\varphi(\theta)\|^2 d\theta$, for some constants $0 < \alpha_1 \leq \alpha_2$,
2) $\frac{d}{dt} v(x_t(\varphi)) \leq -\beta \int_{-h}^{0} \|x(t+\theta,\varphi)\|^2 d\theta$, for some $\beta > 0$.

*Proof:* Given any initial function $\varphi \in \mathcal{C}([-h,0),\mathcal{R}^n)$, it follows from the Theorem conditions that for $2\alpha = \beta\alpha_2^{-1}$ the following inequality:

$$\frac{d}{dt} v(x_t(\varphi)) + 2\alpha v(x_t(\varphi)) \leq 0, \forall t \geq 0,$$

holds. This inequality leads to

$$v(x_t(\varphi)) \leq e^{-2\alpha t} v(\varphi), \forall t \geq 0.$$

Thus it follows that for $t \geq 0$

$$\alpha_1 \int_{-h}^{0} \|x(t+\theta,\varphi)\|^2 d\theta \leq \alpha_2 e^{-2\alpha t} \int_{-h}^{0} \|\varphi(\theta)\|^2 d\theta. \quad (6)$$

From (1) one gets

$$\begin{aligned} & \|x(t,\varphi) - Ax(t-h,\varphi)\|^2 \\ \leq & \left( m_g \int_{-h}^{0} \|x(t+\theta)\| d\theta \right)^2 \\ \leq & m_g^2 h \int_{-h}^{0} \|x(t+\theta)\|^2 d\theta, \end{aligned} \quad (7)$$

where the last inequality has been obtained by using the Cauchy-Schwarz inequality in $\mathcal{L}^2([-h,0),\mathcal{R})$ and

$$m_g = \sup_{\theta\in[-h,0]} \|G(\theta)\|.$$

Combining the inequalities (6) and (7) one obtains

$$\begin{aligned} & \alpha_1 \|x(t,\varphi) - Ax(t-h,\varphi)\|^2 \\ \leq & m_g^2 h \alpha_2 e^{-2\alpha t} \int_{-h}^{0} \|\varphi(\theta)\|^2 d\theta, \end{aligned}$$

which yields the inequality

$$\|x(t,\varphi) - Ax(t-h,\varphi)\| \leq \sqrt{\frac{\alpha_2}{\alpha_1}} m_g h \|\varphi\|_h e^{-\alpha t}.$$

This inequality implies

$$x(t,\varphi) - Ax(t-h,\varphi) = f(t), \quad (8)$$

where $f \in \mathcal{C}([0,\infty),\mathcal{R}^n)$ satisfies

$$\|f(t)\| \leq \mu \|\varphi\|_h e^{-\alpha t}, \forall t \geq 0,$$

with

$$\mu = \sqrt{\frac{\alpha_2}{\alpha_1}} m_g h.$$

Since $A$ is Schur stable, then there exist $\gamma > 0$ and $\sigma > 0$ such that

$$\|A^k\| \leq \gamma e^{-\sigma(kh)}, k = 0,1,2,\ldots.$$

From the Lemma 6 in [7] it follows that the inequality

$$\|x(t,\varphi)\| \leq \eta \|\varphi\|_h e^{-\nu t}, \forall t \geq 0,$$

holds for the solutions $x(t,\varphi)$ of (8) with

$$\begin{aligned} \eta &= \gamma\left(1 + \mu + \frac{\mu}{h e \varepsilon}\right), \\ \nu &= \min\{\sigma,\alpha\} - \varepsilon, \end{aligned}$$

where $\varepsilon \in (0, \min\{\sigma,\alpha\})$. This implies exponential stability of (1). ■

*Remark 2:* In spite of the fact that the state $x_t(\varphi) \in \mathcal{PC}([-h,0),\mathcal{R}^n)$, the Theorem conditions guarantee the exponential stability of (1) by means of *continuous and differentiable* functionals.

## IV. Main Results

In this section, we construct some particular functionals satisfying the conditions of Theorem 2 for the exponential stability of (1).

### A. A general case

We begin with a general case for which a simple-to-check delay-dependent stability condition is derived in the following:

*Proposition 3:* Let system (1) be given and assume that matrix $A$ is Schur stable. If there exist positive definite matrices $W_0$ and $W_1$ such that

$$h\left(\sup_{\theta\in[-h,0]} \|G(\theta)\|\right)^2 < \frac{\lambda_{\min}(W_1)}{\lambda_{\max}\left(P + PAW_0^{-1}A^T P\right)}, \quad (9)$$

with $P$ the positive definite solution of the matrix Lyapunov equation

$$A^T PA - P = -(W_0 + hW_1), \quad (10)$$

then the system (1) is exponentially stable.

*Proof:* Consider the following functional:

$$v(\varphi) = \int_{-h}^{0} \varphi^T(\theta)\left[A^T PA + W_0 + (\theta+h)W_1\right]\varphi(\theta)d\theta, \quad (11)$$

where $P$ is the positive definite solution of (10) and $W_0, W_1$ are positive definite matrices.

The functional (11) satisfies the following inequalities:

$$\alpha_1 \int_{-h}^{0} \|\varphi(\theta)\|^2 d\theta \leq v(\varphi) \leq \alpha_2 \int_{-h}^{0} \|\varphi(\theta)\|^2 d\theta, \quad (12)$$

with $\alpha_1$ and $\alpha_2$ given by

$$\begin{aligned} \alpha_1 &= \lambda_{\min}(W_0), \\ \alpha_2 &= \lambda_{\max}\left(A^T PA + W_0 + hW_1\right). \end{aligned}$$

The integral form of the functional (11) guarantees that along solutions of (1) the function $t \to v(x_t(\varphi))$ is continuous and differentiable for all $t \geq 0$.

The time derivative of the functional (11) along solutions of (1) is

$$\begin{aligned} \frac{dv(x_t)}{dt} &= x^T(t)\left[A^TPA+W_0+hW_1\right]x(t) \\ &-x^T(t-h)\left[A^TPA+W_0\right]x(t-h) \\ &-\int_{-h}^{0} x^T(t+\theta)W_1x(t+\theta)d\theta. \end{aligned}$$

Observing that $A^TPA+W_0+hW_1=P$ and substituting the right-hand side of (1) we have

$$\begin{aligned} \frac{dv(x_t)}{dt} &= x^T(t-h)W_0x(t-h) \\ &+2x^T(t-h)A^TP\int_{-h}^{0}G(\theta)x(t+\theta)d\theta \\ &+\left(\int_{-h}^{0}G(\theta)x(t+\theta)d\theta\right)^T P\left(\int_{-h}^{0}G(\theta)x(t+\theta)d\theta\right) \\ &-\int_{-h}^{0}x^T(t+\theta)W_1x(t+\theta)d\theta. \end{aligned}$$

Using the Jensen inequality (22) (see Appendix) the following inequality:

$$\begin{aligned} &\left(\int_{-h}^{0}G(\theta)x(t+\theta)d\theta\right)^T P\left(\int_{-h}^{0}G(\theta)x(t+\theta)d\theta\right) \\ \le\;& h\int_{-h}^{0}x^T(t+\theta)G^T(\theta)PG(\theta)x(t+\theta)d\theta \end{aligned}$$

holds.

As a consequence we obtain the following upper bound for the derivative:

$$\begin{aligned} \frac{dv(x_t)}{dt} &\le -x^T(t-h)W_0x(t-h) \\ &+2x^T(t-h)A^TP\int_{-h}^{0}G(\theta)x(t+\theta)d\theta \\ &-\int_{-h}^{0}x^T(t+\theta)\left[W_1-hG^T(\theta)PG(\theta)\right]x(t+\theta)d\theta \\ =\;& -\int_{-h}^{0}\left[\begin{array}{cc} x^T(t-h) & x^T(t+\theta)\end{array}\right]\mathcal{N}(\theta)\times \\ &\times\left[\begin{array}{c} x(t-h) \\ x(t+\theta)\end{array}\right]d\theta, \end{aligned}$$

where

$$\mathcal{N}(\theta)=\left[\begin{array}{cc} \frac{1}{h}W_0 & -A^TPG(\theta) \\ -G^T(\theta)PA & W_1-hG^T(\theta)PG(\theta)\end{array}\right].$$

If $\mathcal{N}(\theta)>0, \forall\theta\in[-h,0]$, then there exists $\beta>0$ such that

$$\frac{dv(x_t)}{dt}\le-\beta\int_{-h}^{0}\|x(t+\theta)\|^2 d\theta,$$

and the exponential stability of (1) follows.

By Schur complement, $\mathcal{N}(\theta)>0, \theta\in[-h,0]$, is equivalent to

$$W_1-hG^T(\theta)\left[P+PAW_0^{-1}A^TP\right]G(\theta)>0, \forall\theta\in[-h,0].$$

A sufficient condition for the above inequality is

$$\lambda_{\min}(W_1) -h\lambda_{\max}\left(P+PAW_0^{-1}A^TP\right)\left(\sup_{\theta\in[-h,0]}\|G(\theta)\|\right)^2>0,$$

which leads to (9) and ends the proof. ■

*Remark 3:* $W_0$ and $W_1$ are free positive definite matrices which can be used to improve the right-hand side of the inequality (9).

## *B. A particular case*

Now consider the following class of continuous time difference systems

$$x(t)=Ax(t-h)+\sum_{j=1}^{m}G_j\int_{-h_j}^{0}x(t+\theta)d\theta, \qquad (13)$$

where $0<h_1<h_2<\cdots<h_m=h$, $A\in\mathcal{R}^{n\times n}$ and $G_j\in\mathcal{R}^{n\times n}, j=1,2,\ldots,m.$

The system (13) is a particular case of (1) where

$$G(\theta)=\left\{\begin{array}{ll} G_m, & \theta\in[-h_m,-h_{m-1}), \\ G_m+G_{m-1}, & \theta\in[-h_{m-1},-h_{m-2}), \\ \vdots & \vdots \\ \sum_{j=0}^{m-1}G_{m-j}, & \theta\in[-h_1,0). \end{array}\right.$$

*Proposition 4:* Let system (13) be given and assume that matrix $A$ is Schur stable. If there exists positive definite matrices $W_j, j=0,1,\ldots,m$ such that $\mathcal{M}_j>0, j=1,2,\ldots,m$, where

$$\mathcal{M}_j=\left[\begin{array}{cc} \frac{1}{mh_j}W_0 & -A^TPG_j \\ -G_j^TPA & W_j-mh_jG_j^TPG_j\end{array}\right], \qquad (14)$$

with $P$ the unique positive solution of the matrix Lyapunov equation

$$A^TPA-P=-\left(W_0+\sum_{j=1}^{m}h_jW_j\right), \qquad (15)$$

then the system (13) is exponentially stable.

*Proof:* Consider the functional

$$\begin{aligned} v(\varphi) \;=\; &\int_{-h}^{0}\varphi^T(\theta)\left[A^TPA+W_0\right]\varphi(\theta)d\theta \qquad (16)\\ &+\sum_{j=1}^{m}\int_{-h_j}^{0}\varphi^T(\theta)\left(\theta+h_j\right)W_j\varphi(\theta)d\theta, \end{aligned}$$

where $P$ is the positive definite solution of (15) and $W_j, j=0,1,\ldots,m$ are positive definite matrices.

From (16) we get the following inequalities for the functional:

$$\alpha_1\int_{-h}^{0}\|\varphi(\theta)\|^2d\theta\le v(\varphi)\le\alpha_2\int_{-h}^{0}\|\varphi(\theta)\|^2d\theta,$$

with $\alpha_1$ and $\alpha_2$ given by

$$\begin{aligned} \alpha_1 &= \lambda_{\min}(W_0), \\ \alpha_2 &= \lambda_{\max}\left(A^TPA+W_0\right)+\sum_{j=1}^{m}\lambda_{\max}\left(h_jW_j\right). \end{aligned}$$

Note again that the integral form of the functional (16) guarantees that along solutions of (13) the function $t \to v(x_t(\varphi))$ is continuous and differentiable for all $t \geq 0$.

The time derivative of the functional (16) along solutions of (13) is

$$\frac{dv(x_t)}{dt} = x^T(t)\left[A^TPA + W_0 + \sum_{j=1}^{m} h_j W_j\right]x(t)$$
$$-x^T(t-h)\left[A^TPA + W_0\right]x(t-h)$$
$$-\sum_{j=1}^{m}\int_{-h_j}^{0} x^T(t+\theta)W_j x(t+\theta)d\theta.$$

Using the fact that $P = A^TPA + \left(W_0 + \sum_{j=1}^{m} h_j W_j\right)$ and substituting the right-hand side of (13) we get

$$\frac{dv(x_t)}{dt} = -x^T(t-h)W_0x(t-h)$$
$$+2x^T(t-h)A^TP\sum_{j=1}^{m} G_j \int_{-h_j}^{0} x(t+\theta)d\theta$$
$$-\sum_{j=1}^{m}\int_{-h_j}^{0} x^T(t+\theta)W_j x(t+\theta)d\theta$$
$$+\left(\sum_{j=1}^{m} G_j \int_{-h_j}^{0} x(t+\theta)d\theta\right)^T P \times$$
$$\times\left(\sum_{j=1}^{m} G_j \int_{-h_j}^{0} x(t+\theta)d\theta\right).$$

Using the Jensen inequalities (23) and (22) in the Appendix, in that order, the following inequality:

$$\left(\sum_{j=1}^{m} G_j \int_{-h_j}^{0} x(t+\theta)d\theta\right)^T P \times$$
$$\times\left(\sum_{j=1}^{m} G_j \int_{-h_j}^{0} x(t+\theta)d\theta\right)$$
$$\leq \quad m\sum_{j=1}^{m} h_j \int_{-h_j}^{0} x^T(t+\theta)G_j^TPG_j x(t+\theta)d\theta,$$

holds. As a consequence we obtain the following upper bound for the derivative:

$$\frac{dv(x_t)}{dt} \leq -x^T(t-h)W_0x(t-h)$$
$$+2x^T(t-h)A^TP\sum_{j=1}^{m} G_j \int_{-h_j}^{0} x(t+\theta)d\theta$$
$$-\sum_{j=1}^{m}\int_{-h_j}^{0} x^T(t+\theta)\left[W_j - mh_jG_j^TPG_j\right] \times$$
$$\times x(t+\theta)d\theta$$

which can be rewritten as

$$\frac{dv(x_t)}{dt} \quad\leq\quad -\sum_{j=1}^{m}\int_{-h_j}^{0}\left[\begin{array}{cc} x^T(t-h) & x^T(t+\theta)\end{array}\right]\mathcal{M}_j \times$$
$$\times\left[\begin{array}{c} x(t-h) \\ x(t+\theta)\end{array}\right]d\theta,$$

where $\mathcal{M}_j$, $j = 1, 2, \ldots, m$, are defined by (14).

If $\mathcal{M}_j > 0, j = 1, 2, \ldots, m$, it follows that there exists $\beta > 0$ such that

$$\frac{dv(x_t)}{dt} \leq -\beta\int_{-h}^{0}\|x(t+\theta)\|^2 d\theta,$$

and the exponential stability of (13) is assured. ■

*Remark 4:* Writing the matrix Lyapunov equation (15) as

$$A^TPA - P + \left(W_0 + \sum_{j=1}^{m} h_j W_j\right) < 0 \qquad (17)$$

a linear matrix inequality solver can be used to find a feasible solution $P, W_j, j = 0, 1, \ldots, m$ of the matrix inequalities $\mathcal{M}_j > 0, j = 1, 2, \ldots, m$ and (17). See Example 3 below.

## V. Illustrative Examples

*Example 1:* Consider the system

$$x(t) = Ax(t-h) + G\int_{-h}^{0} x(t+\theta)d\theta, \qquad (18)$$

where

$$A = \left(\begin{array}{cc} 0 & 1 \\ 0.01 & 0\end{array}\right).$$

Matrix $A$ is Schur stable and $\|A\| = 1$. Then, the known results cannot allow us to investigate the stability of (18) since the inequality $\|A\| + h\|G\| < 1$ does not hold for any delay value $h \geq 0$ and matrix $G \in \mathcal{R}^{2\times 2}$.

For this case by using Proposition 3 we are able to get stability conditions for (18). For instance consider that $h = 1$ and select $W_0 = I$ and $W_1 = 1.2I$. For these values, the unique positive definite solution of the matrix Lyapunov equation (10) is

$$P = \left(\begin{array}{cc} 4.4004 & 0 \\ 0 & 2.2004\end{array}\right).$$

From the inequality (9) we get that (18) is exponentially stable for all system matrices $G \in \mathcal{R}^{2\times 2}$ such that $\|G\| < 0.2247$.

*Example 2:* Consider again the system (18) but now with

$$A = \left(\begin{array}{cc} 0.2 & 0 \\ -0.1 & -0.2\end{array}\right).$$

In this case, the matrix $A$ is Schur stable and $\|A\| = 0.2562$.

For $h = 0.5$ the inequality $\|A\| + h\|G\| < 1$ yields stability of (18) for all matrices $G \in \mathcal{R}^{2\times 2}$ such that $\|G\| < 1.4877$. Our Proposition 3 leads to a less restrictive condition $\|G\| < 1.4902$ obtained from the inequality (9) with $W_0 = I$, $W_1 = 7.7I$ and

$$P = \left(\begin{array}{cc} 5.0521 & -0.0972 \\ -0.0972 & 5.1007\end{array}\right)$$

as the positive definite solution of the matrix Lyapunov equation (10).

*Example 3:* Consider now a continuous time difference system (18) where

$$A=\begin{pmatrix} -0.2 & 0 \\ 0.2 & -0.1 \end{pmatrix} \text{ and } G=\begin{pmatrix} -1.1 & -0.2 \\ -0.1 & -1.1 \end{pmatrix}. \tag{19}$$

This system is found in Example 2 of [4] and [10] as a difference operator of a neutral type system with discrete and distributed delay terms. Such neutral system is obtained by a model transformation technique which transform the original neutral system with discrete delay to a neutral system with distributed delay for delay-dependent stability conditions, see [10] for details.

From Proposition 4 and Remark 4 we have that system (18) is exponentially stable if there exists positive definite matrices $P, W_0$ and $W_1$ such that

$$\begin{bmatrix} \frac{1}{h}W_0 & -A^TPG \\ -G^TPA & W_1-hG^TPG \end{bmatrix} > 0, \tag{20}$$

$$A^TPA-P+(W_0+hW_1) < 0. \tag{21}$$

We found a feasible solution of the matrix inequalities (20) and (21) for all delay values $0 \le h \le 0.7435$ that clearly improves the result obtained from the inequality $\|A\|+h\|G\|<1$ which leads to $0 \le h < 0.5658$.

Thus, for $h = 0.7435$ we obtain the following solution for (20) and (21):

$$W_0 = \begin{pmatrix} 6.5250 & -1.4497 \\ -1.4497 & 5.3856 \end{pmatrix},$$
$$W_1 = \begin{pmatrix} 58.7883 & 29.9845 \\ 29.9845 & 65.8883 \end{pmatrix},$$
$$P = \begin{pmatrix} 52.9422 & 20.1481 \\ 20.1481 & 54.9261 \end{pmatrix}.$$

## VI. Concluding Remarks

In this paper, a Lyapunov-Krasovskii functional approach for the exponential stability of linear continuous time difference systems with discrete and distributed delays is introduced. The obtained conditions are less conservative than existing ones. Consequently, the results reported in this contribution can help to improve existing stability conditions of neutral type delay systems having this class of continuous difference systems as difference operators. In particular, the results can serve to reduce the conservatism of delay-dependent stability conditions for neutral type systems with discrete delay which are obtained by means of system transformations.

## VII. Appendix

*Lemma 5:* (Jensen Inequality) For any constant matrix $M \in \mathcal{R}^{n\times n}$, $M = M^T > 0$, vectors $\xi_j \in \mathcal{R}^n, j = 1,2,\ldots,m$, scalar $\gamma > 0$ and vector function $\omega : [0,\gamma] \to \mathcal{R}^n$ such that the integrations concerned are well defined, then

$$\gamma\int_0^\gamma \omega^T(\beta)M\omega(\beta)d\beta \ge \left(\int_0^\gamma \omega(\beta)d\beta\right)^T M \left(\int_0^\gamma \omega(\beta)d\beta\right), \tag{22}$$

$$m\sum_{j=1}^m \xi_j^T M\xi_j \ge \left(\sum_{j=1}^m \xi_j\right)^T M \left(\sum_{j=1}^m \xi_j\right). \tag{23}$$

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