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A new smallest sigma set for the Unscented Transform and its applications on SLAM

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***Abstract*— In this work we propose a new set of sigma points for the Unscented Transform that uses the minimum number of points. We than compare this new set with the symmetric set, the reduced set, and the spherical set. Simulations comparing this sets are done to verify the properties of this set and to verify their transforms. Lastly, we simulate each of these sets in a recursive filter for SLAM. The results show that our set is a better choice for a non symmetric prior distribution and still a good alternative for symmetric prior distributions.**

I. Introduction

Based on the intuition that it is *it is easier to approximate a probability distribution than it is to approximate an arbitrary nonlinear function or transformation* [1], Julier *et all* [1]–[4] proposed a non-linear estimation technique which was lately called the *Unscented Transform* (UT).

The Unscented Transform has been extensively used on the Simultaneous Localization and Map Building (SLAM). To cite some, one can check [5]–[15].

However, the Unscented's computational load is proportional to the number of sigma points which is, in its turn, increasingly related to the state's dimension [16]. SLAM's problems usually have a huge state's dimension, for it is the sum of the dimension of the robot's pose and the dimensions of all the coordinates of the map's landmarks. Moreover, the computational cost can be a tough requirement for a robot, because of it's processing capacities or it's power consumption, since it uses an embedded hardware in a great number of cases. Therefore, reducing the computational cost in a SLAM problem is especially important and, in consequence, reducing the number of sigma points is also mainly important.

As the symmetric sigma sets of Julier use $2n + 1$[1] sigma points, it is extremely desirable to find a set of sigma set that uses less points. In particular, $n + 1$, for it is the minimum number of points that can be used to estimate both mean and covariance matrix [16]. In this direction, [16] proposed a $n + 1$ scheme and [17] offered a $n + 2$ set which $n + 1$ of them lie on a hypersphere [17]. [16] has a specific problem of stability for high values of n [17]. Additionally, both algorithm's have the drawback that their mean and covariance matrix are not equal to the mean and covariance matrix of the *prior* distribution, which is a very important property for the unscented framework, for it implies, with the condition of the differentiability of the non-linear function, that the mean and the covariance matrix of the posterior random variable are estimated up to the second order of their Taylor Series [1].

Our efforts go precisely on the direction of finding a minimum set of sigma points which captures both the mean and covariance matrix of the prior distribution and to apply it in a SLAM's framework. Our set accomplishes these properties and does not present the instability issues of [16]. These properties are verified on simulations. Finally, a comparison is made by simulating the recursive filter using our new set, the symmetric set, the set of [16] and the set of [17]. The results show that our set of $n + 1$ sigma points is a good alternative for the problem of SLAM.

This work is organized as follows: section II provides a background on the Unscented Transform, on the Unscented Kalman Filter and introduces the new sigma set; section III shows the simulations examples and section IV provides the conclusions.

II. The Sigma sets

A. The previous sigma sets

Consider that $X \in \Re^n$ is a vector of random variables and consider that $f : \Re^n \rightarrow \Re^m$ is a transformation that defines Y as follows:

$$Y = f(X).$$

The Unscented Transform approximates the probability density function (pdf) of the prior random variable (RV) - in this case X - by a group of points - called sigma points - which are obtained in a deterministic fashion. From another point of view, one can consider this transformation as an approximation of the random variable's pdf by a probability mass function. Hence, the UT can be seen as a discrete approximation of a random variable.

One important advantage of this non-linear estimation technique is that it does not require the calculations of the Jacobians or Hessians of the nonlinear function as the techniques based on the linearization do [1].

Another important property of the Unscented Transform is that if its sample central moments are equal to the central moments of the prior RV up to the $2k$-th order inclusively, the Taylor Series of the sample mean and of the sample covariance matrix of the transformed points will be equal to the Taylor Series of the mean and of the covariance matrix, respectively, of the posterior RV up to the $2k$-th order inclusively [1].

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[1] n is the state's dimension.

Now, we present three of the sigma points proposed in the literature.

1) The symmetric set: The symmetric set has more than one format, which are the ones of [1]–[4]. Here, we choose the one of [1]. For $i = 1, \ldots, n$, this set can be written as follows:

$$\begin{aligned}
\chi_0 &= \bar{X}, \\
\chi_i &= \bar{X} + \left(\sqrt{\frac{n}{1-w_0}P_{XX}}\right)_i, \\
w_i &= \frac{1-w_0}{2n}, \\
\chi_{i+n} &= \bar{X} - \left(\sqrt{\frac{n}{1-w_0}P_{XX}}\right)_i, \\
w_{i+n} &= \frac{1-w_0}{2n},
\end{aligned} \tag{1}$$

in which $w_0 \neq 1$ and $(A)_i$ represents the i-th column or the line of the matrix A [2].

2) The reduced set: The reduced set of Julier presented in [16] can be described by the following algorithm:

1) Choose w_0, regarding $0 \leq w_0 \leq 1$.
2) Calculate the weights:

$$w_i = \begin{cases} \frac{1-w_0}{2^n} & , for\ i = 1; \\ w_1 & , for\ i = 2; \\ 2^{i-1}w_1 & , for\ i = 3, \ldots, n+1. \end{cases}$$

3) Initiate the vector sequence χ_i^j:

$$\chi_0^1 = [0]; \quad \chi_1^1 = \left[-\frac{1}{\sqrt{2w_1}}\right]; \quad \chi_2^1 = \left[\frac{1}{\sqrt{2w_1}}\right].$$

4) Expand the vector sequence for $j = 1, \ldots, n$ according to

$$\chi_i^{j+1} = \begin{cases} \begin{bmatrix} \chi_0^j \\ 0 \end{bmatrix} & , for\ i = 0; \\ \begin{bmatrix} \chi_i^j \\ -\frac{1}{\sqrt{2w_j}} \end{bmatrix} & , for\ i = 1, \ldots, j; \\ \begin{bmatrix} 0_j \\ \frac{1}{\sqrt{2w_j}} \end{bmatrix} & , for\ i = j+1. \end{cases}$$

3) The spherical set: The $n+2$ spherical set of Julier presented in [17] can be described by the following algorithm:

1) Choose w_0, regarding $0 \leq w_0 \leq 1$.
2) Calculate the weights:

$$w_i = \frac{1-w_0}{n}.$$

3) Initiate the vector sequence χ_i^j:

$$\chi_0^1 = [0]; \quad \chi_1^1 = \left[-\frac{1}{\sqrt{2w_1}}\right]; \quad \chi_2^1 = \left[\frac{1}{\sqrt{2w_1}}\right].$$

4) Expand the vector sequence for $j = 2, \ldots, n$ according to

$$\chi_i^j = \begin{cases} \begin{bmatrix} \chi_0^{j-1} \\ 0 \end{bmatrix} & , for\ i = 0; \\ \begin{bmatrix} \chi_i^{j-1} \\ -\frac{1}{\sqrt{j(j+1)w_1}} \end{bmatrix} & , for\ i = 1, \ldots, j; \\ \begin{bmatrix} 0_{j-1} \\ \frac{1}{\sqrt{j(j+1)w_1}} \end{bmatrix} & , for\ i = j+1. \end{cases}$$

B. The new minimum sigma set

The new transform can be resumed in the following theorem.

Theorem 1 *Let $X \in \Re^n$ be a random variable with mean $\bar{X}$ and covariance matrix $P_{XX} > 0$ with its matrix square root $\sqrt{P_{XX}}$, and let be the set of points and weights, $\{\chi_i, w_i\}, i = 1, \ldots, n+1$ with the following form[3]:*

$$A_\chi = \begin{bmatrix} \chi_0 & \cdots & \chi_n \end{bmatrix} \tag{2}$$

$$:= \begin{bmatrix} -\sqrt{P_{XX}}\frac{[\alpha]_{n\times 1}}{\sqrt{w_0}} & \sqrt{P_{XX}}C\left(\sqrt{W}\right)^{-1} \end{bmatrix} + \left[\bar{X}\right]_{1:n+1}, \tag{3}$$

in which

$$W := \begin{bmatrix} w_1 & 0 & 0 \\ 0 & \ddots & 0 \\ 0 & 0 & w_n \end{bmatrix}, \tag{4}$$

$$\begin{bmatrix} w_1 & \cdots & \sqrt{w_1}\sqrt{w_n} \\ \vdots & \ddots & \vdots \\ \sqrt{w_1}\sqrt{w_n} & \cdots & w_n \end{bmatrix} := w_0\alpha^2 C^{-1}[1]_{n\times n}\left(C^T\right)^{-1}, \tag{5}$$

$$C := \sqrt{I_n - \alpha^2[1]_{n\times n}}, \tag{6}$$

$$\alpha := \sqrt{\frac{1-w_0}{n}}, \tag{7}$$

$$0 < w_0 < 1. \tag{8}$$

Let, still, be the non-linear mapping $f : \Re^n \mapsto \Re^m$ differentiable up, at least, to the second order. Still, let f define the random variable Y according to

$$Y \triangleq f(X),$$

and let be the set of points and weights $\{\gamma_i, w_i | \gamma_i = f(\chi_i)\}$, the following statements are true:

[2]If the matrix square root A of P is of the form $P = A^T A$, then the sigma points are formed from the rows of A. However, if the matrix square root is of the form $P = AA^T$, the columns of A are used.

[3]The notation $[a]_{p\times q}$, with $a \in \Re$, represents a matrix of dimension $p \times q$ which all of its terms are equal to a. In its turn, $[v]_{1:q}$, with $v \in \Re^n$, represents a matrix of dimension $n \times q$ which all the columns are equal to v.

1) *The sample mean of* $\{\chi_i, w_i\}$, $\mu_\chi = \sum_{i=0}^{n} w_i \chi_i$, *is equal to the mean of* X.
2) *The sample covariance matrix of* $\{\chi_i, w_i\}$, $\Sigma_{XX} = \sum_{i=0}^{n} w_i (\chi_i - \mu_\chi)(\chi_i - \mu_\chi)^T$, *is equal to the covariance matrix of* X.
3) *The Taylor Series of the sample mean of* $\{\gamma_i, w_i\}$ *is equal to the Taylor Series of the mean of* Y *up to the second order inclusive.*
4) *The Taylor Series of the sample covariance matrix of* $\{\gamma_i, w_i\}$ *is equal to the Taylor Series of the covariance matrix of* Y *up to the second order inclusive.*

Proof: According to the Taylor Series expansions of the mean and the covariance of $\{\gamma_i, w_i\}$ and of Y, one can see that if the first two items of this theorem are proved, the remaining items will be also proven [1].

Let us than firstly prove item 1 and in sequence item 2. The sample mean of $\{\chi_i, w_i\}$, μ_χ, can be written as:

$$\begin{aligned}
\mu_\chi &= A\chi \begin{bmatrix} w_0 \\ \vdots \\ w_n \end{bmatrix} \\
&+ \left[\bar{X}\right]_{1:n+1} \begin{bmatrix} w_0 \\ \vdots \\ w_n \end{bmatrix} \\
&= -\sqrt{P_{XX}} \frac{[\alpha]_{n\times 1}}{\sqrt{w_0}} w_0 + \sqrt{P_{XX}} C \left(\sqrt{W}\right)^{-1} \begin{bmatrix} w_1 \\ \vdots \\ w_n \end{bmatrix} \\
&+ \bar{X} \\
&= \sqrt{P_{XX}} \left(-\sqrt{w_0 [\alpha]_{n\times 1}} + C \begin{bmatrix} \sqrt{w_1} \\ \vdots \\ \sqrt{w_n} \end{bmatrix} \right) + \bar{X}.
\end{aligned} \tag{9}$$

Substituting (7) in (9) and using (6) the mean's proof will be accomplished..

Now it remains to prove that the sample covariance matrix of $\{\chi_i, w_i\}$, Σ_{XX}, is equal to the covariance matrix of X. For this, define

$$W^a = \begin{bmatrix} w_0 & 0 \\ 0 & W \end{bmatrix}. \tag{10}$$

Than, from the definition of sample covariance matrix:

$$\begin{aligned}
\Sigma_{XX} &\triangleq \\
&= \left[A_\chi - [\mu_\chi]_{1:n}\right] W^a \left[A_\chi - [\mu_\chi]_{1:n}\right]^T. \\
&= \left[A_\chi - \left[\bar{X}_\chi\right]_{1:n}\right] W^a \left[A_\chi - \left[\bar{X}\right]_{1:n}\right]^T.
\end{aligned} \tag{11}$$

Using (3) and (10) on (11)[4]:

$$\begin{aligned}
\Sigma_{XX} &= \left(\left[\begin{array}{cc} -\sqrt{P_{XX}} \frac{[\alpha]_{n\times 1}}{\sqrt{w_0}} & \sqrt{P_{XX}} C \left(\sqrt{W}\right)^{-1} \end{array} \right] \right. \\
&\left. \sqrt{\begin{bmatrix} w_0 & 0 \\ 0 & W \end{bmatrix}} \right) (*)^T \\
&= \sqrt{P_{XX}} C \left(\sqrt{W}\right)^{-1} W \left(\left(\sqrt{W}\right)^{-1}\right)^T C^T \sqrt{P_{XX}}^T \\
&+ \sqrt{P_{XX}} \frac{[\alpha]_{n\times 1}}{\sqrt{w_0}} \frac{[\alpha]_{1\times n}}{\sqrt{w_0}} \sqrt{P_{XX}}^T \\
&= \sqrt{P_{XX}} C C^T \sqrt{P_{XX}}^T + \alpha^2 \sqrt{P_{XX}} [1]_{n\times n} \sqrt{P_{XX}}^T \\
&= \sqrt{P_{XX}} \left[I_n - \alpha^2 [1]_{n\times n}\right] \sqrt{P_{XX}}^T \\
&+ \alpha^2 \sqrt{P_{XX}} [1]_{n\times n} \sqrt{P_{XX}}^T \\
&= P_{XX}.
\end{aligned}$$

■

[4]The expression $(QP)(*)^T$ is equal to $(QP)(QP)^T$.

In the Theorem 1, the condition (7) assures that the weights' sum equals the unit. The weight w_0, which is restricted to (8), gives a degree of freedom on the parameters' choice. A study of a particular choice of this parameter has not been yet studied. One could try to, for example, choose a value that minimizes the difference on the third moments. Furthermore, simulations results made for diverse prior distributions and various functions indicate good robustness of this sigma set (see [18]).

In comparison to the symmetric set, our new sigma set has the advantage of using only $n+1$ sigma points, which is the minimum possible amount. Both these sigma sets share the properties of having their sample mean and covariance matrix equal to the mean and covariance matrix of the prior distribution which result, with the condition of the differentiability of the non-linear function, in the ability to estimate the mean and the covariance matrix of the posterior random variable (RV) up to the second and first order, respectively, of their Taylor Series [1].

When the prior distribution is not symmetric, there is no reason for using the symmetric set instead of the new minimum set, because they offer, for this case, exact estimatives up to the second order but the first requires more computational effort.

For a symmetric prior distribution, the two sets will offer a trade-off choice. The symmetric set will probably offer a better estimative, but with a cost to the computational effort. Meanwhile, the new minimum set will probably offer a poorly estimative, but with a lighter computational effort. However, the new sigma set still gives better estimatives for certain functions even for a Gaussian assumption. We shall see one example on section III-A.

Let us now turn our attention to the other two sets of sigma sets, the ones of [16] and [17].

The set of [16] also uses the minimum amount of sigma points. However, we can see two drawbacks on this set's properties. One is that [16] may be unstable for high values of n (see [17]). The other is that neither the sample mean nor the covariance matrix of the set of [16] equals the mean and covariance matrix respectively of the prior RV when n is greater than one. There is no difficulty on checking it. If one takes the algorithm of section II-A.2 and calculates any set $\{\chi_i^j, w_i\}$ for a j greater than 1 - try 2 - this affirmation will be confirmed.

As for the set of [17] we can see that it uses one extra point, which is not a compromising drawback. However, this set of sigma points also carries the second mentioned drawback for the set of [16] and it can also be verified as easy as for this other set.

C. The Unscented Kalman Filter

The sets of sigma points can be used in a Kalman Filter frame as a recursive filter for non-linear systems resulting in the Unscented Kalman Filter (UKF).

The UKF has been presented mainly in two approaches according to the way the algorithm treats the noise terms [19]. The first one treats both the process noise and the measurement noise as additive. On the other hand, the second Unscented Kalman Filter algorithm incorporates the noise terms into the state vector, creating an augmented state vector for the generation of the sigma points. Here we expose only the augmented UKF (aUKF), for it is the most used and is the one that we are going to use.

Consider the following stochastic non-linear discrete-time dynamic system

$$\begin{aligned} X_k &= f[X_{k-1}, u_k, w_k], \\ Y_k &= h[X_k, u_k, v_k], \end{aligned}$$

in which X_k is the state's vector at time k, Y_k is the vector of the measurements at time k, u_k is the control input at time k and $w_k \sim N(0, Q)$ and $v_k \sim N(0, R)$ are the process' noise and the observation's noise respectively.

First, we must restructure the vectors, covariance matrices and functions as above [1]:

- The augmented prior state vector's media will be:

$$\bar{X}^a_{k-1} = \begin{pmatrix} \bar{X}_{k-1} \\ 0_{n_w} \\ 0_{n_w} \end{pmatrix},$$

  in which 0_{n_w} and 0_{n_w} are vectors of n_w (process noise's dimension) and n_v (measurement noise's dimension) zeros respectively.
- The augmented covariance matrix will be [5]:

$$P^a_{XX,k-1} = \begin{pmatrix} P_{XX,k-1} & 0 & 0 \\ 0 & Q & 0 \\ 0 & 0 & R \end{pmatrix}.$$

- The process and measurements function must be rewritten as functions of $\bar{X}^a_{k-1}$:

$$\begin{aligned} \bar{X}^a_k &= f^a\left[\bar{X}^a_{k-1}, u_k, w_k\right]; \\ Y_k &= h^a\left[\bar{X}^a_k, u_k, v_k\right]. \end{aligned}$$

The aUKF's algorithm is [1]:

1) Generates the augmented sigma points and its weights from the prior distribution:

$$[\chi^{i,a}_{k-1}, w_i] \leftarrow \bar{X}^a_{k-1}.$$

2) Apply the augmented sigma points into the process function to obtain the predicted sigma points:

$$\chi^{i,a}_{k|k-1} \leftarrow f^a\left[\chi^{i,a}_{k-1}, u_k, w_k\right].$$

3) Calculate the predicted augmented media and covariance matrix:

$$\mu^a_{\chi,k|k-1} = \sum_{i=0}^{2n+1} w_i \chi^{i,a}_{k|k-1};$$

$$\begin{aligned} &\Sigma^a_{\chi\chi,k|k-1} \\ &= \sum_{i=0}^{2n+1} w_i \left[\mu^a_{\chi,k|k-1} - \chi^i_{k|k-1}\right]\left[\mu^a_{\chi,k|k-1} - \chi^{i,a}_{k|k-1}\right]^T. \end{aligned}$$

4) Apply the prediction's augmented sigma points into the observation function;

$$\gamma^i_{k|k-1} = h^a\left[\chi^{i,a}_{k|k-1}, u_k, v_k\right].$$

5) Calculate the observation's predicted media and covariance matrix:

$$\mu_{\gamma,k|k-1} = \sum_{i=0}^{2n+1} w_i \gamma^i_{k|k-1};$$

$$\begin{aligned} &\Sigma_{\gamma\gamma,k|k-1} \\ &= \sum_{i=0}^{2n+1} w_i \left[\mu_{\gamma,k|k-1} - \gamma^i_{k|k-1}\right]\left[\mu_{\gamma,k|k-1} - \gamma^i_{k|k-1}\right]^T. \end{aligned}$$

6) Calculate the cross covariance error:

$$\begin{aligned} &\Sigma_{\chi\gamma,k|k-1} \\ &= \sum_{i=0}^{2n+1} w_i \left[\mu_{\chi,k|k-1} - \chi^i_{k|k-1}\right]\left[\mu_{\gamma,k|k-1} - \gamma^i_{k|k-1}\right]^T. \end{aligned}$$

7) Calculate the update using Kalman Filter's equations:

$$\begin{aligned} G_k &= \Sigma_{\chi\gamma,k|k-1}\Sigma^{-1}_{\gamma\gamma,k|k-1}; \\ \nu_k &= y_k - \mu_{\gamma,k|k-1}; \\ \mu_{\gamma,k} &= mu_{\gamma,k} + G_k\nu_k; \\ P_{XX,k} &= \Sigma_{\chi\chi,k|k-1} - G_k\Sigma_{\gamma\gamma,k|k-1}G^T_k. \end{aligned}$$

For more information, see [1].

The only difference between the unscented filters using different sigma sets will be in the first stage of the aUKF algorithm. In the simulations, will shall use this filter in a SLAM frame for each of the sigma sets described in sections II-A and II-B.

III. Examples

A. Simulations of the transforms

In this section we perform a simulation comparing the estimatives of our minimum set of sigma points (MiUT) with the estimatives of the symmetric set (SyUT), of the reduced set of [16] (RUT) and of the spherical set of [17] (SpUT). In all simulations, the prior RV's distributions

[5] We consider that the state and the noises are mutually uncorrelated.

TABLE I

SIMULATIONS RESULTS

	Montecarlo	SyUT	RUT
$\bar{X}$	$\begin{bmatrix} -0.1116 \\ 0.9965 \\ 2.1639 \end{bmatrix}$	$\begin{bmatrix} -0.1116 \\ 0.9965 \\ 2.1639 \end{bmatrix}$	$\begin{bmatrix} -0.2636 \\ 0.8292 \\ 2.3913 \end{bmatrix}$
P_{XX}	$\begin{bmatrix} 100.6877 & 0.7571 & 0.1904 \\ 0.7571 & 48.6841 & -0.7459 \\ 0.1904 & -0.7459 & 299.0401 \end{bmatrix}$	$\begin{bmatrix} 100.6877 & 0.7571 & 0.1904 \\ 0.7571 & 48.6841 & -0.7459 \\ 0.1904 & -0.7459 & 299.0401 \end{bmatrix}$	$\begin{bmatrix} 0.3170 & 0.2382 & 0.6988 \\ 0.2382 & 48.6841 & 0.0668 \\ 0.6988 & 0.0668 & 6.0078 \end{bmatrix}$
$\bar{Y}_1$	454.100	454.100	14.2438
$\bar{Y}_2$	$\begin{bmatrix} 19.1433 \\ 0.1621 \\ 0.1284 \end{bmatrix}$	$\begin{bmatrix} 16.9046 \\ 1.1843 \\ 1.1961 \end{bmatrix}$	$\begin{bmatrix} 2.8786 \\ 0.8943 \\ 0.7037 \end{bmatrix}$
$P_{Y_2Y_2}$	$\begin{bmatrix} 89.1275 & 0.2532 & 0.8015 \\ 0.2532 & 3.4195 & 0.0031 \\ 0.8015 & 0.0031 & 3.0060 \end{bmatrix}$	$\begin{bmatrix} 169.8262 & 0.9615 & -6.2532 \\ 0.9615 & 1.4664 & 0.3953 \\ -6.2532 & 0.3953 & 1.4502 \end{bmatrix}$	$\begin{bmatrix} 2.2908 & 0.3526 & 0.4411 \\ 0.3526 & 1.4695 & 1.199 \\ 0.4411 & 1.199 & 1.7020 \end{bmatrix}$
	SpUT	**MUT**	
$\bar{X}$	$\begin{bmatrix} -0.0930 \\ 1.9930 \\ -10.4422 \end{bmatrix}$	$\begin{bmatrix} -0.1116 \\ 0.9965 \\ 2.1639 \end{bmatrix}$	
P_{XX}	$\begin{bmatrix} 0.0003 & -0.0062 & 0.0324 \\ -0.0062 & 6.6229 & -12.1684 \\ 0.0324 & -12.1684 & 123.5967 \end{bmatrix}$	$\begin{bmatrix} 100.6877 & 0.7571 & 0.1904 \\ 0.7571 & 48.6841 & -0.7459 \\ 0.1904 & -0.7459 & 299.0401 \end{bmatrix}$	
$\bar{Y}_1$	262.0771	454.100	
$\bar{Y}_2$	$\begin{bmatrix} 12.3408 \\ 1.2656 \\ -0.2717 \end{bmatrix}$	$\begin{bmatrix} 20.7854 \\ -0.0136 \\ -0.4013 \end{bmatrix}$	
$P_{Y_2Y_2}$	$\begin{bmatrix} 97.8847 & 0.6102 & -13.4775 \\ 0.6102 & 0.0538 & -0.0231 \\ -13.4775 & -0.0231 & 1.9318 \end{bmatrix}$	$\begin{bmatrix} 25.5601 & 5.5892 & 5.0464 \\ 5.5892 & 3.5939 & 2.3979 \\ 5.0464 & 2.3979 & 2.8100 \end{bmatrix}$	

are a 3 dimension multivariable Gaussian. The errors of the posterior distributions are in relation to a Monte Carlo simulation running 500000 samples. Finally, all the errors are a mean of the errors of 100 simulations.

Table I and Table II contain the three simulations results. In Table II, the first two lines show, respectively, the priors' mean and covariance matrix errors, which are represented as $E_{\bar{X}}$ and $E_{P_{XX}}$. The third line shows the errors on the mean ($E_{\bar{Y}_1}$) of the posterior RV transformed by a second order polynomial function (f_1 below). The fourth and fifth lines present, respectively, the errors on the mean ($E_{\bar{Y}_2}$) and on the covariance matrix ($E_{P_{Y_2Y_2}}$) of the posterior RV transformed by a Cartesian to polar transformation (f_2 below).

The two simulated functions, f_1 and f_2, are the following:

$$f_1(X) = x_1^2 + x_2^2 + x_3^3.$$

$$f_2(X) = \begin{bmatrix} \sqrt{x_1^2 + x_2^2 + x_3^3} \\ \arctan\left(\frac{x^2}{x^1}\right) \\ \arctan\left(\frac{x^3}{x^1}\right) \end{bmatrix}.$$

From the results of the first two lines, we can see that neither [16] nor [17] matched the prior's RV mean and covariance matrix, while our new set and the symmetric set did. From the third line, one can check that our set and the symmetric set really estimated precisely the mean of the posterior RV for a second order polynomial function. Finally, the last two lines showed that our new set provided the best estimative for the mean of the posterior RV and second best one for the covariance matrix, even using only $n+1$ sigma points.

TABLE II

PERCENT ERRORS IN RELATION TO MONTE CARLO

	SyUT	RUT	SpUT	MUT
$E_{\bar{X}}$	1.8×10^{-18}	0.1356	5.2715	5.3×10^{-16}
$E_{P_{XX}}$	3.0×10^{-19}	0.9800	0.5923	2.2×10^{-16}
$E_{\bar{Y}_1}$	2.1×10^{-12}	0.9686	0.4269	1.1×10^{-14}
$E_{\bar{Y}_2}$	1.3148	0.8467	0.3630	0.0911
$E_{P_{Y_2Y_2}}$	0.2642	0.9735	0.1961	0.7507

B. SLAM's simulations

In this section, we consider the same system and UKF-SLAM algorithm of Dr. Tim Bailey[6]. The algorithm performed each of the four sigma sets (MUT, SyUT, SpUT, RUT) running in a Matlab R2009a.

The motion function f and the measurement function h are the following[7]:

$$f\left(x_R\left[k\right], u_k\right)$$

[6]Available at $www-personal.acfr.usyd.edu.au/tbailey/softwares$ /$slam_simulations.htm$.

[7]For convenience, we will use a new notation: $x_R[k] = x_R^k$.

TABLE III

PATH ERRORS FOR THE SLAM'S SIMULATIONS

	SyUT	MUT	RUT	SpUT
RMSE (m)	1.9772	1.8345	unstable	unstable

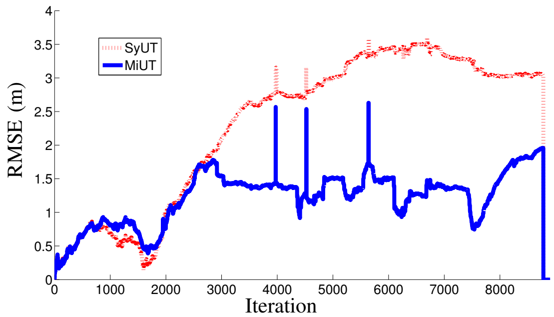

Fig. 1. Path errors in each iteration

$$= \begin{bmatrix} x_R^1[k] + u_k^1 \cos\left(u_k^2 + x_R^3[k]\right)\Delta t \\ x_R^2[k] + u_k^1 \sin\left(u_k^2 + x_R^3[k]\right)\Delta t \\ x_R^3[k] + u_k^1 \sin\left(u_k^2\right)\frac{\Delta t}{d} \end{bmatrix},$$

$$h(x[k]) = \begin{bmatrix} \sqrt{\left(x_{fi}^1[k] - x_R^1[k]\right)^2 + \left(x_{fi}^2[k] - x_R^2[k]\right)^2} \\ \arctan\left(\frac{x_{fi}^2[k]-x_R^2[k]}{x_{fi}^1[k]-x_R^1[k]}\right) - x_R^3[k] \end{bmatrix}.$$

where $x_R[k|k-1]$ is the predicted state at time k, $u_k = \begin{bmatrix} u_k^1 & u_k^2 \end{bmatrix}^T$ is the control input, Δt is the time step, d is the distance between the wheel of the robot, $x_R^i[k]$ i-th scalar component of $x_R[k]$, and x_{fi} is the coordinates of the i-th feature;

Table III contains the root mean square error (RMSE) of the robot's poses in meters using each sigma set. It can be seen that the filters that used the reduced set of [16] and the spherical set of [17] present numerical instability. It happened because state's covariance matrix tended to loss its positiveness.

Fig. 1 shows the errors of the robot's poses in each iteration for the symmetric set and for the new minimum set of Theorem 1. Analysing both Table III and Fig. 1, we can see the SLAM algorithm embedded with the new minimum sigma set, even using less points, gave a better estimative in comparison to the symmetric set.

IV. CONCLUSION

We develop a new set of sigma points for non-linear estimation which uses the minimum amount of points. This set's sample mean and sample covariance matrix are equal to the prior's RV mean and covariance matrix respectively. Numerical examples showed that the estimatives based on the proposed sigma set outperform the estimatives based on the most commonly used sigma sets.

Finally, we test the same sigma sets for a recursive filter application in a SLAM algorithm.

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