2011 50th IEEE Conference on Decision and Control and
European Control Conference (CDC-ECC)
Orlando, FL, USA, December 12-15, 2011

Filter Design for $\mathcal{L}_1$ Adaptive Output–Feedback Controller

Kwang Ki Kevin Kim, Evgeny Kharisov and Naira Hovakimyan

***Abstract*— In this paper we exploit the connection between disturbance observer and $\mathcal{L}_1$ adaptive control theories. We consider $\mathcal{L}_1$ adaptive output–feedback control framework, for which the $\mathcal{L}_1$ reference controller is equivalent to the disturbance observer. Using this fact, we investigate several properties of the disturbance observer architecture, leading to various filter design methods towards verification of the stability conditions for the $\mathcal{L}_1$ adaptive output–feedback controller.**

I. INTRODUCTION

In [1], [2], $\mathcal{L}_1$ adaptive output feedback controllers are presented, which ensure uniform transient and steady–state tracking for uncertain systems in the presence of bounded disturbances. The transient performance of the closed–loop adaptive system is quantified both for the system output and the input by uniform performance bounds with respect to an $\mathcal{L}_1$ reference system, which incorporates a lowpass filter. The performance bounds can be arbitrarily improved by increasing the adaptation gain. The $\mathcal{L}_1$ reference control law contains system uncertainties and, therefore, is usually used only for the analysis purposes. In [3], we have shown that for numerous $\mathcal{L}_1$ adaptive control architectures, the $\mathcal{L}_1$ reference system can be rewritten in the form free of the system uncertainties. For output–feedback architectures, this representation of the reference controller is structurally equivalent to disturbance observer (DOB).

DOB is a widely used method for designing a two degree of freedom control architecture to achieve robustness to modeling errors and provide disturbance rejection [4]–[6]. Several DOB–based controllers have been employed in industrial applications for classes of parametric uncertain systems, [5], [7]–[9], and in the presence of unmodeled dynamics, [10]–[12]. However, these methods were application–oriented and did not have general treatment of theoretical issues. Analytical investigation of the architecture is given in [13], [14].

In this paper we address the problem of selection of the lowpass filter satisfying the $\mathcal{L}_1$–norm sufficient condition of stability using the equivalence between the $\mathcal{L}_1$ reference system and the DOB control architecture. Whereas the prior results in $\mathcal{L}_1$ adaptive output feedback control were shown only to achieve uniform transient and steady–state performance without specifying the *desired performance*, [15], [16], with the proposed tools in this paper for parametrization of the lowpass filters we address stability and *desired performance* of $\mathcal{L}_1$ adaptive output feedback controllers simultaneously. We borrow a theoretical result from [14] to show that for minimum–phase systems with parametric uncertainties the design requirements of $\mathcal{L}_1$ adaptive output feedback control can be always satisfied, and we modify the DOB structure in [13] to solve the design problems for the case of nonminimum–phase systems. In the presence of unmodeled dynamics, *affecting the relative degree of the unknown system*, we resort to different methods for analysis and synthesis of robust stability and performance, in particular, structured singular value (SSV) framework and μ–synthesis (see, for example, [17]–[19] and references therein for general results on μ–synthesis). However, due to the special structure of $\mathcal{L}_1$ adaptive output feedback controller, application of these methods needs to be done with special consideration.

This paper is organized as follows. Section II gives a review of the $\mathcal{L}_1$ adaptive output feedback control solution, and defines the DOB architecture. In Section III, we perform a transition of some of the robust control design methods to $\mathcal{L}_1$ adaptive control theory. In particular we investigate the robust stability and robust performance of the $\mathcal{L}_1$ reference system, and suggest guidelines for selection of lowpass filters for different classes of systems. Section IV concludes the paper.

II. $\mathcal{L}_1$ ADAPTIVE OUTPUT–FEEDBACK CONTROLLER AND DISTURBANCE OBSERVER

In this section we first give a brief review of the $\mathcal{L}_1$ adaptive output–feedback control architecture with piecewise constant adaptation laws. Then we define the $\mathcal{L}_1$ reference system for this adaptive architecture and give the equivalent DOB architecture.

A. $\mathcal{L}_1$ Adaptive Output–Feedback Control

Consider the following SISO system:

$$y(s) = A(s)(u(s) + d(s))\,, \tag{1}$$

where $y(s)$ and $u(s)$ are the Laplace transforms of the system's output and input signals, respectively, $A(s)$ is a strictly proper unknown transfer function with known relative degree d_r, $d(s)$ is the Laplace transform of the time–varying uncertainties $d(t) = f(t, y(t))$, while f is an unknown map, for which there exist constants $L > 0$ and $L_0 > 0$, such that

$$|f(t, y_1) - f(t, y_2)| \leq L|y_1 - y_2|\,,\ |f(t, y)| \leq L|y| + L_0\,,$$

for arbitrary $y, y_1, y_2 \in \mathbb{R}$ uniformly in $t \in \mathbb{R}_0^+$. We also assume that there exist constants $L_1,\ L_2,\ L_3 > 0$, such that

$$|\dot{d}(t)| \leq L_1|\dot{y}(t)| + L_2|y(t)| + L_3\,, \quad \forall t \in \mathbb{R}_0^+\,.$$

Let the control specifications be given via the *ideal system* $M(s)$, which is assumed to be minimum–phase and stable.

We notice that the system in (1) can be rewritten in terms of the ideal system $M(s)$:

$$y(s) = M(s)(u(s) + \sigma(s))\,,$$

where $\sigma(s) \triangleq \frac{(A(s)-M(s))u(s)+A(s)d(s)}{M(s)}$. Let $(A_m \in \mathbb{R}^{n\times n},\ b_m \in \mathbb{R}^n,\ c_m^\top \in \mathbb{R}^{1\times n})$ be the minimal realization of $M(s)$. Further, since A_m is Hurwitz, there exists $P =$

This work was supported by AFOSR under Contract FA9550–09–1–0265 and NASA under Contract NNX08BA65A.

K. Kim and E. Kharisov are with the Department of Aerospace Engineering, University of Illinois at Urbana–Champaign, Urbana, IL, 61801, USA, {kkim78,evgeny}@illinois.edu

N. Hovakimyan is with the Department of Mechanical Science and Engineering, University of Illinois at Urbana–Champaign, Urbana, IL, 61801, USA, nhovakim@illinois.edu.

$P^\top > 0$ that satisfies the algebraic Lyapunov equation $A_m^\top P + PA_m = -Q$, for arbitrary $Q = Q^\top > 0$. From the properties of P, it follows that there exits nonsingular $\sqrt{P}$ such that $P = (\sqrt{P})^\top \sqrt{P}$. Given the vector $c_m^\top(\sqrt{P})^{-1}$, let $D \in \mathbb{R}^{(n-1)\times n}$ contain the null–space of $c_m^\top(\sqrt{P})^{-1}$, i.e. $D(c_m^\top(\sqrt{P})^{-1})^\top = 0$, and further let $\Lambda \triangleq [c_m \quad (D\sqrt{P})^\top]^\top$. Finally, let $T_s \in \mathbb{R}^+$ be the sampling rate of the available CPU, and $\mathbf{1}_1 \triangleq [1,\ 0,\ \ldots,\ 0]^\top \in \mathbb{R}^n$. The elements of $\mathcal{L}_1$ adaptive controller are introduced next:

The *state–predictor* is given by

$$\begin{aligned} \dot{\hat{x}}(t) &= A_m\hat{x}(t) + b_m u(t) + \hat{\sigma}(t)\,, \quad \hat{x}(0) = 0\,, \\ \hat{y}(t) &= c_m^\top \hat{x}(t)\,, \end{aligned} \tag{2}$$

where $\hat{\sigma}(t) \in \mathbb{R}^n$ is the vector of adaptive parameters propagated according to the following *adaptation law*:

$$\hat{\sigma}(t) = -\Phi^{-1}(T_s)\mu(iT_s)\,, \quad t \in [iT_s,\ (i+1)T_s)\,, \tag{3}$$

where $i = 0, 1, 2, \ldots$, $\tilde{y}(t) \triangleq \hat{y}(t) - y(t)$, and

$$\begin{aligned} \Phi(T_s) &\triangleq \Lambda A_m^{-1}\left(\mathbb{I} - \mathrm{e}^{A_m(T_s-\tau)}\right)\,, \\ \mu(iT_s) &\triangleq \mathrm{e}^{\Lambda A_m \Lambda^{-1} T_s}\mathbf{1}_1\tilde{y}(iT_s)\,, \quad i = 0, 1, 2, \ldots\,. \end{aligned}$$

The *control law* is defined as follows:

$$u(s) = C(s)r(s) - \frac{C(s)c_m^\top(s\mathbb{I} - A_m)^{-1}}{c_m^\top(s\mathbb{I} - A_m)^{-1}b_m}\hat{\sigma}(s)\,, \tag{4}$$

where $r(t) \in \mathbb{R}$ is the reference command, and $C(s)$ is a stable strictly–proper lowpass filter of relative degree d_r, with $C(0) = 1$.

The development of $\mathcal{L}_1$ adaptive output–feedback controller proceeds by considering the $\mathcal{L}_1$ *reference system*, based on partial compensation of uncertainties within the bandwidth of the given control channel:

$$\begin{aligned} y_{\mathrm{rf}}(s) &= M(s)(u_{\mathrm{rf}}(s) + \sigma_{\mathrm{rf}}(s))\,, \\ u_{\mathrm{rf}}(s) &= C(s)(r(s) - \sigma_{\mathrm{rf}}(s))\,, \end{aligned} \tag{5}$$

where $\sigma_{\mathrm{rf}}(s) \triangleq \frac{(A(s)-M(s))u_{\mathrm{rf}}(s)+A(s)d_{\mathrm{rf}}(s)}{M(s)}$, $d_{\mathrm{rf}}(s)$ is the Laplace transform of $d_{\mathrm{rf}}(t) \triangleq f(t, y_{\mathrm{rf}}(t))$. The reference system (5) is stable if the lowpass filter $C(s)$ and the desired model $M(s)$ are chosen to verify the following bound:

$$\|G(s)\|_{\mathcal{L}_1} L < 1\,, \tag{6}$$

where

$$G(s) \triangleq \frac{A(s)M(s)(1 - C(s))}{C(s)A(s) + (1 - C(s))M(s)}\,. \tag{7}$$

The main result on the performance bounds is summarized in the following theorem.

Theorem 1 (*Theorem 1, [2]*). Consider the system in (1) and the $\mathcal{L}_1$ adaptive controller in (2), (3), and (4), subject to the $\mathcal{L}_1$–norm condition in (6). If we choose T_s to ensure

$$\gamma_0(T_s) < \bar{\gamma}_0\,,$$

where $\gamma_0(T_s)$ is a computable bound, dependent upon the sampling time T_s, and $\bar{\gamma}_0$ is an arbitrary positive constant, then

$$\begin{gathered} \|\tilde{y}\|_{\mathcal{L}_\infty} < \bar{\gamma}_0\,, \\ \|y_{\mathrm{rf}} - y\|_{\mathcal{L}_\infty} \le \gamma_1(T_s)\,, \quad \|u_{\mathrm{rf}} - u\|_{\mathcal{L}_\infty} \le \gamma_2(T_s)\,, \end{gathered}$$

in which $\gamma_1(T_s)$, and $\gamma_2(T_s)$ are computable bounds that can be arbitrarily reduced by decreasing the sampling time T_s.

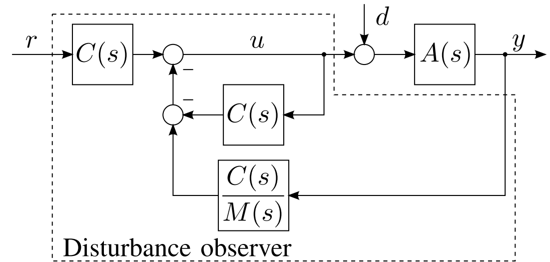

Fig. 1: Disturbance observer architecture

B. Disturbance Observer

We notice that the $\mathcal{L}_1$ reference system achieves the same control objective as the DOB given in Figure 1 [14]. In [3] we have shown that for linear output–feedback systems the $\mathcal{L}_1$ reference controller is equivalent to the DOB. Therefore the robust control design theory used for DOB can be also applied to the design of the $\mathcal{L}_1$ adaptive controller. In the next section of the paper we use the equivalence between the architectures and introduce modifications of the robust control methods and μ–analysis and synthesis to perform design of the lowpass filter, which achieves robust stability and robust performance for the $\mathcal{L}_1$ adaptive controller.

Remark 1. Notice that in [3] we have shown that the architectures of the $\mathcal{L}_1$ reference system and DOB for nonlinear systems are different. Namely, they both perform the ideal system inversion in the feedback loop, however the system inversion is performed before filtering of the measured system state in the $\mathcal{L}_1$ reference system, while in the DOB the inversion is performed after the filtering. Also notice that for linear reference systems these architectures are identical, since the linearity of the system permits an interchange in the order of filtering and inverting.

III. Robust Filter Design for $\mathcal{L}_1$ Adaptive Output–Feedback Control

We start by considering the systems with known relative degree and parametric type of uncertainties. In this case we extend the result from DOB literature to ensure robust stability and performance of the $\mathcal{L}_1$ reference system along with satisfaction of the $\mathcal{L}_1$–norm stability condition. Next we consider the case of systems with unknown relative degree or nonparametric type of uncertainties. For these systems we propose application of μ–analysis and synthesis theory.

A. Filter Design in the Presence of Parametric Uncertainty

Let $A(s)$ belong to the following set of transfer functions:

$$\Pi_p \triangleq \left\{ P(s) = \frac{\sum_{i=0}^{n-d_r} b_i s^i}{\sum_{i=0}^{n} a_i s^i} \,\middle|\, a_i \in [a_{li}, a_{ui}],\ b_i \in [b_{li}, b_{ui}] \right\},$$

where a_{li}, a_{ui}, b_{li}, b_{ui} are known conservative bounds, and a_n, b_{n-d_r} are assumed to be nonzero and such that the relative degrees of the transfer functions from this set are fixed and equal to d_r. We consider the filter design problem for minimum–phase and nonminimum–phase systems separately. In the first case we show the existence of a lowpass filter satisfying the $\mathcal{L}_1$–norm stability condition, and we propose a systematic methodology for filter design satisfying the $\mathcal{L}_1$–norm condition. In the second case of nonminimum–phase systems we again show the existence of the lowpass filter, which ensures stability of $H(s)$, but the design has a limitation, which does not allow for reducing the $\mathcal{L}_1$–norm of the transfer function $G(s)$ in (6) arbitrarily. We introduce the following notation: let $C(s;\tau) \triangleq C(\tau s)$[1] be the lowpass filter.

Remark 2. Notice that there are similar results on the filter design in literature. Namely, in [1], [20] it has been shown that one can ensure the stability of $H(s)$ for $A(s)$ with relative degree 1 and 2. In this paper we consider arbitrary relative degree of the system.

1) Filter Design for Minimum–phase Systems: The next lemma states a sufficient condition for the existence of the filter, which ensures stability of $H(s)$.

Lemma 1 (*Robust stability [14]*). Consider the transfer function $H(s)$ given in (7). If $M(s)$ and $A(s) \in \Pi_p$ are minimum phase and have the same relative degree d_r, $M(s)$ is stable, and the transfer function $C(s;1) = \frac{C_n(s;1)}{C_d(s;1)}$ ensures that the polynomial

$$\chi_f(s) \triangleq C_d(s;1) + \left(\lim_{s\to\infty} \frac{A(s)}{M(s)} - 1\right) C_n(s;1) \quad (8)$$

is Hurwitz, then there exists $\tau^* > 0$, such that the transfer function $H(s)$ is stable with any $C(s;\tau)$ for $\tau \in (0,\tau^*]$.

The following theorem justifies that increasing the bandwidth of the lowpass filter $C(s)$ ensures satisfaction of the $\mathcal{L}_1$–norm condition (6).

Theorem 2 (*Robust performance*). Assume that all the conditions of Lemma 1 hold. Then the following limit holds:

$$\lim_{\tau\to 0} \|G(s)\|_{\mathcal{L}_1} = 0.$$

Proof. For notation convenience, we use $G(s;\tau)$ for $G(s)$ and it can be rewritten as $G(s;\tau) = G(s;0) + \frac{\partial G(s;\tau)}{\partial \tau}\Big|_{\tau=0} \tau + o(\tau)$, where $o(\tau)$ satisfies $\lim_{\tau\to 0} \frac{o(\tau)}{\tau} = 0$ uniformly, i.e., for all $s \in \mathbb{C}$. Since $G(s;\tau)$ can be rewritten as $\frac{\tau s F(s;\tau) C_d(s;\tau) A_n(s) M_n(s)}{C_n(s;\tau) A_n(s) M_d(s) + \tau s F(s;\tau) M_n(s)}$, where $F(s;\tau) \triangleq (C_d(s;\tau) - C_n(\tau;s))/(\tau s)$, we have $G(s;0) = 0$ and $\frac{\partial G(s;\tau)}{\partial \tau}\Big|_{\tau=0} = sF(s;0)M(s)$, where $F(s;0)$ is a nonzero constant. Thus, the inverse Laplace transform of $G(s;\tau)$ becomes $g(t;\tau) = \bar{g}'(t)\tau + o(\tau)$, where $\bar{g}(t)$ denotes the inverse Laplace transform of $\bar{G}(s) \triangleq F(s;0)M(s)$, such that its time derivative $\bar{g}'(t)$ is uniformly bounded for all time t. Notice that from the fact $\lim_{\tau\to 0} \frac{\int o(\tau;s) s^{st} ds}{\tau} = \int \lim_{\tau\to 0} \frac{o(\tau;s)}{\tau} s^{st} ds = 0$ it follows that the inverse Laplace transform of $o(\tau;s)$ is $o(\tau;t)$, where s and t are used to denote the frequency and time domains, respectively. Therefore, we conclude that $\lim_{\tau\to 0} \|G(s;\tau)\|_{\mathcal{L}_1} = \lim_{\tau\to 0} \int_0^\infty |g(t;\tau)| dt = \int_0^\infty \lim_{\tau\to 0} |g(t;\tau)| dt = 0$. □

Remark 3. Notice that the Hurwitz condition on the polynomial in (8) is equivalent to the stability of the negative feedback interconnection of the transfer function $C(s;\tau)$ and the bounded constant $\upsilon_0 \triangleq \lim_{s\to\infty} \frac{A(s)}{M(s)} - 1 \in [\upsilon^-, \upsilon^+]$. The filter satisfying the latter requirement can be designed using well established tools from linear systems theory.

[1] τ corresponds to the scaling of the cut–off frequency of $C(s)$ and in this paper we often use the notation $C(s;\tau)$ for the lowpass filter $C(s)$ to emphasize its dependence on τ

Assume that we designed a filter $C(s)$ satisfying (8). Lemma 1 and Theorem 2 indicate that there exists a parameter τ_0, for which the transfer function $H(s)$ is stable and the $\mathcal{L}_1$ norm condition in (6) holds. To find the value of τ_0 we may use the bisection method. For this purpose we state the following theorem and the corollary.

Lemma 2 (*Kharitonov theorem, [21]*). The polynomial family $P(s,\mathcal{I}_a) = \{p(s,a) = \sum_{i=0}^n a_i s^i : a \in \mathcal{I}_a\}$, $a_n > 0$, is stable if and only if the following four polynomials are all stable:

$$\begin{aligned}
\kappa_1(s) &= a_{u0} + a_{l1}s + a_{l2}s^2 + a_{u3}s^3 + a_{u4}s^4 + \cdots, \\
\kappa_2(s) &= a_{u0} + a_{u1}s + a_{l2}s^2 + a_{l3}s^3 + a_{u4}s^4 + \cdots, \\
\kappa_3(s) &= a_{l0} + a_{u1}s + a_{u2}s^2 + a_{l3}s^3 + a_{l4}s^4 + \cdots, \\
\kappa_4(s) &= a_{l0} + a_{l1}s + a_{u2}s^2 + a_{u3}s^3 + a_{l4}s^4 + \cdots,
\end{aligned} \quad (9)$$

where $\mathcal{I}_a \triangleq \{a \in \mathbb{R}^n : a_i \in [a_{li}, a_{ui}],\ i = 1,\ldots,n\}$. The polynomials (9) are called Kharitonov polynomials.

Consider the characteristic polynomial of the transfer function $H(s)$ in (7):

$$H_d(s) \triangleq M_d(s)A_n(s)C_n(s) + A_d(s)M_n(s)\left(C_d(s) - C_n(s)\right),$$

where the subscripts n and d denote the numerator and the denominator of the transfer functions respectively. Lemma 2 implies that to check stability of $H(s)$, we need to check if the polynomials $\kappa_1(s), \cdots, \kappa_4(s)$ are Hurwitz. We use this test in the bisection method, which helps us to compute the maximum value of τ^* given in Lemma 1. Next, decreasing the value of $\tau \in (0,\ \tau^*]$, according to Theorem 2 we can always find the value of τ_0, which leads to the filter choice satisfying the $\mathcal{L}_1$–norm condition in (6). The next example illustrates the filter design procedure.

Example 1. Consider the ideal system model

$$M(s) = \omega_n^2/(s^2 + 2\zeta\omega_n s + \omega_n^2), \quad (10)$$

where ζ and ω_n are given parameters. Let

$$A(s) = c/(s^2 + as + b),$$

where $a \in [a_l,\ a_u]$, $b \in [b_l,\ b_u]$, and $0 < c \in [c_l,\ c_u]$. In this example we consider the following numeric data: $\omega_n = 1$, $\zeta = 0.7$, $a \in [-1,-3]$, $b \in [1,3]$, $c \in [1,4]$. Consider the following lowpass filter:

$$C(s;\tau) = 1/(\tau s + 1)^3.$$

It is straightforward to see that the polynomial

$$\chi_f(s) = s^3 + 3s^2 + 3s + c/\omega_n^2$$

is robustly stable for every $c > 0$. Next we follow the design procedure using the bisection method and compute $\tau^* \approx 0.0217$ such that $H(s)$ is robustly stable for all $\tau \in (0,\tau^*]$. The root locus of the Kharitonov polynomials for $\tau \in [0.0001, 0.02]$ is shown in Figure 2. One can see that the polynomials are stable for all $\tau \in [0.0001, 0.02]$. Next we compute the value of $\|\hat{G}(s)\|_{\mathcal{L}_1}$, and obtain $\|\hat{G}(s)\|_{\mathcal{L}_1} \approx 0.0029$ for $\tau = 0.001$, where $\hat{G}(s)$ is the transfer function $G(s)$ with $A(s)$ replaced by its nominal model $\hat{A}(s) = \frac{2.5}{s^2 - 2s + 2}$.

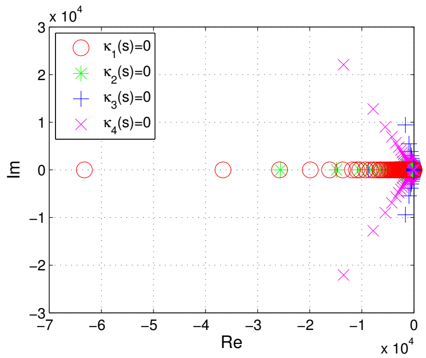

Fig. 2: Root locus of the four Kharitonov polynomials for $\tau \in [0.0001, 0.02]$ (Example 1)

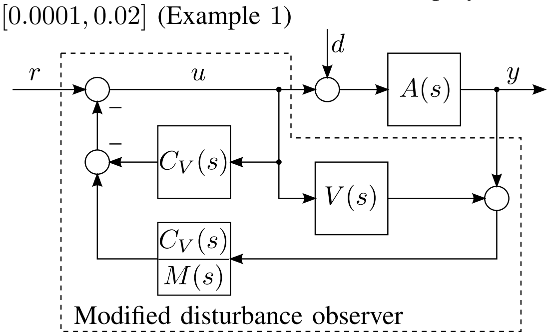

Fig. 3: Modified DOB structure

2) Filter Design for Nonminimum–phase Systems: Consider the characteristic equation $H_d(s) = 0$. It can be rewritten as:

$$M_d(s)A_n(s)\left(C_n(s) - 1\right) + A_d(s)M_n(s)\left(C_d(s) - C_n(s)\right) = -A_n(s)M_d(s)\,.$$

We see from the above equation that the solutions of the characteristic equation $H_d(s) = 0$ converge to the zeros of the transfer function $A(s)$ as $\tau \to 0$. From the continuity property, if there exists a RHP zero for $A(s)$, then the transfer function $H(s)$ cannot be stabilized by any choice of $C(s;\tau)$ with small value of τ. This shows the limitations of the methods considered in the previous section in the case of nonminimum–phase systems. In this section we consider a modification of the DOB configuration for nonminimum–phase systems. A systematic design methodology, specialized to the proposed DOB structure, is given with the help of Youla parameterization (or Q–parameterization) of stabilizing controllers.

We proceed with modifying the DOB controller structure in Figure 1, and introduce an additional feed–forward loop with a transfer function $V(s)$ as shown in Figure 3. Next, we rewrite the transfer functions $H(s)$ and $G(s)$ in terms of $C_V(s)$ and $V(s)$:

$$\begin{aligned} H(s) &= \frac{A(s)(M(s)+V(s))}{C_V(s)(A(s)+V(s)) + (1-C_V(s))(M(s)+V(s))}\,, \\ G(s) &= \frac{A(s)M(s)(1-C_V(s)) + A(s)V(s)}{C_V(s)(A(s)+V(s)) + (1-C_V(s))(M(s)+V(s))}\,. \end{aligned} \tag{11}$$

The main result of this section is given in the following theorem.

Theorem 3. Consider the transfer function $H(s)$ given in (11). If $M(s)$ is stable, and there exists a stable transfer function $V(s)$ such that $A(s) + V(s)$ is minimum phase for all $A(s) \in \Pi_p$, and the transfer function $C_V(s;1)$ ensures that the polynomial

$$\chi_g(s) \triangleq C_{Vd}(s;1) + \left(\lim_{s\to\infty} \frac{A(s)+V(s)}{M(s)+V(s)} - 1\right) C_{Vn}(s;1) \tag{12}$$

is Hurwitz, then there exists $\tau^* > 0$ such that the transfer function $H(s)$ is stable with any $C_V(s;\tau)$ for $\tau \in (0,\tau^*]$. Furthermore, if $C_{Vn}(s)$ is Hurwitz and independent of τ, then it also achieves

$$\lim_{\tau\to 0} \|G(s)\|_{\mathcal{L}_1} \le \sup_{A(s)\in\Pi_p} \left\| \frac{A(s)}{V^{-1}(s)A(s)+1} \right\|_{\mathcal{L}_1}\,. \tag{13}$$

Proof. First, if $C_V(s;\tau)$ satisfies the stability condition for $\chi_g(s)$ in (12), then it directly follows from Lemma 1 that there exists τ^* such that $H(s)$ is robustly stable for every $A(s) \in \Pi_p$ with any $C_V(s;\tau)$ for $\tau \in (0,\tau^*]$. Next, we prove the inequality (13). The transfer function $G(s)$ in (11) can be rewritten as the sum of two transfer functions $G_1(s) \triangleq H(s)(1 - C_V(s))$ and $G_2(s) \triangleq A(s)V(s)C_V(s)/G_d(s)$. Then from Theorem 2, for any $\epsilon_1 > 0$ there exists a $\tau_1 \in (0,\tau^*]$ such that $\|G_1(s)\|_{\mathcal{L}_1} \le \epsilon_1$ for $C_V(s;\tau_1)$. In addition, if $C_{Vn}(s)$ does not depend on τ, then $G_2(s)$ can be rewritten as $\frac{A_n(s)C_{Vn}(s)V_n(s)M_d(s)}{M_d(s)C_{Vn}(s)(V_d(s)A_n(s)+V_n(s)A_d(s))+\tau\tilde{G}_d(s)}$, where $\tilde{G}_d(s)$ is a polynomial with $\tilde{G}_d(0) = 0$. From the continuity of the induced norm of a transfer function in its coefficients, for any $\epsilon_2 > 0$ there exists a $\tau_2 \in (0,\tau^*]$, such that $\|G_2(s) - \frac{A_n(s)C_{Vn}(s)V_n(s)M_d(s)}{M_d(s)C_{Vn}(s)(V_d(s)A_n(s)+V_n(s)A_d(s))}\|_{\mathcal{L}_1} \le \epsilon_2$ for $C_V(s;\tau_2)$. Since $C_{Vn}(s)$ and $M_d(s)$ are assumed to be stable, we conclude that for any $\epsilon_1, \epsilon_2 > 0$ there exists $\tau_m \triangleq \min\{\tau_1,\tau_2\} \in (0,\tau^*]$, such that $\|G(s)\|_{\mathcal{L}_1} \le \|A(s)/(V^{-1}(s)A(s)+1)\|_{\mathcal{L}_1} + \epsilon_1 + \epsilon_2$ for $C_V(s;\tau_m)$. This completes the proof. □

The next lemma proposes a method to select $V(s)$ that achieves minimum–phase $A(s)+V(s)$ for every $A(s) \in \Pi_p$.

Lemma 3 (*Achievement of minimum–phase property*). If a stable transfer function $V_Q(s)$ stabilizes the transfer function $A(s) - M(s)$ for all $A(s) \in \Pi_p$, then $A(s) + V(s)$ is minimum phase for any $A(s) \in \Pi_p$ with

$$V^{-1}(s) \triangleq \frac{V_Q(s)}{1 - M(s)V_Q(s)}\,. \tag{14}$$

Proof. It can be easily verified that $A(s)+V(s)$ is minimum phase for any $A(s) \in \Pi_p$ if and only if the transfer function $J(s) \triangleq \frac{V^{-1}(s)A(s)}{V^{-1}(s)A(s)+1}$ is stable for any $A(s) \in \Pi_p$. In addition, we choose $V(s)$ according to Youla parametrization of all stabilizing controllers for the nominal transfer function $\frac{V^{-1}(s)M(s)}{V^{-1}(s)M(s)+1}$. This leads to $V^{-1}(s)$ in (14). Then we rewrite $J(s)$ as $\frac{V_Q(s)A(s)}{V_Q(s)(A(s)-M(s))+1}$. The proof is complete. □

Remark 4. Alternatively, instead of using the model–based parametrization of stabilizing controllers, we may consider a simple PID structure for the additional filter $V(s)$:

$$V^{-1}(s) = \frac{k_i + k_p s + k_d s^2}{s}\,,$$

where (k_p, k_i, k_d) are the user–defined design parameters to be determined.

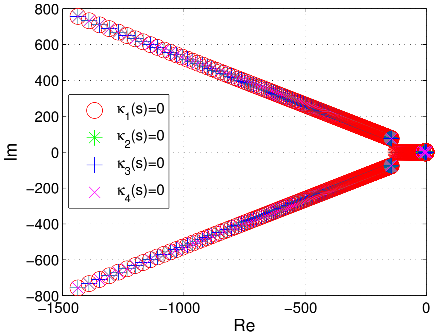

Fig. 4: Root locus of the four Kharitonov polynomials for $\tau \in [0.001, 0.01]$ (Example 2)

Once we obtain $V(s)$, which makes $A(s) + V(s)$ minimum phase, we perform the same design procedure as in Section III-A.1 to find the transfer function $C_V(s)$ satisfying the design requirements. The next example illustrates this design procedure.

Example 2. Consider the ideal system model

$$M(s) = \omega_n^2/(s^2 + 2\zeta\omega_n s + \omega_n^2)\,,$$

where $\omega_n = 1$, $\zeta = 0.7$. Let the plant be given by

$$A(s) = (d - s)/(s^3 + as^2 + b^s + c)\,,$$

where $a = 6$, $b = 5$, $c = 1$ are known, but $0 < d \in [d_l, d_u]$ is unknown with known bounds $d_l = 3$, $d_u = 5$. Note that $A(s)$ is nonminimum phase. We select $V(s) = \frac{2s}{s^3+as^2+b^s+c}$ so that $A(s)+V(s) = \frac{s+d}{s^3+as^2+b^s+c}$ is minimum phase for every $0 < d \in [d_l, d_u]$ and the lowpass filter $C_V(s;\tau) = \frac{1}{(\tau s+1)^3}$. It is straightforward to verify that $\chi_g(s) = s^3+3s^2+3s+\frac{1}{\omega_n^2+2}$ is stable for all $0 < d \in [d_l, d_u]$. Then there exists τ^* such that $H(s)$ is stable for all $\tau \in (0, \tau^*]$. Next, we follow the design procedure presented in this section and obtain $\tau^* \approx 0.057$. Furthermore, the nominal performance of the closed–loop system is given by $\|\hat{G}(s)\|_{\mathcal{L}_1} \approx 0.67$ for $\tau = 0.001$, where $\hat{G}(s)$ is the transfer function $G(s)$ with the replacement of $A(s)$ by its nominal model $\hat{A}(s) = \frac{4-s}{s^3+6s^2+5s+1}$. Figure 4 shows the root–locus of the four Kharitonov polynomials in (9). We see that this design achieves robust stability of $H(s)$ for all $\tau \in (0, \tau^*]$.

B. Filter Design Using μ Analysis and Synthesis

While under limited information on the actual plant the analysis and synthesis for robust stability and performance based on the SSV framework might not be completely informative, we will resort to this well–developed and systematic method to design a lowpass filter that guarantees the robust stability of the closed–loop reference system in (5) and the robust performance given by (6).

Since the step of rewriting an uncertain system as the standard $N - \Delta$ configuration, known as linear fractional transformation (LFT), has been shown in literature (see [19], for example), we assume that the system transfer function $A(s)$ is parameterized by a set of possible plants $\Pi \triangleq \{\mathcal{F}_u\left(\bar{A}(s), \Delta\right) : \Delta \in \Delta_c\}$, i.e., $A(s) \in \Pi$, where Δ_c denotes the set of block diagonal perturbation matrices, which reflects the structure of the uncertainties. To demonstrate how robust control theory can be applied to the problem

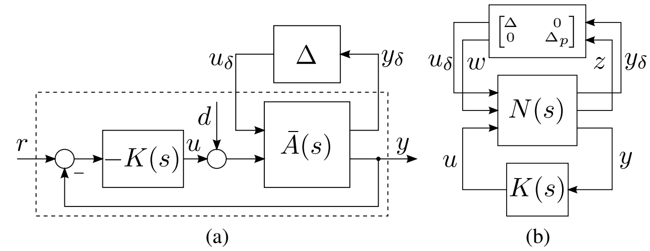

Fig. 5: LFT representation of the closed–loop reference system

at hand, consider a special case of unstructured uncertain system, where $A(s) = M(s)\,(1 + \Delta(s))\,, \Delta(s) \in \mathcal{RH}_\infty$. Then from the Nyquist stability criterion, if $C(s)$ satisfies

$$\left|\frac{C(j\omega)}{1 - C(j\omega)}\,(1 + \Delta(j\omega))\right| < 1 \Leftrightarrow |C(j\omega)| < \frac{1}{|\Delta(j\omega)|}\ \forall\omega \in \mathbb{R},$$

then $H(s)$ is robustly stable. However, this sufficient condition might be too conservative, as it does not use complete information about the uncertainties. Next, we rely on the SSV approaches to reduce the possible conservatism and utilize more of the available information about the uncertainties.

Figure 5a shows a reconfiguration of the closed–loop reference system in (5), and Figure 5b provides an equivalent LFT description of Figure 5a. The next two theorems propose methods of analyzing the robust stability and performance of the closed–loop reference system in (5), by adopting the SSV framework for Figure 5.

Theorem 4 (*Robust stability: Analysis*). Consider the uncertain system $A(s) \in \Pi$ and the transfer function $H(s)$ given in (7). If there exists a transfer function $K_I(s)$, such that the inequality $\mu_{\Delta_c}(N_{11}(j\omega)) < 1$ holds for all $\omega \in \mathbb{R} \cup \infty$ with

$$N(s) \triangleq \begin{bmatrix}1 & -\bar{A}_{12}(s)K(s)\\ 0 & 1 - \bar{A}_{22}(s)K(s)\end{bmatrix}^{-1} \begin{bmatrix}\bar{A}_{11}(s) & \bar{A}_{12}(s)C_H(s)\\ \bar{A}_{21}(s) & \bar{A}_{22}(s)C_H(s)\end{bmatrix},$$

where $K(s) \triangleq \frac{1}{s}K_I(s)$, $C(s) \triangleq \frac{-M(s)K_I(s)}{s-M(s)K_I(s)}$, $C_H(s) \triangleq \frac{C(s)}{1-C(s)}$, and $K_I(s)$ is a transfer function, which has no zeros in the origin, then the transfer function $H(s)$ is stable for all $A(s) \in \Pi$. Furthermore,

$$N_{11}(s) = \mathcal{F}_\ell(\bar{A}, K) = \mathcal{F}_\ell(\bar{A}_I, K_I)\,,$$

where

$$\bar{A}_I(s) \triangleq \begin{bmatrix}\bar{A}_{11}(s) & \bar{A}_{12}(s)\\ \bar{A}_{21}(s)\frac{1}{s} & \bar{A}_{22}(s)\frac{1}{s}\end{bmatrix}.$$

We omit the proof due to space considerations.

Theorem 5 (*Robust stability and performance: Analysis*). Consider the uncertain system $A(s) \in \Pi$ and the transfer function $H(s)$, given in (7). Let $\bar{\Delta}_c^\gamma \triangleq \{\text{diag}[\Delta, \gamma\Delta_p] : \Delta \in \Delta_c\}$, where γ is a predetermined performance bound, and Δ_p is a full square matrix with its dimension equal to the number of outputs. Also define

$$\bar{N}(s) \triangleq \begin{bmatrix}N_{11}(s) & N_{12}(s)\\ N_{21}(s) & (1 - C(s))N_{22}(s)\end{bmatrix},$$

where $N(s)$, $K(s)$, $C(s)$, $K_I(s)$, and $C_H(s)$ are the same as in Theorem 4. If there exists a transfer function $K_I(s)$ such that the inequality $\mu_{\bar{\Delta}_c^\gamma}(\bar{N}(j\omega)) < 1$ holds for all $\omega \in \mathbb{R}\cup\infty$, then the transfer function $H(s)$ is stable and $\|G(s)\|_\infty \le 1/\gamma$ for all $A(s) \in \Pi$.

We omit the proof due to space considerations.

In addition to the analysis of robust stability and performance, we also propose a design method for the lowpass filter of $\mathcal{L}_1$ adaptive controller. Due to the unit DC–gain requirement for $C(s)$, the proposed optimal design reduces to the problem of finding a transfer function $K_I(s)$ that minimizes $\|G(s)\|_{\mathcal{L}_1}$. However, since the $\mathcal{L}_1$–norm minimization problem is not convex, we resort to minimization of the $\mathcal{H}_\infty$–norm to exploit the existing iterative algorithms and off–the–shelf softwares [22], [23]. The relation between the $\mathcal{L}_1$ and the $\mathcal{H}_\infty$–norm of a transfer function is

$$\|G(s)\|_\infty \le \|G(s)\|_{\mathcal{L}_1} \le (2n+1)\|G(s)\|_\infty \,,$$

where n denotes the number of states in the minimal realization of $G(s)$.

Next, we suggest application of the μ synthesis problem for filter design of $\mathcal{L}_1$ adaptive controller. Namely, computation of $C(s)$ that minimizes the $\mathcal{H}_\infty$–norm of $G(s)$ reduces to the synthesis problem $\min_{K_I} \mu_\Delta \left(\mathcal{F}_\ell(\bar{N}_I, K_I)\right)$, under the assumption of nominal stability of the system. While this synthesis problem is not yet fully solved, a common approach is the so–called $D-K$ iteration:

$$\min_{K_I} \inf_{D, D^{-1} \in \mathcal{D}} \|D\mathcal{F}_\ell(\bar{N}_I, K_I)D^{-1}\|_\infty \,, \tag{15}$$

where $\mathcal{D}$ is a set of commutative $\mathcal{H}_\infty$ operators to the uncertainty Δ. One needs to solve (15) for K_I and D. For this, the off–the–shelf software [22] can be used.

Example 3. Consider the ideal system model in (10) and the actual plant with an input multiplicative uncertainty:

$$A(s) = \hat{A}_\delta(s)(1 + \Delta_m(s)) \,,$$

where $\hat{A}_\delta(s) = 1/(s^2 - 4s + \delta_1)$, $\delta_1 \in [1.6, 5.6]$, and $\Delta_m(s) \in \Delta_c$. In particular, we consider the unmodeled dynamics $\Delta_m(s)$, about 10% at low–frequency, rising to 100% at 50 radians/second. If we consider the standard internal model controller for $K(s)$ with a lowpass filter $C(s;\tau,k) = \frac{1}{(\tau s+1)^k}$, it gives the following transfer functions of consideration:

$$\bar{A}_I(s) = \begin{bmatrix} 0 & 1 \\ \frac{1}{s(s-1)^2} & \frac{1}{s(s-1)^2} \end{bmatrix} \,,\; K_I(s) = \frac{s^2 + 2\zeta\omega_n s + \omega_n^2}{((\tau s+1)^k - 1)\omega_n^2} \,,$$

$$N_{11}(s) = \frac{s^2 + 2\zeta\omega_n s + \omega_n^2}{\omega_n^2 (s-1)^2((\tau s+1)^k - 1) - (s^2 + 2\zeta\omega_n s + \omega_n^2)} \,.$$

Using the Matlab Toolboxes [22], [23], the robust stability and performance margins are obtained as $\gamma_s = 1.500$ and $\gamma_p = 1.497$, respectively. Furthermore, its nominal performance is given by $\|\hat{G}(s)\|_{\mathcal{L}_1} \approx 0.028$ for $\tau = 0.01$ and $k = 3$, where $\hat{G}(s)$ is the transfer function $G(s)$ with the replacement of $A(s)$ by its nominal model $\hat{A}(s) = \frac{1}{s^2-4s+4}(1 + \frac{1}{s^2+20s+10})$.

IV. Conclusion

In this paper we exploited the connection between the $\mathcal{L}_1$ adaptive output–feedback control and the disturbance observers, which helped us to show that the methods for compensator design from DOB and robust control literature can be used towards the filter design in $\mathcal{L}_1$ adaptive output feedback controller.

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