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Kalman Filtering for a Class of Degenerate Systems with Intermittent Observations

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Abstract— **This paper addresses the performance of a Kalman filter when measurements are intermittently available, i.e., network transmission problems. More specifically, we present a method to determine whether the expected value of the estimation error covariance is bounded for a given stochastic network model. The method applies to very general network models and for a class of degenerate systems. It can be easily adapted to non-degenerate systems, recovering known results on the critical value. The main result follows from the convergence conditions on a series that describes the bounds on the expected error covariance.**

I. Introduction

The performance of a Kalman filter when measurements are intermittently available is studied in this paper. The problem has attracted great interest in the recent years, partly due to the development of communications technologies, which today, allows distributed control and monitoring in a great range of applications. When measurements sent through a communication channel are subject to random losses, the estimation accuracy of the Kalman filter will deteriorate. In [1], the authors established the mathematical foundations for the basic problem and pointed out that the covariance of the estimation error does not reach a steady state. Since then, several authors have studied different aspects of the problem, using different assumptions on network models and protocols.

The most common entity studied in this context is the Error Covariance (EC) matrix. When a Kalman filter is subject to intermittent measurements, its EC becomes a random variable and its statistical properties are studied. Bounds on the expected value of the EC are given in [2], [3], [4], [5]. In [3], [6], higher order statistics of the EC are addressed, while in [7], [8], [9] the distribution function of the EC is studied.

The question of whether the Expected value of the EC (EEC) is bounded by a constant matrix or unbounded is the topic of this paper. The answer depends on the system under consideration and on the given stochastic network model. Two of the most popular network models used are the independent and identically distributed (i.i.d.) and the Gilbert-Elliot [10] models. Under the i.i.d. model assumption, the only network parameter is the probability that a given measurement arrives at the estimator. The smallest probability that yields a finite EEC is called the critical value. In the case of more elaborated models, such as Gilbert-Elliot, there might be more than one parameter that controls the dichotomy of behaviors.

Even when the i.i.d. network model is used, finding the critical value for a general system is still an open problem. In [1], the authors showed that there exists a critical value, i.e., the EEC is bounded if and only if the arrival probability is greater than the critical value. They also provided lower and upper bounds on measurement arrival probability in order for the EEC to be bounded, and for the particular case that the observation matrix C is invertible, these bounds are tight, i.e., the critical value is given. This condition is relaxed to only requiring the invertibility of the part of the matrix C corresponding to the observable subspace in [11]. In [12] the conditions under which the critical value is known were expanded to the case when the eigenvalues of the system have distinct absolute values.

As an attempt to model effects like the fading of the communication channel, the Gilbert-Elliot network model was introduced. It assumes that the availability of a measurement depends on the availability of previous ones. The problem was first introduced in [13], where necessary conditions for the stability of the peak covariance were developed. In [14], these conditions were further improved, providing less conservative results for systems with observability index of two. Again, to the best of our knowledge, there is no analytic solution to find the critical parameters for the Gilbert-Elliot network model in the general case.

A class of systems that still lacks on the knowledge of the critical value includes the degenerate systems. We present a formal definition of degenerate systems in section II, which is equivalent to the one in [6]. We point out that most of the results available in the literature apply only to non-degenerate systems.

In this paper, we study a class of discrete-time linear degenerate systems in which all eigenvalues have the same magnitude, but different phases. We also require the differences on the phases to be rational numbers and the system dynamics' matrix to be diagonalizable. Although our results are for degenerate systems, they can be easily modified to account for the simpler case of non-degenerate systems, recovering known results in the literature. The main result follows from the convergence conditions on a series that describes bounds on the expected error covariance. A necessary and sufficient condition for the finiteness of the EEC is presented in terms of the probability to observe a given class of sequences. This probability is then derived in a later section. We point out that the results presented here are preliminary and that they can be extended relatively

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easily to more general systems.

II. Problem Statement

Consider the discrete-time linear system:

$$\left\{ \begin{array}{lcl} x_{t+1} & = & Ax_t + w_t \\ y_t & = & Cx_t + v_t \end{array} \right. \tag{1}$$

where the state vector $x_t \in \mathbb{R}^n$ has initial condition $x_0 \sim N(0, P_0)$, $y_t \in \mathbb{R}^p$ is the measurement, $w_t \sim N(0, Q)$ is the process noise and $v_t \sim N(0, R)$ is the measurement noise. It was shown in [1] that even when the measurements are subject to random losses, the standard Kalman filter still obtains the best estimate $\hat{x}_t$ of the state x_t. In this case, however, the covariance P_t of the state estimation error becomes a random matrix.

We assume that the measurements y_t are sent to the Kalman estimator through a network subject to random packet losses, and that there is no delay in the transmission. Let γ_t be a binary random variable describing the arrival of a measurement at time t, i.e., $\gamma_t = 1$ when y_t is received at the estimator and $\gamma_t = 0$ otherwise.

The update equation for P_t depends on the availability of measurements. When a measurement is available, both steps, prediction and update, are performed. When a measurement is not available, only the prediction step can be computed. The equation for P_t can then be written as follows:

$$P_{t+1} = \left\{ \begin{array}{ll} \Phi_1(P_t), & \gamma_t = 1 \\ \Phi_0(P_t), & \gamma_t = 0 \end{array} \right. \tag{2}$$

with

$$\begin{aligned} \Phi_1(P_t) &= AP_tA' + Q + \\ & \quad -AP_tC'(CP_tC' + R)^{-1}CP_tA' \qquad (3) \\ \Phi_0(P_t) &= AP_tA' + Q. \qquad (4) \end{aligned}$$

We point out that when all measurements are available, and the Kalman filter reaches its steady state, the EC is given by the solution of the following algebraic Riccati equation

$$\underline{P} = A\underline{P}A' + Q - A\underline{P}C'(C\underline{P}C' + R)^{-1}C\underline{P}A'. \tag{5}$$

In [15], the authors introduce the definition of a degenerate system, which applies to systems that can be written in a diagonal standard form. A diagonal system in the form (1) is said to be degenerate if it contains at least one sub-system (i.e., a pair $\tilde{A} = \operatorname{diag}\{\alpha_{i_1}, \cdots, \alpha_{i_J}\}$, $\tilde{C} = [c_{i_1}, \cdots, c_{i_J}]$, where α_{i_j} and c_{i_j}, $j = 1, \cdots, J$, denote the i_j-th diagonal entry of A and the i_j-th column of C, respectively), whose eigenvalues have the same absolute value, and such that $\tilde{C}$ does not have full column rank.

Notice that degenerate systems arise in systems containing more than one eigenvalue with the same absolute value and a wide matrix C. An example of such a system is

$$A = \begin{bmatrix} 0 & 1 \\ -1 & 1 \end{bmatrix}, \quad C = \begin{bmatrix} 1 & 1 \end{bmatrix}. \tag{6}$$

In this paper, we present a necessary and sufficient condition for the EEC of a class of degenerate systems to be finite. More precisely, the proposed criterion requires the following assumption:

Assumption 2.1: A is diagonalizable and there exists $T \in \mathbb{N}$ such that $A^T = \nu^T I$, for some $\nu \in \mathbb{C}$. Also, the pair (A, C) is observable.

Notice that Assumption 2.1 implies that all the eigenvalues of A have the same magnitude. We point out that if A has eigenvalues with distinct magnitudes, the proposed method can be used to obtain a necessary condition for the finiteness of the EEC. This is done by applying it to each sub-system that contains eigenvalues with the equal absolute values.

We use the following notation. For A satisfying Assumption 2.1, α denotes the magnitude of its eigenvalues. For given $N \in \mathbb{N}$ and $0 \le m \le 2^N - 1$, the symbol S_m^N denotes the binary sequence of length N formed by the binary representation of m. We also use $S_m^N(i)$, $i = 1, \cdots, N$ to denote the i-th entry of the sequence, i.e.,

$$S_m^N = [S_m^N(1),\ S_m^N(2), \ldots,\ S_m^N(N)] \tag{7}$$

and

$$m = \sum_{k=1}^{N} 2^{k-1} S_m^N(k). \tag{8}$$

For a given sequence S_m^N, and a matrix $P \in \mathbb{R}^{n \times n}$, we define the map

$$\phi(P, S_m^N) = \Phi_{S_m^N(N)} \circ \Phi_{S_m^N(N-1)} \circ \ldots \Phi_{S_m^N(1)}(P) \tag{9}$$

where $\circ$ denotes the composition of functions (i.e. $f \circ g(x) = f(g(x))$). Notice that if m is such that S_m^N represents the sequence of measurements available from $t = 0$ to $t = N-1$, then

$$P_N = \phi(P_0, S_m^N). \tag{10}$$

We use $\mathbb{P}(S_m^N)$ to denote the probability that the sequence of available measurements in the last N sampling times is as in S_m^N.

Notice that the expected value of $\|P_N\|$ can be written as

$$\mathbb{E}(\|P_N\|) = \sum_{m=0}^{2^N-1} \mathbb{P}\left(S_m^N\right) \|\phi(P_0, S_m^N)\|. \tag{11}$$

We are interested in finding the conditions for

$$\lim_{N \to \infty} \mathbb{E}(\|P_N\|) < \infty. \tag{12}$$

III. Necessary and Sufficient Condition for the Finiteness of the EEC

We define the observability matrix corresponding to the sequence S_m^N as

$$O(S_m^N) \triangleq R(S_m^N) \begin{bmatrix} C \\ CA \\ \vdots \\ CA^{N-1} \end{bmatrix} \tag{13}$$

where $R(S_m^N)$ is the matrix that, when pre-multiplying, removes the rows corresponding to lost measurements. To simplify the notation, we will often omit the argument of O when it is clear from the context.

We can write the vector Y_N containing all the available measurements as

$$Y_N = Ox_0 + FW + V, \tag{14}$$
$$x_N = A^N x_0 + GW, \tag{15}$$

where

$$Y_N = R(S_m^N)\begin{bmatrix} y_0 \\ \vdots \\ y_N \end{bmatrix}, W = \begin{bmatrix} w_0 \\ \vdots \\ w_{N-1} \end{bmatrix},$$
$$V = \begin{bmatrix} v_0 \\ \vdots \\ v_{N-1} \end{bmatrix}, G = \left[A^{N-1}, \cdots, A, I\right],$$
$$F = R(S_m^N)\begin{bmatrix} 0 & 0 & \cdots & 0 \\ C & 0 & \cdots & 0 \\ CA & C & \ddots & \vdots \\ \vdots & \vdots & \ddots & 0 \\ CA^{N-1} & CA^{N-2} & \cdots & C \end{bmatrix}.$$

From [16, p. 39], we have that the estimation of x_N conditioned on Y_N produces the error covariance

$$P_N = \Sigma_x - \Sigma_{xY}\Sigma_Y^{-1}\Sigma_{xY}^*, \tag{16}$$

where

$$\Sigma_x = A^N P_0 A^{N*} + G\Sigma_W G^* \tag{17}$$
$$\Sigma_{xY} = A^T P_0 O + G\Sigma_W F^* \tag{18}$$
$$\Sigma_Y = OP_0O^* + F\Sigma_W F^* + \Sigma_V, \tag{19}$$

with $\Sigma_W = Q \otimes I$ and $\Sigma_V = R \otimes I$, where $\otimes$ denotes the Kronecker product.

Define

$$\mathcal{R}^N \triangleq \{m : \ O(S_m^N) \text{ does not have FCR}\}. \tag{20}$$

The following two lemmas state an upper bound for the growing rate of $\|P_N\|$, when $m \in \overline{\mathcal{R}}^N$, and a lower bound for the case when $m \in \mathcal{R}^N$.

Lemma 3.1: Let m be such that S_m^N is the sequence of measurements received from time 0 to $N-1$. If the system satisfies Assumption 2.1, and $O(S_m^N)$ has FCR (i.e., $m \in \overline{\mathcal{R}}^N$), then, there exists a constant p_N, independent of P_0, such that

$$\|P_N\| \leq p_N. \tag{21}$$

Proof: Consider the following sub-optimal estimator and its associated error:

$$\begin{aligned} \hat{x}_N &= A^T O^\dagger Y \qquad (22)\\ &= A^T O^\dagger\left(OA^{-N}x_N + (F - OA^{-N}G)W + V\right)\\ \tilde{x}_N &= x_N - \hat{x}_N \qquad (23)\\ &= \left(G - A^T O^\dagger F\right)W - A^T O^\dagger V \end{aligned}$$

The covariance of the estimation error is given by

$$\begin{aligned} \overline{P}_N &= \mathbb{E}(\tilde{x}_N\tilde{x}_N^*)\\ &= \left(G - A^T O^\dagger F\right)\Sigma_W\left(G - A^T O^\dagger F\right)^* +\\ &\quad \left(A^T O^\dagger\right)\Sigma_V\left(A^T O^\dagger\right)^*. \qquad (24) \end{aligned}$$

Notice that since the estimator is sub-optimal, we have $\|P_N\| \leq \|\overline{P}_N\|$. Also, $\|\overline{P}_N\|$ does not depend on the initial error covariance P_0 and its maximum value can be obtained evaluating all the sequences of length N, that result in $O(S_m^N)$ having FCR. The proof is concluded by making

$$p_N = \max_m \|\overline{P}_N\|. \tag{25}$$

■

Lemma 3.2: Let m be such that S_m^N is the sequence of measurements received from time 0 to $N-1$. If $O(S_m^N)$ does not have FCR, then

$$\|P_N\| \geq \alpha^{2N}\|P_0^{-1}\|^{-1}. \tag{26}$$

Proof: Suppose that the noises w_t and v_t are known for $0 \leq t < N$, i.e., $\Sigma_W = 0$ and $\Sigma_V = 0$, and let $\underline{P}_N$ be the resulting estimation error covariance. Notice that $P_N \geq \underline{P}_N$, and therefore $\|P_N\| \geq \|\underline{P}_N\|$. Let p be the smallest eigenvalue of P_0, i.e., $p \triangleq \|P_0^{-1}\|^{-1}$. We have

$$\begin{aligned} \underline{P}_N &= A^N P_0 A^{N*} - A^N P_0 O^*\left(OP_0O^*\right)^{-1}\left(A^N P_0 O^*\right)^*\\ &\geq p\left(A^N A^{N*} - A^N O^*\left(OO^*\right)^{-1}\left(A^N O^*\right)^*\right) \qquad (27)\\ &= pA^N\left(I - O^*\left(OO^*\right)^{-1}O\right)A^{N*} \qquad (28)\\ &= pA^N\left(I - O^\dagger O\right)A^{N*}. \qquad (29) \end{aligned}$$

Notice that $I - O^\dagger O$ is a projection, hence

$$\begin{aligned} \|\underline{P}_N\| &\geq p\|A^N\left(I - O^\dagger O\right)\|^2 \qquad (30)\\ &= p\alpha^{2N}\|\left(I - O^\dagger O\right)\|^2 \qquad (31)\\ &= p\alpha^{2N} \qquad (32)\\ &= \alpha^{2N}\|P_0^{-1}\|^{-1} \qquad (33) \end{aligned}$$

■

The following lemma is required to show the main result of the section.

Lemma 3.3: Define

$$\xi^N(x) \triangleq \sum_{m=1}^{2^N}\|\phi(xI, S_m^N)\|\mathbb{P}(S_m^N) \tag{34}$$
$$x_N^\star \triangleq \mathrm{sol}_x\left\{x = \xi^N(x)\right\}. \tag{35}$$

If there exists a finite solution $x_N^\star$, then it is an upper bound for the EEC.

$$x_N^\star \geq \lim_{N\to\infty}\mathbb{E}(\|P_N\|) \tag{36}$$

Proof: From the monotonicity of $\phi(\cdot, S_m^N)$ (see [1]), we have that

$$\phi(P_0, S_m^N) \leq \phi(\|P_0\|I, S_m^N). \tag{37}$$

Substituting (37) into (11), we have

$$\begin{aligned} \mathbb{E}(\|P_N\|) &\leq \sum_{m=1}^{2^N}\left\|\phi(\|P_0\|I, S_m^N)\right\|\mathbb{P}(S_m^N) \qquad (38)\\ &= \xi^N(\|P_0\|). \qquad (39) \end{aligned}$$

It follows from the concavity of $\phi(\cdot, S_m^N)$ (see [8]) that

$$\xi^{kN}(x) \leq \xi^{N^{(k)}}(x) \tag{40}$$

where $\xi^{N^{(k)}}(\cdot)$ is the composition of $\xi^N(\cdot)$ k times.

The proof is concluded by noting that

$$\begin{aligned}\lim_{N\to\infty}\mathbb{E}(\|P_N\|) &\leq \lim_{N\to\infty}\xi^N(\|P_0\|) &(41)\\ &= \lim_{k\to\infty}\xi^{kN}(\|P_0\|) &(42)\\ &\leq \lim_{k\to\infty}\xi^{N^{(k)}}(\|P_0\|) &(43)\\ &= x_N^\star. &(44)\end{aligned}$$

■

The following theorem states the main result of the section, namely, a necessary and sufficient condition for the EEC to be finite.

Theorem 3.1: Consider a system satisfying Assumption 2.1. If

$$\alpha^2 \limsup_{N\to\infty}\mathbb{P}\left\{\mathcal{R}^N\right\}^{1/N} > 1, \quad (45)$$

then

$$\lim_{N\to\infty}\mathbb{E}(\|P_N\|) = +\infty. \quad (46)$$

Also, if

$$\alpha^2 \limsup_{N\to\infty}\mathbb{P}\left\{\mathcal{R}^N\right\}^{1/N} < 1, \quad (47)$$

then

$$\lim_{N\to\infty}\mathbb{E}(\|P_N\|) < \infty \quad (48)$$

Proof: **Necessity:** From (11), we have that

$$\begin{aligned}\mathbb{E}(\|P_N\|) &= \sum_{m=0}^{2^N-1}\mathbb{P}\left\{S_m^N\right\}\left\|\phi(P_0,S_m^N)\right\|\\ &\geq \sum_{m\in\mathcal{R}^N}\mathbb{P}\left\{S_m^N\right\}\left\|\phi(P_0,S_m^N)\right\|\\ &\geq \sum_{m\in\mathcal{R}^N}\mathbb{P}\left\{S_m^N\right\}\alpha^{2N}\left\|P_0^{-1}\right\|^{-1}\\ &= \alpha^{2N}\left\|P_0^{-1}\right\|^{-1}\mathbb{P}\left\{\mathcal{R}^N\right\}\\ &= \left(\alpha^2\mathbb{P}\left\{\mathcal{R}^N\right\}^{1/N}\right)^N\left\|P_0^{-1}\right\|^{-1},\end{aligned}$$

and the result follows.

Sufficiency: From (34), we have

$$\begin{aligned}\xi^N(x) &= \sum_{m\in\mathcal{R}^N}\|\phi(xI,S_m^N)\|\mathbb{P}(S_m^N)\\ &\quad+\sum_{m\notin\mathcal{R}^N}\|\phi(xI,S_m^N)\|\mathbb{P}(S_m^N). &(49)\end{aligned}$$

Notice that

$$\begin{aligned}\|\phi(xI,S_0^N)\| &= \|A^N x A'^N + \sum_{j=0}^{N-1}A^j Q A'^j\| &(50)\\ &\leq \|\mathbf{A}\|^{2N}x + \sum_{j=0}^{N-1}\|\mathbf{A}\|^{2j}\|\mathbf{Q}\| &(51)\\ &= \alpha^{2N}x + \sum_{j=0}^{N-1}\alpha^{2j}\|\mathbf{Q}\|. &(52)\end{aligned}$$

Since S_0^N is a sequence with all measurements lost, (52) provides an upper bound for any sequence.

Define

$$\overline{\xi}^N(x) \triangleq \left(\alpha^{2N}\mathbb{P}(\mathcal{R}^N)\right)x + \beta \quad (53)$$

where $\beta = \mathbb{P}(\mathcal{R}^N)\sum_{j=0}^{N-1}\alpha^{2j}\|Q\| + p_N\mathbb{P}(\overline{\mathcal{R}}^N)$. Using (52) and (21) in (49), we have that $\overline{\xi}^N(x)\geq\xi^N(x)$.

We have that

$$\lim_{N\to\infty}\alpha^{2N}\mathbb{P}\left\{\mathcal{R}^N\right\} = \lim_{N\to\infty}\left(\alpha^2\mathbb{P}\left\{\mathcal{R}^N\right\}^{1/N}\right)^N = 0.$$

Then, there exists N_0 such that, for all $N > N_0$, $\alpha^{2N}k\mathbb{P}(\mathcal{R}^N) < 1$. This in turn implies that there exists x_0 such that $x_0 = \overline{\xi}^{N_0}(x_0)$, and the result follows from Lemma 3.3. ■

IV. COMPUTING $\limsup_{N\to\infty}\mathbb{P}(\mathcal{R}^N)^{1/N}$

As pointed out in [6], when A is diagonalizable, we can assume without loss of generality that it is diagonal. Consider assumption 2.1, and let T be the smallest integer such that

$$A^T = \alpha^T\exp(i\theta T)I, \quad (54)$$

for some and $\theta\in(-\pi,\pi]$. Therefore, if the measurement y_t is available, the measurement y_{t+T} will not increase the rank of the observability matrix.

We define the cumulative arrival sequence as

$$G^T(S_m^N) = \{g_1,g_2,\ldots,g_T\} \quad (55)$$

with

$$g_j = \begin{cases}0, & S_m^N(kT+j)=0,\forall k=0,\cdots,(N-j)/T\\ 1, & \text{otherwise.}\end{cases} \quad (56)$$

It follows that

$$\mathrm{rank}(O(S_m^N)) = \mathrm{rank}(O(G^T(S_m^N))). \quad (57)$$

The next lemma computes the probability that the observability matrix corresponding to the random sequence S_m^N has FCR.

Lemma 4.1: Let $\mathcal{S}^T$ denote the set of sequences of length T, whose associated observability matrix does not have FCR, i.e.,

$$\mathcal{S}^T \triangleq \{S_m^T:\ m\in\mathcal{R}^T\}. \quad (58)$$

Let $[\mathcal{S}^T]_j$, $j=1,\ 2,\ \ldots,\ J$ denote the j-th element of the set $\mathcal{S}^T$, with $[\mathcal{S}^T]_1 = S_0^T$. Define the matrix M such that its (i,j)-th entry $[M]_{i,j}$ is given by

$$[M]_{i,j} \triangleq \mathbb{P}\left(G^T\{S_m^N,\gamma\} = [\mathcal{S}^T]_i | G^T(S_m^N) = [\mathcal{S}^T]_j\right), \quad (59)$$

where $\{S_m^N,\gamma\}$ is the sequence with length $N+1$ whose first N elements are as in S_m^N and the last element is γ. We have

$$\mathbb{P}(\mathcal{R}^N) = uM^N z \quad (60)$$

with

$$u = [1\ 1\ \ldots\ 1] \text{ and } z = [1\ 0\ \ldots\ 0]' \quad (61)$$

of appropriate dimensions.

Proof: For time N, define the vector W_N containing the probabilities of $G^T(S_m^N)$ taking values in $\mathcal{S}^T$, i.e.,

$$W_N = \begin{bmatrix} \mathbb{P}\left(G^T(S_m^N) = [\mathcal{S}^T]_1\right) \\ \mathbb{P}\left(G^T(S_m^N) = [\mathcal{S}^T]_2\right) \\ \vdots \\ \mathbb{P}\left(G^T(S_m^N) = [\mathcal{S}^T]_J\right) \end{bmatrix}. \tag{62}$$

We can write a recursive expression for W_N as

$$W_{N+1} = MW_N. \tag{63}$$

Hence, for a given $N > 0$, the distribution W_N is given by

$$W_N = M^N W_0. \tag{64}$$

Since $[\mathcal{S}^T]_1$ is the empty sequence, the initial distribution is given by

$$W_0 = z. \tag{65}$$

Finally, we obtain the probability that $O(S_m^N)$ does not have full column rank by adding all the entries of the vector W_N, i.e., by pre-multiplying u to W_N. ■

Consider the following factorization of the matrix M:

$$M = V \begin{bmatrix} B_1 & 0 & \dots & 0 \\ 0 & B_2 & \dots & 0 \\ \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & \dots & B_J \end{bmatrix} V^{-1} \tag{66}$$

where $B_j, j = 1, \dots, J$ are the Jordan blocks of M. Define

$$U = \begin{bmatrix} U_1 & U_2 & \dots & U_J \end{bmatrix} = uV \tag{67}$$

and

$$Z = \begin{bmatrix} Z_1 \\ Z_2 \\ \vdots \\ Z_J \end{bmatrix} = V^{-1} z \tag{68}$$

such that

$$uM^N z = \sum_{j=1}^{J} U_j B_j^N Z_j. \tag{69}$$

Let $\lambda_j \exp(i\theta_j)$ be the eigenvalue associated with the Jordan block $B_j \in \mathbb{R}^{N_j \times N_j}$, with $\lambda_j, \theta_j \in \mathbb{R}$. We have

$$U_j B_j^N Z_j = \lambda_j^k \psi_j(N) \tag{70}$$

where

$$\psi_j(N) = \sum_{n=1}^{N_j} \sum_{m=1}^{N_j} [U_j]_n [Z_j]_m a_{j,n,m}(N), \tag{71}$$

and $a_{j,n,m}(N)$ is a polynomial in N given by

$$a_{j,n,m}(N) = \begin{cases} \binom{N}{m-n} \lambda_j^{n-m} e^{i\theta_j(n-m+N)}, & n \le m \\ 0, & n > m. \end{cases} \tag{72}$$

We now group the Jordan blocks whose eigenvalues have the same magnitude, to obtain

$$uM^N z = \sum_{l=1}^{L} \Lambda_l^N \Psi_l(N) \tag{73}$$

with $\Lambda_1 > \Lambda_2 > \dots, \Lambda_L \in \mathbb{R}$ and

$$\Psi_l(N) = \sum_{\substack{j=0: \\ \lambda_j = \Lambda_l}}^{J} \psi_j(N). \tag{74}$$

Notice that $\Psi_l(N) \neq 0$ for some $l = 1, \cdots, L$. We can then state the main result of this section.

Theorem 4.1: Let M be defined as in (59), u, z as in (61) and $\Lambda_l, \Psi_l(N)$ as in (73). Let l_0 be the smallest integer such that $\Psi_{l_0}(N) \neq 0$ for some N. Then,

$$\limsup_{N \to \infty} \mathbb{P}(\mathcal{R}^N)^{1/N} = \Lambda_{l_0}. \tag{75}$$

Proof: The proof is divided in six steps.

1) Define

$$G(N) \triangleq \Psi_{l_0}(N) + \sum_{l=l_0+1}^{L} \left(\frac{\Lambda_l}{\Lambda_{l_0}}\right)^N \Psi_l(N) \tag{76}$$

and notice that

$$uM^N z = \Lambda_{l_0}^N G(N). \tag{77}$$

From (60), we have

$$\limsup_{N \to \infty} \mathbb{P}\left(\mathcal{R}^N\right)^{1/N} = \limsup_{N \to \infty} \left(\Lambda_{l_0}^N G(N)\right)^{1/N} \tag{78}$$

$$= \Lambda_{l_0} \limsup_{N \to \infty} \left(G(N)\right)^{1/N} \tag{79}$$

2) Let p_0 be the greatest power of N in $\Psi_{l_0}(N)$. It is straightforward to verify that there exist $K \in \mathbb{N}$ and $g \in \mathbb{R}$ such that

$$gN^{p_0} \ge |G(N)|, \text{ for all } N > K. \tag{80}$$

Now, since

$$\lim_{N \to \infty} |gN^{p_0}|^{1/N} = 1 \tag{81}$$

it follows that

$$\lim_{N \to \infty} \sup |G(N)|^{1/N} \le 1. \tag{82}$$

3) Notice that for every $\epsilon > 0$, there exists a K_1 such that

$$\left| \sum_{l=l_0+1}^{L} \left(\frac{\Lambda_l}{\Lambda_{l_0}}\right)^N \Psi_l(N) \right| < \epsilon, \ \forall N > K_1. \tag{83}$$

4) Write $\Psi_{l_0}(N)$ as

$$\Psi_{l_0}(N) = \sum_{\substack{j=0 \\ \lambda_j = \Lambda_{l_0}}}^{J} \psi_j(N) = \sum_{p=0}^{p_0} N^p \beta_p(N) \tag{84}$$

$$= N^{p_0} \left(\beta_{p_0}(N) + \sum_{p=0}^{p_0-1} N^{p-p_0} \beta_p(N) \right)$$

where $\beta_p(N), \ p = 0, \dots, p_0$ are linear combinations of complex exponential functions. Then, we have that for every $\epsilon > 0$, there exists K_2 such that

$$\left| \sum_{p=0}^{p_0-1} N^{p-p_0} \beta_p(N) \right| < \epsilon, \ \forall N > K_2. \tag{85}$$

5) Now, since $\beta_{p_0}(N)$ is a finite linear combination of complex exponentials, it follows from [17, Section VI.5] that $\beta_{p_0}(N)$ is an almost-periodic function. Hence, for every $0 < \epsilon < \sup_{N\in\mathbb{N}} \beta_{p_0}(N)/2$, there exists an infinite sequence $T_j \in \mathbb{N}$ such that

$$|\beta_{p_0}(T_j)| \geq 2\epsilon. \tag{86}$$

Now, define the increasing sequence $\tilde{T}_j \in \mathbb{N}$, by taking from the sequence $T_j \in \mathbb{N}$, the values that are greater than $\max(K_1, K_2)$. Substituting (83), (85) and (86) in (76), we have that, for all $N \in \tilde{T}_j$

$$|G(N)| \geq (N^{p_0} - 1)\epsilon. \tag{87}$$

Hence, we have

$$\lim_{N\to\infty} \sup |G(N)|^{1/N} \geq 1. \tag{88}$$

6) From (82) and (88), it follows that

$$\lim_{N\to\infty} \sup |G(N)|^{1/N} = 1, \tag{89}$$

and the result follows by substituting (89) in (79). ■

Combining Theorems 3.1 and 4.1, we have the following corollary.

Corollary 4.1: Consider the system (1) satisfying Assumption 2.1. Let M be defined as in (59), u, z as in (61) and $\Lambda_l, \Psi_l(k)$ as in (73). Let l_0 be the smallest integer such that $\Psi_{l_0}(k) \neq 0$ for some k. Then,

$$\Lambda_{l_0}\alpha^2 > 1 \quad \Rightarrow \quad \lim_{N\to\infty} \mathbb{E}(\|P_N\|) = \infty \tag{90}$$

$$\Lambda_{l_0}\alpha^2 < 1 \quad \Rightarrow \quad \lim_{N\to\infty} \mathbb{E}(\|P_N\|) < \infty. \tag{91}$$

V. Conclusion and Future Work

In this paper we studied the state estimation error covariance produced by a Kalman filter whose measurements are subject to random losses. We did so considering a class of degenerate systems. We provided a necessary and sufficient condition for the limit of the expected value of the norm of the error covariance to be finite.

We pointed out how the presented result can be used to derive a necessary condition for any arbitrary system. In a future work, we aim to extend this to provide a necessary and sufficient condition for general systems.

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