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Canonical Forms for Nonlinear Discrete Time Control Systems

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***Abstract*— In this paper we provide a canonical form for discrete-time control systems whose linear approximation around an equilibrium is controllable and prove that two systems are feedback equivalent if and only if their canonical forms coincide. This is a nice generalization of results obtained for continuous time control systems. We also compute the homogeneous invariants under the action of a homogeneous feedback group. Consequently, as for the continuous systems, we deduce that the discrete time systems in consideration do not admit nontrivial symmetries, i.e., a map preserving the dynamics.**

Keywords: discrete-time, normal forms, homogeneous transformations.

I. Introduction

The study of normal forms of vector fields (differential dynamical systems) and maps (discrete-time systems) via a formal approach can be traced back to the works of Cartan and Poincaré. Poincaré in his Ph.D. thesis (see [16]) proposed a formal approach which consists of expanding the dynamics of the vector field or map via Taylor series and looking for a change of coordinates (called formal transformation) that simplifies, step by step, the terms of same degree of the system. For a vector field $\nu(x)$ or equivalently, the dynamical system (resp. map)

$$\dot{x} = \nu(x), \quad (\text{resp.} \quad x^+ = \nu(x))$$

around an equilibrium point $x_e = 0$, i.e., $\nu(0) = 0$, we associate the Taylor series expansion for dynamical systems

$$\dot{x} = \nu(x) = \nu^{[1]}(x) + \nu^{[2]}(x) + \cdots = \sum_{m=1}^{\infty} \nu^{[m]}(x),$$

respectively for maps,

$$x^+ = \nu(x) = \nu^{[1]}(x) + \nu^{[2]}(x) + \cdots = \sum_{m=1}^{\infty} \nu^{[m]}(x),$$

where for any $m \geq 1$, each component of the vector field $\nu^{[m]}(x)$, say $\nu_j^{[m]}(x), j = 1, \ldots, n$, is a homogeneous polynomial of degree m. For a change of coordinates $z = \varphi(x)$ we consider its Taylor series expansion

$$z = \varphi(x) = \varphi^{[1]}(x) + \varphi^{[2]}(x) + \cdots = \sum_{m=1}^{\infty} \varphi^{[m]}(x).$$

The first problem addressed by Poincaré is whether a formal transformation $z = \varphi(x)$ exists that transforms the dynamical system (resp. map) into a linear differential equation (resp. linear map)

$$\dot{z} = \varphi_* \nu(z) = Az, \quad (\text{resp.} \quad z^+ = \varphi(\nu(\varphi^{-1}(z))) = Az.)$$

We refer to the literature for conditions on linearization of vector fields which are strictly related to the eigenvalues of the matrix A.

For continuous time control systems

$$\dot{x} = f(x) + g(x)u, x \in \mathbb{R}^n, u \in \mathbb{R}^m$$

Krener was the first to adapt Poincaré's classical method to control systems and was followed by a vast literature on normal forms [10], [11], [12], [13], [18].

The continuous time method was extended to discrete time control systems with various normal forms obtained in [2], [5], [14] for quadratic and cubic terms. Normal forms for all degrees was obtained by [9] for linearly controllable discrete control systems and recently another treatment appeared in [15]. Linearization and/or approximate linearization of discrete time control systems have been addressed in several papers [1], [15] and the references therein.

Let us acknowledge that the formal approach has proved to be very useful for both continuous time and discrete time systems. Stabilization of systems with uncontrollable linearization, in continuous and discrete-time, were studied in [3], [4], [6], [7], [8], [13], [14], a complete description of symmetries around equilibrium [17], [21], and a characterization of systems equivalent to feedforward forms obtained in [19], [20].

In this paper, we generalize the results of [18] by providing a canonical form for discrete time control systems. The main result states the fact that two discrete time control systems are feedback equivalent if and only if their canonical forms coincide. As a consequence of this canonical form, we also deduce that single-input discrete-time systems with controllable linearization do not admit symmetries (see [21] for continuous-time systems).

The paper is organized as following: we first recall briefly our result on normal forms [9] and in Section III, we construct a canonical form for discrete-time nonlinear control systems whose linear approximation is controllable followed by an illustrative example. The proofs are given in Section IV. In the last section we extend the results of [21] to single-input discrete-time systems whose linear approximation is controllable, showing that if the system is not truly linearizable, then it admits no symmetries preserving the equilibrium.

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II. Normal forms

We briefly recall here some results obtained for normal forms (see [9] for details) and [15] for a different approach.

Consider a discrete-time nonlinear control system

$$\Pi : x^+ = f(x,u), \quad x(\cdot)\in\mathbb{R}^n \quad u(\cdot)\in\mathbb{R},$$

where $x^+(k) = x(k+1)$, and $f(x,u) = f(x(k),u(k))$ for any $k\in\mathbb{N}$, and a feedback transformation of the form

$$\Upsilon : \begin{cases} z = \varphi(x) \\ u = \gamma(z,v). \end{cases}$$

The transformation Υ brings Π to the system

$$\tilde{\Pi} : z^+ = \tilde{f}(z,v) ,$$

whose dynamics are given by

$$\tilde{f}(z,v) = \varphi(f(\varphi^{-1}(z),\gamma(z,v))) .$$

Conversely, if systems Π and $\tilde{\Pi}$ are given, we say that they are *feedback equivalent* if there is a feedback transformation Υ that maps Π into $\tilde{\Pi}$ as above. We suppose that $(0,0)\in\mathbb{R}^n\times\mathbb{R}$ is an equilibrium point, i.e., $f(0,0) = 0$, and we expand the system via Taylor series

$$\Pi^\infty : x^+ = Fx + Gu + \sum_{m=2}^{\infty} f^{[m]}(x,u).$$

Few important questions are addressed. What is the simplest form the map Π can take after action of feedback transformation Υ? Is that form unique? Does there exist feedback transformations Υ that leave invariant the map Π, that is, such that $\tilde{f}(z,v) = f(z,v)$? We would call the diffeomorphism $z = \varphi(x)$ a symmetry of Π if there is a feedback $u = \gamma(x,v)$ so that the feedback transformation Υ leaves Π invariant. We obtained the following [9].

Theorem II.1 *The control system Π^∞ is feedback equivalent, by a formal feedback transformation Υ^∞ of the form*

$$\Upsilon^\infty : \begin{cases} z = \varphi(x) = Tx + \sum_{m=2}^{\infty}\varphi^{[m]}(x) \\ u = \gamma(x,v) = Kx + Lv + \sum_{m=2}^{\infty}\gamma^{[m]}(x,v) \end{cases}$$

to the normal form

$$\Pi^\infty_{NF} : z^+ = Az + Bv + \sum_{m=2}^{\infty}\bar{f}^{[m]}(z,v) ,$$

where for any $m\geq 2$, we have

$$\bar{f}^{[m]}_j(z,v) = \begin{cases} \sum_{i=j+2}^{n+1} z_1 z_i P^{[m-2]}_{j,i}(\bar{z}_i) & \text{if } 1\leq j<n \\ 0 & \text{if } j=n. \end{cases} \tag{II.1}$$

Above, $z_{n+1}\triangleq v$ denotes the control, $\bar{z}_i = (z_1,\cdots,z_i)$, and the pair (A,B) is in Brunovský canonical form.

The formal transformation Υ^∞ is viewed as a composition $\Upsilon^\infty = \cdots\circ\Upsilon^m\circ\cdots\circ\Upsilon^1$, where for

$$\Upsilon^1 : \begin{cases} z = Tx \\ u = Kx + Lv \end{cases}$$

and $\forall m\geq 2$ the homogeneous feedback transformation

$$\Upsilon^m : \begin{cases} z = x + \varphi^{[m]}(x) \\ u = v + \gamma^{[m]}(x,v) \end{cases}$$

act on the corresponding homogeneous part of the system as

Proposition II.2 *The homogeneous feedback transformation Υ^m leaves invariant all terms of Π^∞ of degree smaller than m, and transforms the homogeneous part $f^{[m]}(x,u)$ as*

$$\tilde{f}^{[m]}(x,u) = f^{[m]}(x,u) + \varphi^{[m]}(Ax+Bu) - A\varphi^{[m]}(x) + B\gamma(x,u)$$

or equivalently, for all $1\leq j\leq n-1$,

$$\begin{aligned} \varphi^{[m]}_j(Ax+Bu) - \varphi^{[m]}_{j+1}(x) &= \tilde{f}^{[m]}_j(x,u) - f^{[m]}_j(x,u) \\ \varphi^{[m]}_n(Ax+Bu) + \gamma^{[m]}(x,u) &= \tilde{f}^{[m]}_n(x,u) - f^{[m]}_n(x,u). \end{aligned} \tag{II.2}$$

A. m-Invariants

First, an invariant under a feedback group transformation is an object (property, function, vector function, relationship) that is preserved by the action of the group. In other words all elements of the same equivalence group share that same object. In this section we investigate potential invariants related to the action of the feedback transformation Υ^m.

Let us introduce some notation. For convenience we will put $u\triangleq x_{n+1}$, and for any $1\leq k\leq i\leq n+1$, we will write

$$\bar{x}^i_k = (x_k,\dots,x_i,0,\dots,0)^T\in\mathbb{R}^{n+1}.$$

Notice that any homogeneous function $h^{[m]}(x_1,\dots,x_{n+1})$ can be decomposed uniquely as following

$$h^{[m]}(x_1,\dots,x_{n+1}) = \sum_{1\leq k\leq i\leq n}^{n+1}\int_0^{x_k}\int_0^{x_i} h^{k[m-2]}_i(\bar{x}^i_k)ds_i ds_k,$$

where in the integrand, the variables x_k and x_i are respectively replaced by s_k and s_i.

Now consider the degree m homogeneous part $f^{[m]}(x,u)$ of Π^∞, and decompose each component $f^{[m]}_j(x,u)$ as:

$$f^{[m]}_j(x,u) = \sum_{k=1}^{n+1}\sum_{i=k}^{n+1}\int_0^{x_k}\int_0^{x_i} f^{k[m-2]}_{j,i}(\bar{x}^i_k)\, ds_i ds_k. \tag{II.3}$$

Define the homogeneous polynomials $a^{[m-2]}_{j,i}(\bar{x}_i)$ as

$$a^{[m-2]}_{j,i}(\bar{x}_i) = \sum_{k=1}^{n-i+2} f^{k[m-2]}_{j+k-1,i+k-1}(\bar{x}_i) \tag{II.4}$$

for any $1\leq j\leq n-1$ and any $j+2\leq i\leq n+1$. We claim that the homogeneous polynomials $a^{[m-2]}_{j,i}(\bar{x}_i)$ are invariants under the action of the homogeneous group transformation Υ^m. This fact is stated in the following proposition.

Proposition II.3 (a) *Consider a system Π^∞ and let $\tilde{\Pi}^\infty$ be its transform via a homogeneous feedback transformation Υ^m. Then we have $a_{j,i}^{[m-2]} = \tilde{a}_{j,i}^{[m-2]}$, that is,*

$$\sum_{k=1}^{n-i+2} f_{j+k-1,i+k-1}^{k[m-2]}(\bar{x}_i) = \sum_{k=1}^{n-i+2} \tilde{f}_{j+k-1,i+k-1}^{k[m-2]}(\bar{x}_i)$$

for all $1 \le j \le n-1$ and all $j+2 \le i \le n+1$.

(b) *The normal form $\bar{f}^{[m]}(z,v)$ given by (II.1) is such that*

$$\frac{\partial^2 \bar{f}_j^{[m]}}{\partial z_1 \partial z_i}\Big|_{W_i(z)} = a_{j,i}^{[m-2]}(\bar{z}_i),\ 3 \le j+2 \le i \le n+1, \quad \text{(II.5)}$$

where $W_i(z) = \left\{ z \in \mathbb{R}^{n+1} : z_{i+1} = \cdots = z_{n+1} = 0 \right\}$.

The proof of the proposition is given in the section IV.

III. Canonical forms.

The objective of this section is to give a canonical form for discrete-time control systems. Indeed, the normal form Π_{NF}^∞ given by is not unique under feedback transformations Υ^∞.

Let m_0 be the degree of the first non linearizable homogeneous terms of the system Π^∞. Without loss of generality, we can suppose that the system is of the form

$$\Pi^\infty : x^+ = Ax + Bu + \bar{f}^{[m_0]}(x,u) + \sum_{m=m_0+1}^{\infty} f^{[m]}(x,u), \quad \text{(III.1)}$$

where the components of $\bar{f}^{[m_0]}(x,u)$ are given by

$$\bar{f}_j^{[m_0]}(z,v) = \begin{cases} \sum_{i=j+2}^{n+1} z_1 z_i P_{j,i}^{[m_0-2]}(\bar{z}_i) & \text{if } 1 \le j < n \\ 0 & \text{if } j = n. \end{cases} \quad \text{(III.2)}$$

Let $1 \le j_* \le n-2$ be the largest integer such that $\bar{f}_{j_*}^{[m_0]} \neq 0$. Take $(i_1, \cdots, i_{n+1})$, with $i_1 + \cdots + i_{n+1} = m_0$, to be the largest $(n+1)$-tuple of nonnegative integers such that

$$\frac{\partial^{m_0} \bar{f}_{j_*}^{[m_0]}}{\partial z_1^{i_1} \cdots \partial z_{n+1}^{i_{n+1}}} \neq 0. \quad \text{(III.3)}$$

Now, we can state our main result.

Theorem III.1 *The control system Π^∞ is feedback equivalent via a formal feedback transformation Υ^∞ to the system*

$$\Pi_{CF}^\infty : z^+ = Az + Bv + \sum_{m=m_0}^{\infty} \bar{f}^{[m]}(z,v),$$

where for any $m \ge m_0$, we have

$$\bar{f}_j^{[m]}(z,v) = \begin{cases} \sum_{i=j+2}^{n+1} z_1 z_i P_{j,i}^{[m-2]}(\bar{z}_i) & \text{if } 1 \le j < n \\ 0 & \text{if } j = n; \end{cases} \quad \text{(III.4)}$$

Additionally for any $m \ge m_0 + 1$ we have

$$\text{(i)}\ \frac{\partial^{m_0} \bar{f}_{j_*}^{[m_0]}}{\partial z_1^{i_1} \cdots \partial z_{n+1}^{i_{n+1}}} = \pm 1;\ \text{(ii)}\ \frac{\partial^{m_0} \bar{f}_{j_*}^{[m]}}{\partial z_1^{i_1} \cdots \partial z_{n+1}^{i_{n+1}}}\Big|_{W_1} = 0. \quad \text{(III.5)}$$

The system Π_{CF}^∞ defined by (III.4)-(III.5) will be called the *canonical form* of Π^∞, and this name is justified by the following theorem.

Theorem III.2 *Two discrete time control systems Π_1^∞ and Π_2^∞ are feedback equivalent if and only if their canonical forms $\Pi_{1,CF}^\infty$ and $\Pi_{2,CF}^\infty$ coincide.*

Proposition II.3 will play a crucial role in the proof of Theorem III.1.

IV. Proofs

In this section we will prove our main results, that is, Proposition II.3, Theorem III.1, and III.2.

A. Proof of Proposition II.3

(a) It is enough to show the equality when the system Π^∞ is transformed into a normal form $\bar{\Pi}^\infty$. The general case follows from the following commutative diagram

$$\begin{tikzcd} \Pi^\infty \arrow[rr, "(\tilde{\Upsilon}^m)^{-1} \circ \Upsilon^m"] \arrow[dr, "\Upsilon^m"'] & & \tilde{\Pi}^\infty \arrow[dl, "\tilde{\Upsilon}^m"] \\ & \Pi_{NF}^\infty & \end{tikzcd}$$

Indeed, on one hand side $a_{j,i}^{[m-2]} = \bar{a}_{j,i}^{[m-2]}$ and on the other $\tilde{a}_{j,i}^{[m-2]} = \bar{a}_{j,i}^{[m-2]}$ which implies $a_{j,i}^{[m-2]} = \tilde{a}_{j,i}^{[m-2]}$.

(b) Let $1 \le j \le n-1$. Notice that

$$\varphi_j^{[m]}(x) = \sum_{k=1}^{n} \sum_{i=k}^{n} \int_0^{x_k} \int_0^{x_i} \varphi_{j,i}^{k[m-2]}(\bar{x}_k^i)\, ds_i ds_k,$$

from which we deduce that

$$\varphi_j^{[m]}(Ax+Bu) = \sum_{k=2}^{n+1} \sum_{i=k}^{n+1} \int_0^{x_k} \int_0^{x_i} \varphi_{j,i-1}^{k-1[m-2]}(\bar{x}_k^i)\, ds_i ds_k.$$

Decomposition (II.3) and Proposition II.2 (II.2) imply

$$\sum_{k=2}^{n+1} \sum_{i=k}^{n+1} \int_0^{x_k} \int_0^{x_i} \varphi_{j,i-1}^{k-1[m-2]}(\bar{x}_k^i) - \sum_{k=1}^{n} \sum_{i=k}^{n} \int_0^{x_k} \int_0^{x_i} \varphi_{j+1,i}^{k[m-2]}(\bar{x}_k^i)$$
$$= \bar{f}_j^{[m]}(x,u) - \sum_{k=1}^{n+1} \sum_{i=k}^{n+1} \int_0^{x_k} \int_0^{x_i} f_{j,i}^{k[m-2]}(\bar{x}_k^i)\, ds_i ds_k$$

for $1 \le j \le n-1$. Let $j+2 \le i \le n+1$. Differentiating twice with respect to x_k and x_i yields the following system

$$\text{(i)}\ \frac{\partial^2 \bar{f}_j^{[m]}}{\partial x_1 \partial x_{n+1}} = f_{j,n+1}^{1[m-2]}, \qquad (k=1, i=n+1)$$

$$\text{(ii)}\ -\varphi_{j+1,i}^{1[m-2]} = \frac{\partial^2 \bar{f}_j^{[m]}}{\partial x_1 \partial x_i} - f_{j,i}^{1[m-2]}, \qquad 1 \le i \le n$$

$$\text{(iii)}\ \varphi_{j,i-1}^{k-1[m-2]} - \varphi_{j+1,i}^{k[m-2]} = -f_{j,i}^{k[m-2]}, \qquad 2 \le k \le i \le n$$

$$\text{(iv)}\ \varphi_{j,n}^{k-1[m-2]} = -f_{j,n+1}^{k[m-2]}, \qquad 2 \le k \le n+1$$

From (i) we see that (II.5) holds for $i = n+1$. From (ii) we have

$$\frac{\partial^2 \bar{f}_j^{[m]}}{\partial x_1 \partial x_i} = f_{j,i}^{1[m-2]} - \varphi_{j+1,i}^{1[m-2]}$$

Substitute j by $j+1$, i by $i+1$, and $k=2$ in (iii), to get

$$\varphi_{j+1,i}^{1[m-2]} = \varphi_{j+2,i+1}^{2[m-2]} - f_{j+1,i+1}^{2[m-2]}$$

and hence

$$\frac{\partial^2 \bar{f}_j^{[m]}}{\partial x_1 \partial x_i} = f_{j,i}^{1[m-2]} + f_{j+1,i+1}^{2[m-2]} - \varphi_{j+2,i+1}^{2[m-2]}.$$

Using (iii) repeatedly, and at last (iv), we arrive to

$$\frac{\partial^2 \bar{f}_j^{[m]}}{\partial x_1 \partial x_i} = f_{j,i}^{1[m-2]} + f_{j+1,i+1}^{2[m-2]} + \cdots + f_{j-i+n+1,n+1}^{n-i+2[m-2]} = a_{j,i}^{[m-2]}.$$

All expressions above are restricted to the set $W_i(x)$.

Now for $i \leq j+1$ or $2 \leq k \leq i \leq n+1$ we already have

$$\frac{\partial^2 \bar{f}_j^{[m]}}{\partial x_k \partial x_i} = 0$$

because of the normal form (III.4) (see [9] for proof).

To complete the proof we need to show that

$$\bar{a}_{j,i}^{[m-2]}(\bar{x}_i) = \frac{\partial^2 \bar{f}_j^{[m]}}{\partial x_1 \partial x_i}\Big|_{W_i(x)}.$$

Using the decomposition

$$\bar{f}_j^{[m]}(x,u) = \sum_{k=1}^{n+1}\sum_{i=k}^{n+1} \int_0^{x_k}\int_0^{x_i} \bar{f}_{j,i}^{k[m-2]}(\bar{x}_k^i)\, ds_i ds_k$$

and the fact that $\bar{f}_j^{[m]}(x,u) = \sum_{i=j+2}^{n+1} x_1 x_i P_{j,i}^{[m-2]}(\bar{x}_i)$ we deduce that $\bar{f}_{j,i}^{k[m-2]}(\bar{x}_k^i) = 0$ for $k \geq 2$. Thus

$$\bar{f}_j^{[m]}(x,u) = \sum_{i=1}^{n+1} \int_0^{x_1}\int_0^{x_i} \bar{f}_{j,i}^{1[m-2]}(\bar{x}_1^i)\, ds_i ds_1.$$

Hence, differentiating twice the above expression, we have

$$\begin{aligned} \frac{\partial^2 \bar{f}_j^{[m]}}{\partial x_1 \partial x_i}\Big|_{W_i(x)} &= \bar{f}_{j,i}^{1[m-2]}(\bar{x}_1^i) \\ &= \sum_{k=1}^{n-i+2} \bar{f}_{j+k-1,i+k-1}^{k[m-2]}(\bar{x}_i) \\ &= \bar{a}_{j,i}^{[m-2]}(\bar{x}_i) = a_{j,i}^{[m-2]}(\bar{x}_i). \end{aligned}$$

This achieves the proof of Proposition II.3.

B. Proof of Theorem III.1

Let us consider the system

$$\Pi^\infty : x^+ = Ax + Bu + \bar{f}^{[m_0]}(x,u) + \sum_{m=m_0+1}^{\infty} f^{[m]}(x,u), \tag{IV.1}$$

where the components of the vector fields $\bar{f}^{[m_0]}(x,u)$ are of the form (III.2). A linear feedback of the form $z = \lambda x, w = \lambda u$ takes the system (IV.1) into

$$\Pi^\infty : z^+ = Az + Bw + \frac{\bar{f}^{[m_0]}(z,w)}{\lambda^{m_0-1}} + \sum_{m=m_0+1}^{\infty} \frac{f^{[m]}(z,w)}{\lambda^{m-1}}.$$

We can thus choose λ so that (III.5) is satisfied.

Let us suppose that the system (IV.1) is transformed, via a polynomial feedback, to the form

$$\Pi^\infty : x^+ = Ax + Bu + \sum_{m=m_0}^{m_0+l-1} \bar{f}^{[m_0]}(x,u) + \sum_{m=m_0+l}^{\infty} f^{[m]}(x,u) \tag{IV.2}$$

for some $l \geq 1$, where for any $m_0 \leq m \leq m_0+l-1$, the components of the vector fields $\bar{f}^{[m]}(x,u)$ satisfy the conditions (III.4), (III.5).

We will apply the homogeneous feedback transformation

$$\Upsilon^{l+1} : \begin{cases} z = x + \varphi^{[l+1]}(x) \\ u = v + \gamma^{[l+1]}(x,v), \end{cases}$$

whose components are given by

$$\left\{ \begin{array}{rcl} \varphi_1^{[l+1]}(x) &=& a_{l+1} x_1^{l+1} \\ \varphi_2^{[l+1]}(x) &=& \varphi_1^{[l+1]}(Ax+Bu) = a_{l+1} x_2^{l+1} \\ & \cdots & \\ \varphi_n^{[l+1]}(x) &=& \varphi_{n-1}^{[l+1]}(Ax+Bu) = a_{l+1} x_n^{l+1} \\ \gamma^{[l+1]}(x,w) &=& \varphi_n^{[l+1]}(Ax+Bu) = a_{l+1} u^{l+1}. \end{array} \right. \tag{IV.3}$$

It is straightforward from Proposition II.2 that the feedback transformation Υ^{l+1}, defined above, leaves invariant all terms of degree less or equal to $l+1$ of system (IV.2). Moreover, it transforms (IV.2) into

$$\Pi^\infty : z^+ = Az + Bv + \sum_{m=m_0}^{m_0+l-1} \bar{f}^{[m_0]}(z,v) + \sum_{m=m_0+l}^{\infty} \tilde{f}^{[m]}(z,v), \tag{IV.4}$$

where

$$\tilde{f}^{[m_0+l]}(z,v) = f^{[m_0+l]}(z,v) + \left[\bar{f}^{[m_0]}(z,v), \varphi^{[l+1]}(z)\right]. \tag{IV.5}$$

Without loss of generality we can suppose that the components of $f^{[m_0+l]}(z,v)$ are of the form (III.4). Now, if we denote by $\hat{f}^{[m_0+l]}(z,v) = \left[\bar{f}^{[m_0]}(z,v), \varphi^{[l+1]}(z)\right]$ with the components given by

$$\hat{f}_j^{[m_0+l]}(z,v) = a_{l+1}\left[(l+1) z_j^l \bar{f}_j^{[m_0]}(z,v) - \sum_{k=1}^{n+1} z_k^{l+1} \frac{\partial \bar{f}_j^{[m_0]}}{\partial z_k}\right]$$

for $1 \leq j \leq n$. The m-invariants $\hat{a}_{j,i}^{[m_0+l-2]}$ associated with the homogeneous part $\hat{f}_j^{[m_0+l]}(z,v)$ are given by

$$\hat{a}_{j,i}^{[m_0+l-2]} = \hat{f}_{j,i}^{1[m_0+l-2]}(\bar{z}_i) + \cdots + \hat{f}_{n+j-i+1,n+1}^{n-i+2[m_0+l-2]}(\bar{z}_i)$$

and for $j = j_*$, $i = n+1$ reduces to (recall definition of j_*)

$$\hat{a}_{j_*,n+1}^{[m_0+l-2]} = \hat{f}_{j_*,n+1}^{1[m_0+l-2]}(\bar{z}_{n+1})$$

from which we have

$$\frac{\partial^{m_0+l-2} \hat{a}_{j_*,n+1}^{[m_0+l-2]}}{\partial z_1^{i_1+l-1} \partial z_2^{i_2} \cdots \partial z_{n+1}^{i_{n+1}-1}} = -a_{l+1}(l+1)! \frac{\partial^{m_0} \bar{f}_{j_*}^{[m_0]}}{\partial z_1^{i_1} \cdots \partial z_{n+1}^{i_{n+1}}}.$$

By the superposition principle of invariants, we deduce from (IV.5) the identity

$$\tilde{a}_{j,i}^{[m_0+l-2]}(\bar{z}_i) = a_{j,i}^{[m_0+l-2]}(\bar{z}_i) + \hat{a}_{j,i}^{[m_0+l-2]}(\bar{z}_i)$$

and we can choose a_{l+1} so that $\frac{\partial^{m_0+l-2} \tilde{a}_{j_*,n+1}^{[m_0+l-2]}}{\partial z_1^{i_1+l-1} \partial z_2^{i_2} \cdots \partial z_{n+1}^{i_{n+1}-1}} = 0.$

C. Proof of Theorem III.2

Consider two systems Π^∞ and $\tilde{\Pi}^\infty$ and suppose they are feedback equivalent. Let

$$\Pi^\infty_{CF} : z^+ = Az + Bv + \sum_{m=m_0}^{\infty} \bar{f}^{[m]}(z,v),$$

and

$$\tilde{\Pi}^\infty_{CF} : \tilde{z}^+ = A\tilde{z} + B\tilde{v} + \sum_{m=\tilde{m}_0}^{\infty} \bar{\tilde{f}}^{[m]}(\tilde{z},\tilde{v})$$

be their respective canonical forms with $\bar{f}^{[m]}(z,v)$ and $\bar{\tilde{f}}^{[m]}(\tilde{z},\tilde{v})$ as in (III.4)-(III.5). Necessarily, $m_0 = \tilde{m}_0$. Otherwise if $m_0 > \tilde{m}_0$ the homogeneous terms of degree $\tilde{m}_0$ of Π^∞ being zero implies (Proposition II.3) that the corresponding invariants are also zero. Thus $\bar{\tilde{f}}^{[\tilde{m}_0]}(z,v) = 0$ which contradicts the definition of $\tilde{m}_0$. The argument works similarly if $m_0 < \tilde{m}_0$ by inverting the role of the systems. Consequently $\bar{f}^{[m_0]}(z,v) = \bar{\tilde{f}}^{[m_0]}(\tilde{z},\tilde{v})$.

Assume that for $l \geq 1$ we have $\bar{f}^{[m]}(z,v) = \bar{\tilde{f}}^{[m]}(\tilde{z},\tilde{v})$ for $m_0 \leq m \leq m_0 + l - 1$. Then the transformation

$$\Upsilon : \begin{cases} \tilde{z} = z + \sum\limits_{m=2}^{\infty} \varphi^{[m]}(z) \\ v = \tilde{v} + \sum\limits_{m=2}^{\infty} \gamma^{[m]}(z,\tilde{v}), \end{cases}$$

mapping Π^∞_{CF} into $\tilde{\Pi}^\infty_{CF}$ should preserve all terms of degree less or equal to $m_0 + l - 1$ and transform the terms $\bar{f}^{[m_0+l]}(z,v)$ into $\bar{\tilde{f}}^{[m_0+l]}(\tilde{z},\tilde{v})$. It is easy to see that the components of Υ are given by

$$\begin{cases} \varphi_1^{[m]}(z) &=& a_m z_1^m \\ \varphi_2^{[m]}(z) &=& \varphi_1^{[m]}(Az + B\tilde{v}) = a_m z_2^m \\ & \cdots & \\ \varphi_n^{[m]}(z) &=& \varphi_{n-1}^{[m]}(Az + B\tilde{v}) = a_m z_n^m \\ \gamma^{[m]}(z,\tilde{v}) &=& \varphi_n^{[m]}(Az + B\tilde{v}) = a_m \tilde{v}^m \end{cases}$$

for $m_0 \leq m \leq m_0 + l - 1$. Moreover, the action of Υ implies the following equality

$$\bar{\tilde{f}}^{[m_0+l]}(z,v) = \bar{f}^{[m_0+l]}(z,v) + \left[\bar{f}^{[m_0]}(z,v), \varphi^{[l+1]}(z)\right]$$

from which we deduce (see steps above) that

$$\frac{\partial^{m_0} \bar{\tilde{f}}_{j_*}^{[m_0+l]}}{\partial z_1^{i_1} \cdots \partial z_{n+1}^{i_{n+1}}} = \frac{\partial^{m_0} \bar{f}_{j_*}^{[m_0+l]}}{\partial z_1^{i_1} \cdots \partial z_{n+1}^{i_{n+1}}} - a_{l+1}(l+1)! \frac{\partial^{m_0} \bar{f}_{j_*}^{[m_0]}}{\partial z_1^{i_1} \cdots \partial z_{n+1}^{i_{n+1}}}.$$

Taking the restriction on the subset

$$W_1(z) = \left\{(z_1, \ldots, z_{n+1}) \in \mathbb{R}^{n+1} \mid z_2 = \cdots = z_{n+1} = 0\right\}$$

and using the fact that $\bar{f}_{j_*}^{[m_0+l]}$ and $\bar{\tilde{f}}_{j_*}^{[m_0+l]}$ satisfy (III.5)(ii), we deduce that $a_{l+1} = 0$ and thus $\bar{\tilde{f}}^{[m_0+l]}(z,v) = \bar{f}^{[m_0+l]}(z,v)$. This completes the proof of Theorem III.2.

V. EXAMPLES

Consider the Bressan and Rampazzo's variable length pendulum (see [18]) described by the equations

$$\begin{cases} \dot{x}_1 &=& x_2 \\ \dot{x}_2 &=& -g \sin x_3 + x_1 u^2 \\ \dot{x}_3 &=& u, \end{cases}$$

where x_1 denotes the length of the pendulum, x_2 its velocity, x_3 the angle of the pendulum with respect to the horizontal, u its angular velocity, and g the gravity constant.

We discretize the system by taking

$$\dot{x}_1 = x_1^+ - x_1, \dot{x}_2 = x_2^+ - x_2, \dot{x}_3 = x_3^+ - x_3.$$

The system above rewrites

$$\begin{cases} x_1^+ &=& x_1 + x_2 \\ x_2^+ &=& x_2 - g \sin x_3 + x_1 u^2 \\ x_3^+ &=& x_3 + u. \end{cases}$$

The change of coordinates

$$\begin{aligned} \tilde{x}_1 &= x_1 \\ \tilde{x}_2 &= x_2 + x_1 \\ \tilde{x}_3 &= -g \sin x_3 + 2x_2 + x_1 \\ \tilde{u} &= \tilde{x}_3^+. \end{aligned}$$

takes the system into the form

$$\begin{cases} \tilde{x}_1^+ &=& \tilde{x}_2 \\ \tilde{x}_2^+ &=& \tilde{x}_3 + \tilde{x}_1 h^2(\tilde{x}_1, \tilde{x}_2, \tilde{x}_3, \tilde{u}) \\ \tilde{x}_3^+ &=& \tilde{u}. \end{cases}$$

Actually the function $h^2(\tilde{x}_1, \tilde{x}_2, \tilde{x}_3, \tilde{u})$ can be decomposed as

$$h^2(\tilde{x}_1, \tilde{x}_2, \tilde{x}_3, \tilde{u}) = h_1(\tilde{x}_1, \tilde{x}_2, \tilde{x}_3) + \tilde{u} h_2(\tilde{x}_1, \tilde{x}_2, \tilde{x}_3, \tilde{u})$$

where the 1-jet at 0 of h_l is zero and $h_2(0) = 0$. Put $H_1(\tilde{x}_1, \tilde{x}_2, \tilde{x}_3) = \tilde{x}_1 h_1(\tilde{x}_1, \tilde{x}_2, \tilde{x}_3)$

The objective is to show that we can get rid of the terms $H_1(\tilde{x}_1, \tilde{x}_2, \tilde{x}_3)$. Let us suppose that the k-jet at 0 of $H_1(\tilde{x}_1, \tilde{x}_2, \tilde{x}_3)$ is zero. Consider the change of coordinates $z_1 = \tilde{x}_1, z_2 = \tilde{x}_2, z_3 = \tilde{x}_3 + H_1(\tilde{x}_1, \tilde{x}_2, \tilde{x}_3), v = z_3^+$. This change of coordinates takes the system into the form

$$\begin{cases} z_1^+ &=& z_2 \\ z_2^+ &=& z_3 + \tilde{H}_1(z_1, z_2, z_3) + z_1 v \tilde{H}_2(z_1, z_2, z_3, v) \\ z_3^+ &=& v. \end{cases}$$

where $\tilde{H}_1(z_1, z_2, z_3)$ and $\tilde{H}_2(z_1, z_2, z_3)$ are some smooth functions. It is enough to remark that the (k+2)-jet at 0 of $\tilde{H}_2(z_1, z_2, z_3)$ is zero because the 2-jet of $z_1 v \tilde{H}_2(z_1, z_2, z_3, v)$ is zero. Then by iteration we can cancel terms $\tilde{H}_1(z_1, z_2, z_3)$ and put the system into the desired normal form

$$\begin{cases} z_1^+ &=& z_2 \\ z_2^+ &=& z_3 + z_1 v P(z_1, z_2, z_3, v) \\ z_3^+ &=& v. \end{cases}$$

Since the linear approximation of the transformation above is such that $z_1 = x_1, z_2 = x_1 + x_2, z_3 \approx x_1 + 2x_2 - g x_3$, we have $x_3 \approx \frac{1}{g}(z_1 + 2z_2 - z_3)$ and we can thus show that the degree of the first non linearizable terms is 3, i.e., $m_0 = 3$, and the largest 4-tuple is $(i_1, i_2, i_3, i_4) = (2, 0, 0, 1)$. In other words $z_1 v P(z, v)$ can be expanded in Taylor series as

$$z_1 v P(z,v) = z_1 v \left(c_1 z_1 + c_2 z_2 + c_3 z_3 + c_4 v + P^{[\geq 2]}(z,v) \right)$$

where $c_1 \neq 0$. By a linear change $\tilde{z} = \lambda z$ with $\lambda = \sqrt{|c_1|}$ the coefficient of the term $z_1^2 v$ becomes equal to $\mathrm{sign}(c_1)$. A change quadratic change of coordinates of the form

$$\begin{cases} \tilde{z}_1 &= z_1 + a_2 z_1^2, \\ \tilde{z}_2 &= z_2 + a_2 z_2^2, \\ \tilde{z}_3 &= z_3 + a_2 z_3^2 \end{cases}$$

whose inverse is in the form

$$\begin{cases} z_1 &= \tilde{z}_1 - a_2 \tilde{z}_1^2 + \varphi_1^{[\geq 3]}(\tilde{z}_1), \\ z_2 &= \tilde{z}_2 - a_2 \tilde{z}_2^2 + \varphi_2^{[\geq 3]}(\tilde{z}_2), \\ z_3 &= \tilde{z}_3 - a_2 \tilde{z}_3^2 + \varphi_3^{[\geq 3]}(\tilde{z}_3) \end{cases}$$

yields $\tilde{z}_1^+ = \tilde{z}_2$ and

$$\begin{aligned} \tilde{z}_2^+ &= z_2^+ + a_2 (z_2^+)^2 = z_3 + z_1 v P(z,v) + a_2 (z_3 + z_1 v P(z,v))^2 \\ &= z_3 + a_2 z_3^2 + z_1 v \left(\mathrm{sign}(c_1) z_1 + c_2 z_2 + c_3 z_3 + c_4 v \right. \\ &\quad \left. + z_1 v P^{[\geq 2]}(z,v) \right) \\ &= \tilde{z}_3 + \mathrm{sign}(c_1)(\tilde{z}_1 - a_2 \tilde{z}_1^2)^2 (\tilde{v} - a_2 \tilde{v}^2) + \tilde{z}_1 \tilde{v} \tilde{P}(\tilde{z}, \tilde{v}), \end{aligned}$$

where $\tilde{v} = \tilde{z}_3^+ = v + a_2 v^2$. Notice that this system is still in normal form as the nonlinear terms are still in the form $\tilde{z}_1 \tilde{v} \tilde{Q}(\tilde{z}, \tilde{v})$. The expansion of $(\tilde{z}_1 - a_2 \tilde{z}_1^2)^2 (\tilde{v} - a_2 \tilde{v}^2)$ contains the term $-2 a_2 \tilde{z}_1^3 \tilde{v}$ and the value of a_2 can be chosen to cancel the corresponding term in $\tilde{z}_1 \tilde{v} \tilde{P}(\tilde{z}, \tilde{v})$. Repeating the same process we show that we can eliminate all terms $z_1^{l+1} v$ with $l \geq 3$ and put the system in the canonical form

$$\begin{cases} z_1^+ &= z_2 \\ z_2^+ &= z_3 + z_1 v \bar{P}(z_1, z_2, z_3, v) \\ z_3^+ &= v, \end{cases}$$

where $\left. \dfrac{\partial^3 [z_1 v \bar{P}(z,v)]}{\partial z_1^2 \partial v} \right|_{z_1 = 0} = 0$ or equivalently

$$\bar{P}(z,v) = \mathrm{sign}(c_1) + z_2 \bar{P}_1(\bar{z}_2) + z_3 \bar{P}_2(\bar{z}_3) + v \bar{P}_3(z,v).$$

Symmetries

Consider $\Pi^\infty : x^+ = f(x,u) = Fx + Gu + \sum_{m=2}^{\infty} f^{[m]}(x,u)$ and let $\mathcal{A}(x) = \{f(x,u), u \in \mathbb{R}\}$ be its field of velocities. A diffeomorphism $z = \sigma(x)$ is called a symmetry of Π^∞ if $\sigma_* \mathcal{A}(x) = \mathcal{A}(\sigma(x))$, where $\sigma_* \mathcal{A}(x) = \{\sigma(f(x,u)), u \in \mathbb{R}\}$. In other words $z = \sigma(x)$ is a symmetry of Π^∞ if there is a feedback $u = \gamma(x,v)$ such that the feedback transformation $\Upsilon : (\sigma(x), \gamma(x,v))$ transforms Π^∞ into itself. Following [21], we can show, using the commutative diagram

$$\begin{tikzcd} \Pi^\infty \arrow[r, "\sigma"] \arrow[d, "\varphi"'] & \Pi^\infty \arrow[d, "\varphi"] \\ \Pi^\infty_{CF} \arrow[r, "\bar{\sigma}"'] & \Pi^\infty_{CF} \end{tikzcd}$$

that the system do not admit nontrivial symmetries preserving the equilibrium. This is due to the uniqueness of the feedback transformation Υ^∞ that takes a system into its canonical form. Indeed, any symmetry σ of Π^∞ gives rise to a symmetry $\bar{\sigma}$ of the canonical form Π^∞_{CF}.

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