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# Cooperative Control Design for Circular Flocking of Underactuated Hovercrafts

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***Abstract*— This paper introduces a Lyapunov-based control design for circular flocking with collision avoidance of underactuated hovercrafts. The desired flocking is achieved by means of consensus on a coordinate-dependent transformation of vehicle linear velocity in surge and finiteness of a coordination function. The design makes use of linear velocity in surge and angular velocity in yaw as virtual controls in a backstepping scheme so that the desired flocking algorithm is obtained in a systematic way, and both convergence and collision avoidance are proved with mathematical rigor.**

## I. INTRODUCTION

Flocking motion of autonomous agents has been a matured topic in systems and control. After the early development of flocking algorithms inspired by modeling and simulation studies in biophysics and computer graphics [1]–[3] for agents of point mass type, e.g., [4], [5], recent development aims at either deriving cooperative control systematically [6] or studying flocking algorithms for agents of real vehicles such as nonholonomic mobile robots [7], [8]. Yet, flocking in nonlinear geometric shapes is of increasing interest [8]–[10]. Towards further development, the current paper takes the goals of systematic derivation and convergence analysis of cooperative control for circular flocking with collision avoidance of underactuated hovercrafts.

In comparison to reported results, specifying the state of circular flocking for hovercraft systems appeals for a serious attention. Indeed, the circularity in geometric shape does not admit consensus on even vehicle linear speed as in, e.g, [5], [7] since, in a circular flocking, inner vehicles might move at a speed lower than speeds of outer vehicles. Furthermore, the second-order nonholonomy in hovercraft systems makes the unit speed assumption for cooperative control of mobile robots [6], [8]–[10] no longer relevant.

On the other hand, achieving collision avoidance in flocking of nonholonomic vehicles of either mobile robot type or underactuated hovercraft type is still of current challenge. Though collision avoidance has been considered in [7], [8], these works are still in the extent of improving collision avoidance ability by incorporating an additional term to the formation control. A systematic derivation for flocking control achieving simultaneously collision avoidance and formation remains open.

In this paper, we resolve the above issues in an intrinsic approach of formulating, in Section II, a relevant control problem whose solution is to be achieved, in Section III, in a systematic way of Lyapunov-based design. We consider a coordinate-dependent transformation of hovercraft linear velocity in surge that feature both flocking geometry and curvature-dependent motion. In such a way, the control goals for circular flocking become consensus on the transformation among vehicles and collision avoidance.

To achieve collision avoidance, it is customary to have finiteness for a coordination function usually called potential function. In this paper, we explicitly incorporate this function into Lyapunov function candidate and develop a backstepping design from the kinematic level of coordinate variables to dynamic level of linear velocities and angular velocity. Due to reduced actuators, we achieve the desired specification on potential function by using the linear velocity in surge as virtual control and then cancel its effect in sideways motion coupled with linear velocity in sway by the control torque in yaw. Using this approach, we obtain the collision avoidance term for the final control that actually guarantees collision avoidance for the resulting flocking motion, and, at the same time, the consensus term for the final control is obtained by a direct matching design.

The theoretical interest of the above approach is the enabled mathematical proof of collision avoidance and cohesion maintenance in flocking of nonholonomic vehicles which shall be presented in Section IV. To illustrate the novelty of the proposed control, we present in Section V simulation result. And finally, Section VI concludes the results presented.

## II. PROBLEM FORMULATION

Given a collective system of $N$ identical underactuated hovercrafts labeled by numbers $1, \dots, N$ whose respective equations of motion are

$$\begin{aligned} \dot{x}_i &= u_i \cos\psi_i - v_i \sin\psi_i \\ \dot{y}_i &= u_i \sin\psi_i + v_i \cos\psi_i \\ \dot{\psi}_i &= r_i \\ \dot{u}_i &= v_i r_i + \tau_{u,i} \\ \dot{v}_i &= -u_i r_i \\ \dot{r}_i &= \tau_{r,i} \end{aligned} \quad , \ i = 1, \dots, N \tag{1}$$

where, as shown in Fig. 1, for the $i$-th vehicle, $q_i = [x_i, y_i]^T \in \mathbb{R}^2$ is the position vector, $\psi_i \in \mathbb{R}$ is the

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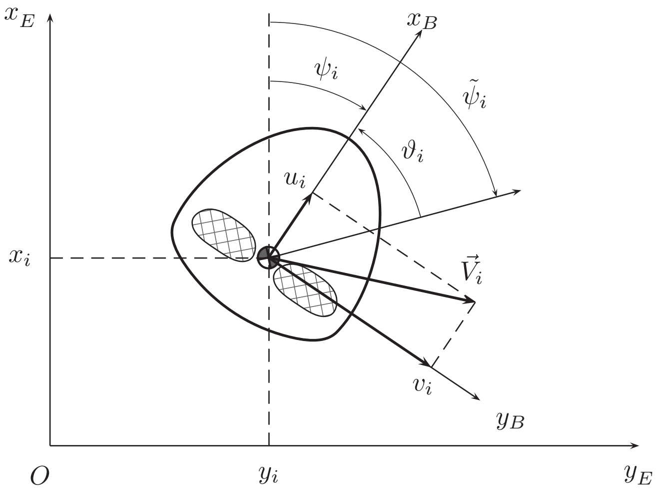


Fig. 1. Configuration of the $i$-th hovercraft: $\tilde{\psi}_i = \psi_r(q_i, q_C) - \delta_\psi(\chi_i)$

orientation, $u_i \in \mathbb{R}$ and $v_i \in \mathbb{R}$ are linear velocities in surge and sway, respectively, $r_i \in \mathbb{R}$ is the angular velocity, $\tau_{u,i} \in \mathbb{R}$ is the control force in surge, and $\tau_{r,i} \in \mathbb{R}$ is the control torque in yaw. The derivation for the model (1) is available in [11].

To continue, let us clarify the following notations.

*Notations*: for $q = [q_1, \ldots, q_n]^T$, $\nabla_q$ is the del operator $\nabla_q = [\partial/\partial q_1, \ldots, \partial/\partial q_n]^T$ [12]; for two vectors $a$ and $b$, $a \cdot b$ is their scalar product, and $\mathrm{col}(a, b)$ is the vector $[a^T, b^T]^T$; $|\cdot|$ is the absolute value of scalars; and $\|\cdot\|$ is the Euclidean norm of vectors.

The control goal for the collective system (1) is to design state-feedback controls $\tau_i \stackrel{\text{def}}{=} [\tau_{u,i}, \tau_{r,i}]^T$ of the same structure such that resulting collective motion of the $N$ vehicles eventually forms a ring centered at their center of mass

$$q_C \stackrel{\text{def}}{=} \begin{bmatrix} x_C \\ y_C \end{bmatrix} = \frac{1}{N} \sum_{i=1}^{N} q_i. \tag{2}$$

Towards this goal, our preliminary work is to configure the desired flocking motion by specifications that are technically achievable.

At the individual level, the two primitive vehicle behaviors are orientation and coordination. Accordingly, we shall specify the desired circular flocking motion by reference orientation and coordination function for either vehicle relative positioning or flock positioning. Assuming a circular orbit centered at the center of mass of the $N$ vehicles, the reference orientation for the $i$-th vehicle can be given by the angle

$$\psi_r(q_i, q_C) = \varphi(q_i, q_C) - \frac{\pi}{2}, \tag{3}$$

where $\varphi(q_i, q_C)$ is the argument of the vector $q_i - q_C$. Clearly, the circular shape for collective motion of all vehicles corresponds to the consensus on orientation mismatches $\psi_i - \psi_r(q_i, q_C)$. This in combination with the nonholonomic structure, for which the desired hovercraft velocity cannot be created directly, motivates us aim at consensus on the following modified orientation mismatches

$$\vartheta_i = \psi_i - \psi_r(q_i, q_C) + \delta_\psi(\chi_i), \tag{4}$$

where $\chi_i = \mathrm{col}(q_i, q_C, \psi_i)$ and $\delta_\psi$ is a design function.

On the other hand, it is customary to have vehicle linear speed varies with the trajectory curvature. Such property is of design interest in circular flocking for which the centering vehicles might move slower than the vehicles at the boundary. However, in view of the reduced number of actuators, we shall achieve this property with respect to the linear velocity in surge $u_i$. Thus, we are also interested in consensus on

$$\nu_i(q_i, q_C, u_i) = \kappa(q_i, q_C) u_i = \frac{1}{\varepsilon + \|q_i - q_C\|} u_i \tag{5}$$

where $\kappa(q_i, q_C)$ can be considered as the *mollified* curvature of the circle centered at $q_C$ and pass $q_i$.

In light of the above elaboration, we shall achieve the circular shape for our desired flocking motion by means of consensus on the following velocity transformation

$$u_{f,i}(q_i, q_C, u_i) = \nu_i(q_i, q_C, u_i) \vec{e}(\vartheta_i), \tag{6}$$

where $\vec{e}(\vartheta_i) = [\cos\vartheta_i, \sin\vartheta_i]^T$.

The remaining specification of flocking motion is about collision avoidance and cohesion maintenance. Let $\bar{q}_N = [q_1^T, \ldots, q_N^T]^T$. As usual, we are interested in the following potential function with binary structure

$$U(\bar{q}_N) = \frac{1}{N} \sum_{i=1}^{N} U_0(q_i, q_C) + \frac{1}{2} \frac{1}{N} \sum_{i=1}^{N} \frac{1}{N} \sum_{j=1}^{N} U_1(q_i, q_j). \tag{7}$$

Usually, the specification of $U_0$ and $U_1$ is such that the desired configuration $\bar{q}_N$ corresponds to the minimal value of $U(\bar{q}_N)$. In this way, we have the control problem of driving $\nabla_{q_i} U(\bar{q}_N)$ converge to zero. Nevertheless, due to reduced number of actuators, we do not have sufficient degree of freedom to drive $\nabla_{q_i} U(\bar{q}_N)$ converge to zero by direct matching design. To overcome this difficulty, we shall achieve this goal indirectly by firstly use the linear velocity in surge $u_i$ as virtual control to drive the following quantity to zero

$$\vec{e}(\psi_i) \cdot \nabla_{q_i} U(\bar{q}_N), \tag{8}$$

with $\vec{e}(\psi_i) = [\cos\psi_i, \sin\psi_i]^T$ the unit vector in surge, and then by a proper design of $\delta_{\psi,i}$, the hovercraft orientation $\vec{e}(\psi_i)$ is continuously forced to turn away the direction of the normal vector of $\nabla_{q_i} U(\bar{q}_N)$ until $\nabla_{q_i} U(\bar{q}_N) = 0$.

In summary, we have the following control problem to achieve the desired flocking algorithm.

*Property 2.1:* The function $U(\bar{q}_N)$ is bounded away from zero and has the structure (7). The function $U_1$ symmetric, i.e. $U_1(q_1, q_2) = U_1(q_2, q_1), \forall q_1, q_2 \in \mathbb{R}^2$. Furthermore, the functions $U_0(q_1, q_2)$ and $U_1(q_1, q_2)$ are continuously differentiable except at $q_1 = q_2$, receive zero value at $q_1 = q_2$

and has the following properties

$$\limsup_{\|q_1-q_2\|\to 0} U_i(q_1,q_2)=\infty$$
$$\limsup_{\|q_1-q_2\|\to \infty} U_i(q_1,q_2)=\infty, i=0,1. \tag{9}$$

*Problem 2.1 (Flocking Control Problem):* Given the collective system (1) of $N$ hovercrafts and state-dependent functions $U(\bar{q}_N), \psi_r, \delta_\psi$ with $U(\bar{q}_N)$ satisfying Property 2.1. Find, for system (1), the state-feedback controls $\tau_i \overset{\text{def}}{=} [\tau_{u,i}, \tau_{r,i}]^T, i=1,\dots N$ such that the following properties hold along the trajectory of the closed-loop system:

i) $U(\bar{q}_N)$ remains finite;
ii) $\vartheta_i(t)-\vartheta_j(t)\to 0, t\to\infty$;
iii) $\nu_i(t)-\nu_j(t)\to 0, t\to\infty$; and
iv) $\vec{e}(\psi_i(t))\cdot\nabla_{q_i}U(\bar{q}_N(t))\to 0, t\to 0$

for all $i,j=1,\dots,N$.

*Remark 2.1:* As this is an early development, we do not limit hovercrafts' sensing ranges, a case that usually requires the incorporation of the graph theory [4]. Nevertheless, by property (9), condition i) in the above problem is to have both cohesion maintenance and collision avoidance. The conditions ii)–iv) are for desired flocking geometry.

*Remark 2.2:* The inclusion of $U_0$ in (7) is to make the hovercrafts generate flocking motion circulating around a desired point $q_C$. As can be seen in the following derivation, $q_C$ can be replaced by a fixed point or a moving reference point $q_r(t)$ whose derivative is computable.

## III. Control Design

In this section, we present a systematic design procedure achieving solution to the Problem 2.1 formulated in the previous section. As we are using the linear velocity in surge $u_i$ and the angular velocity in yaw $r_i$ as virtual control, let us consider the following changes of variable:

$$\zeta_i=\nu_i-\kappa(q_i,q_C)\alpha_i$$
$$\eta_i=r_i-\dot{\psi}_r(q_i,q_C)+\dot{\delta}_\psi(\chi_i)-\beta_i, \tag{10}$$

where $\chi_i=\mathrm{col}(q_i,q_C,\psi_i)$, $\nu_i$ and $\kappa$ are given by (5), and $\alpha_i$ and $\beta_i$ are design functions to be specified.

In view of (10), the consensus on $\zeta_i$ coupled with vanishing $\alpha_i$ implies consensus on $\nu_i$. Accordingly, we shall achieve the property ii) in Problem 2.1 by means of consensus on $\zeta_i$ and convergence of $\alpha_i$. Defining $\kappa_i=\kappa(q_i,q_C)$ and using (1), (5), and (10), we have

$$\dot{\zeta}_i=\kappa_i(v_ir_i+\tau_{u,i})+u_i\dot{\kappa}_i-\frac{d}{dt}(\kappa_i\alpha_i)$$
$$\dot{\eta}_i=\tau_{r,i}-\frac{d}{dt}(\beta_i+\dot{\psi}_r(q_i,q_C)-\dot{\delta}_\psi(\chi_i)). \tag{11}$$

We have the following initial design.

### A. Initial Design

In view of (11), let us consider the following structure for the control $\tau_{u,i}$ and $\tau_{r,i}$:

$$\tau_{u,i}=\tau^\circ_{u,i}+\frac{1}{\kappa_i}\tau^1_{u,i}$$
$$\tau_{r,i}=\tau^\circ_{r,i}+\tau^1_{r,i}, \tag{12}$$

and make the initial design

$$\tau^\circ_{u,i}=-v_ir_i-\frac{1}{\kappa_i}\Big(u_i\dot{\kappa}_i-\frac{d}{dt}(\kappa_i\alpha_i)\Big)$$
$$\tau^\circ_{r,i}=\frac{d}{dt}\Big(\beta_i+\dot{\psi}_r(q_i,q_C)-\dot{\delta}_\psi(\chi_i)\Big). \tag{13}$$

We note that $\alpha_i$ and $\beta_i$ are design functions of $x_i,y_i,\psi_i$ to be given by (24) and (27). The design of $\alpha$ and $\beta$ is independent of $\tau^\circ_{u,i}$ and $\tau^\circ_{r,i}$ so that there will arise no circular argument while the control (13) is computable explicitly using the dynamic equation of $x_i,y_i,\psi_i$ given by (1).

Let $q_i=[x_i,y_i]^T$ and $\vec{e}(\psi_i)=[\cos\psi_i,\sin\psi_i]^T$. Then, under the control (12)-(13) and changes of variable (4) and (10), the collective system (1) becomes

$$\begin{aligned}
\dot{q}_i&=\Big(\frac{\zeta_i}{\kappa_i}+\alpha_i\Big)\vec{e}(\psi_i)+v_i\vec{e}^\perp(\psi_i)\\
\dot{\vartheta}_i&=\eta_i+\beta_i\\
\dot{\zeta}_i&=\tau^1_{u,i} \qquad\qquad i=1,\dots,N\\
\dot{\eta}_i&=\tau^1_{r,i}\\
\dot{v}_i&=-\Big(\frac{\zeta_i}{\kappa_i}+\alpha_i\Big)\Big(\eta_i+\beta_i+\dot{\psi}_r(q_i,q_C)-\dot{\delta}_\psi(\chi_i)\Big).
\end{aligned} \tag{14}$$

We note that, in view of (4), $\psi_i$ in (14) is now a function of $\vartheta_i$ and $q_i$. Using (14), we have the following Lyapunov-based design for $\alpha$, $\beta$, $\tau^1_{u,i}$ and $\tau^1_{r,i}$.

### B. Lyapunov-based Control Design

Consider the following Lyapunov function candidate

$$V=\frac{1}{2}\sum_{i=1}^N\zeta_i^2+\frac{1}{2}\sum_{i=1}^N\vartheta_i^2+U(\bar{q}_N)\frac{1}{2}\sum_{i=1}^N\eta_i^2, \tag{15}$$

where $U(\bar{q}_N)$ is the design function (7).

The time derivative of $V$ along trajectory of (14) is

$$\begin{aligned}
\dot{V}=&\sum_{i=1}^N\zeta_i\tau^1_{u,i}+\sum_{i=1}^N\vartheta_i(\eta_i+\beta_i)+U(\bar{q}_N)\sum_{i=1}^N\eta_i\tau^1_{r,i}\\
&+\frac{\eta^*}{2N}\sum_{i=1}^N\nabla_{q_i}U_0(q_i,q_C)\cdot\big(u_i\vec{e}(\psi_i)+v_i\vec{e}^\perp(\psi_i)\big)\\
&+\frac{\eta^*}{2N}\sum_{i=1}^N\nabla_{q_C}U_0(q_i,q_C)\cdot\frac{1}{N}\sum_{j=1}^N\big(u_j\vec{e}(\psi_j)+v_j\vec{e}^\perp(\psi_j)\big)\\
&+\frac{\eta^*}{4N}\sum_{i=1}^N\frac{1}{N}\sum_{j=1}^N\nabla_{q_i}U_1(q_i,q_j)\cdot\big(u_i\vec{e}(\psi_i)+v_i\vec{e}^\perp(\psi_i)\big)\\
&+\frac{\eta^*}{4N}\sum_{i=1}^N\frac{1}{N}\sum_{j=1}^N\nabla_{q_j}U_1(q_i,q_j)\cdot\big(u_j\vec{e}(\psi_j)+v_j\vec{e}^\perp(\psi_j)\big),
\end{aligned} \tag{16}$$

where

$$\eta^*=\sum_{i=1}^N\eta_i^2 \quad\text{and}\quad u_i=\frac{\zeta_i}{\kappa_i}+\alpha_i. \tag{17}$$

By the symmetry of $U_1(q_i,q_j)$, we have $U_1(q_i,q_j)=U_1(q_j,q_i)$ and hence

$$\nabla_{q_j}U_1(q_i,q_j)=\nabla_{q_j}U_1(q_j,q_i). \tag{18}$$

Accordingly, we have the following expression for the last term of (16)

$$\begin{aligned}&\frac{\eta^*}{4N}\sum_{i=1}^N\frac{1}{N}\sum_{j=1}^N\nabla_{q_j}U_1(q_i,q_j)\cdot\big(u_j\vec{e}(\psi_j)+v_j\vec{e}^{\perp}(\psi_j)\big)\\&=\frac{\eta^*}{4N}\sum_{i=1}^N\frac{1}{N}\sum_{j=1}^N\nabla_{q_j}U_1(q_j,q_i)\cdot\big(u_j\vec{e}(\psi_j)+v_j\vec{e}^{\perp}(\psi_j)\big)\\&=\frac{\eta^*}{4N}\sum_{j=1}^N\frac{1}{N}\sum_{i=1}^N\nabla_{q_i}U_1(q_i,q_j)\cdot\big(u_i\vec{e}(\psi_i)+v_i\vec{e}^{\perp}(\psi_i)\big),\end{aligned}\tag{19}$$

where we have relabeled the running indices $i\to j$ and $j\to i$ for the third line of (19). Clearly, the last term of (19) is exactly the term in the fourth line of (16).

Also, by relabeling the running indices $i\to j$ and $j\to i$, the term in the third line of (16) is

$$\begin{aligned}&\frac{\eta^*}{2N}\sum_{i=1}^N\nabla_{q_C}U_0(q_i,q_C)\cdot\frac{1}{N}\sum_{j=1}^N\big(u_j\vec{e}(\psi_j)+v_j\vec{e}^{\perp}(\psi_j)\big)\\&=\frac{\eta^*}{2N}\sum_{j=1}^N\nabla_{q_C}U_0(q_j,q_C)\cdot\frac{1}{N}\sum_{i=1}^N\big(u_i\vec{e}(\psi_i)+v_i\vec{e}^{\perp}(\psi_i)\big)\\&=\frac{\eta^*}{2N}\frac{1}{N}\sum_{i=1}^N\sum_{j=1}^N\nabla_{q_C}U_0(q_j,q_C)\cdot\big(u_i\vec{e}(\psi_i)+v_i\vec{e}^{\perp}(\psi_i)\big)\end{aligned}\tag{20}$$

Substituting (19) and (20) into (16) yields

$$\begin{aligned}\dot V=&\sum_{i=1}^N\zeta_i\tau_{u,i}^1+\sum_{i=1}^N\vartheta_i(\eta_i+\beta_i)+U(\bar q_N)\sum_{i=1}^N\eta_i\tau_{r,i}^1\\&+\frac{\eta^*}{2N}\sum_{i=1}^N\nabla_{q_i}U_0(q_i,q_C)\cdot\big(u_i\vec{e}(\psi_i)+v_i\vec{e}^{\perp}(\psi_i)\big)\\&+\frac{\eta^*}{2N}\frac{1}{N}\sum_{i=1}^N\sum_{j=1}^N\nabla_{q_C}U_0(q_j,q_C)\cdot\big(u_i\vec{e}(\psi_i)+v_i\vec{e}^{\perp}(\psi_i)\big)\\&+\frac{\eta^*}{2N}\sum_{i=1}^N\frac{1}{N}\sum_{j=1}^N\nabla_{q_i}U_1(q_i,q_j)\cdot\big(u_i\vec{e}(\psi_i)+v_i\vec{e}^{\perp}(\psi_i)\big).\end{aligned}\tag{21}$$

Defining

$$\begin{aligned}W=&\sum_{i=1}^N\nabla_{q_i}U_0(q_i,q_C)\cdot\vec{e}^{\perp}(\psi_i)v_i\\&+\frac{1}{N}\sum_{i=1}^N\sum_{j=1}^N\nabla_{q_C}U_0(q_j,q_C)\cdot\vec{e}^{\perp}(\psi_i)v_i\\&+\frac{1}{N}\sum_{i=1}^N\sum_{j=1}^N\nabla_{q_i}U_1(q_i,q_j)\cdot\vec{e}^{\perp}(\psi_i)v_i\\\vec{W}_i=&\nabla_{q_i}U_0(q_i,q_C)+\frac{1}{N}\sum_{j=1}^N\nabla_{q_C}U_0(q_j,q_C)\\&+\frac{1}{N}\sum_{j=1}^N\nabla_{q_i}U_1(q_i,q_j)\end{aligned}\tag{22}$$

and using expression (17) for $\eta^*$ and $u_i$, the expression (21) becomes

$$\begin{aligned}\dot V=&\sum_{i=1}^N\zeta_i\tau_{u,i}^1+\sum_{i=1}^N\vartheta_i(\eta_i+\beta_i)+U(\bar q_N)\sum_{i=1}^N\eta_i\tau_{r,i}^1\\&+\frac{\eta^*}{2N}\sum_{i=1}^N\Big(\frac{\zeta_i}{\kappa_i}+\alpha_i\Big)\vec{W}_i\cdot\vec{e}(\psi_i)+\frac{1}{2N}\sum_{i=1}^N\eta_i^2W\end{aligned}\tag{23}$$

As $U(\bar q_N)$ is bounded away from zero by the hypothesis of Problem 2.1, we have the following design from (23)

$$\begin{aligned}\tau_{r,i}^1&=-\frac{1}{U(\bar q_N)}\Big(\vartheta_i+\frac{\eta_i}{2N}W\Big)\\\tau_{u,i}^1&=\tau'_{u,i}-\frac{\eta^*}{2N}\frac{1}{\kappa_i}\vec{W}_i\cdot\vec{e}(\psi_i)\\\alpha_i&=-k_1\vec{W}_i\cdot\vec{e}(\psi_i)\end{aligned}\tag{24}$$

where $\tau'_{u,i}$ is to be further specified and $k_1>0$ is a design constant. Under (24), the equation (23) becomes

$$\dot V=\sum_{i=1}^N\zeta_i\tau'_{u,i}+\sum_{i=1}^N\vartheta_i\beta_i-k_1\frac{\eta^*}{2N}\sum_{i=1}^N(\vec{W}_i\cdot\vec{e}(\psi_i))^2.\tag{25}$$

Our remaining task is to design $\tau'_{u,i}$ and $\beta_i$ to achieve consensus on $\zeta_i$ and $\vartheta_i$. Such design is straightforward from the identity

$$\sum_{i=1}^N a_i\sum_{j=1}^N(a_i-a_j)=\frac{1}{2}\sum_{i=1}^N\sum_{j=1}^N(a_i-a_j)^2.\tag{26}$$

Let us have the design

$$\begin{aligned}\tau'_{u,i}&=-k_2\sum_{j=1}^N(\zeta_i-\zeta_j)\\\beta_i&=-k_2\sum_{j=1}^N(\vartheta_i-\vartheta_j).\end{aligned}\tag{27}$$

Substituting (27) into (25) and using identity (26), we arrive at

$$\begin{aligned}\dot V=&-\frac{k_1}{2N}\sum_{k=1}^N\eta_k^2\sum_{i=1}^N(\vec{W}_i\cdot\vec{e}(\psi_i))^2\\&-\frac{k_2}{2}\sum_{i=1}^N\sum_{j=1}^N\big((\zeta_i-\zeta_j)^2+(\vartheta_i-\vartheta_j)^2\big).\end{aligned}\tag{28}$$

Finally, substituting (13), (22), (24), and (27) into (12), we obtain the actual control $\tau_{u,i}$ and $\tau_{r,i}$. This completes our design procedure.

## IV. Convergence Analysis

In this section, we present the main theorem asserting that a solution to the Problem 2.1 formulated in Section II is the actual control $\tau_{u,i}$ and $\tau_{r,i}$ designed as (12), (13), (22), (24), and (27) in the previous subsections.

*Theorem 4.1:* Given the collective system (1) and state-dependent functions $U(\bar q_N),\psi_r,\delta_\psi$ with $U(\bar q_N)$ having Property 2.1. Suppose that the design function $\delta_\psi(\chi_i)$ is monotone, has the same sign as $\psi_i-\psi_r(q_i,q_C)$, and is nonzero for $\|\vec{W}_i\|\neq 0$. Then, the trajectory $\bar\chi(t)$ with $\bar\chi=$

$\mathrm{col}(q_1, \psi_1, u_1, v_1, r_1, \ldots, q_N, \psi_N, u_N, v_N, r_N)$ of the system (1) subject to the control $\tau_{u,i}$ and $\tau_{r,i}$ given by (12), (13), (22), (24), and (27) satisfies conditions i)–iv) in the Problem 2.1. In addition, collision avoidance and cohesion maintenance are guaranteed.

*Proof:* As designed in the previous subsection, under the hypothesis of the theorem, the dissipation equality (28) holds true along the trajectory of the closed-loop system. Hence, by LaSalle's invariance principle, the trajectory of the closed-loop system converges to the set $\{\chi : \dot{V} = 0\}$. Thus, we have

$$\lim_{t\to\infty} |\zeta_i(t) - \zeta_j(t)| \to 0 \tag{29}$$

$$\lim_{t\to\infty} |\vartheta_i(t) - \vartheta_j(t)| \to 0 \tag{30}$$

$$\lim_{t\to\infty} \sum_{k=1}^{N} \eta_k^2(t) \sum_{i=1}^{N} (\vec{W}_i(t)\cdot \vec{e}(\psi_i(t)))^2 \to 0. \tag{31}$$

We now use (31) to verify that $\vec{W}_i(t)\cdot\vec{e}(\psi_i(t)) \to 0$ as $t\to\infty$. Indeed, suppose that the converse holds true. Then, there is a constant $\epsilon > 0$ and $T \geq 0$ such that $\vec{W}_i(t)\cdot \vec{e}(\psi_i(t)) \geq \epsilon, \forall t \geq T$. Accordingly, we have

$$\lim_{t\to\infty} \eta_i(t) \to 0, \forall i = 1,\ldots,N, \tag{32}$$

and hence $\dot{\eta}_i(t) \to 0, t\to\infty$. This in combination with the dynamics of $\eta_i$ given by (14) and the design of $\tau_{r,i}^1$ in (24) indicates that

$$\lim_{t\to\infty} \vartheta_i(t) = 0. \tag{33}$$

Furthermore, the design function $\delta_\psi(\chi_i)$ is specified to have the same sign as $\psi_i - \psi_r(q_i, q_C)$. Thus, (33) and definition (4) of $\vartheta_i$ imply that

$$\lim_{t\to\infty} \delta_\psi(\chi_i(t)) = 0. \tag{34}$$

However, as $\delta_\psi(\chi_i)$ is specified to be monotone and have non-zero value for $\|\vec{W}_i\| \neq 0$, (34) further implies that $\|\vec{W}_i(t)\| \to 0, t\to\infty$ and hence

$$\lim_{t\to\infty} \vec{W}_i(t)\cdot\vec{e}(\psi_i(t)) = 0 \tag{35}$$

which is a contradiction.

Thus, we conclude that $\vec{W}_i(t)\cdot\vec{e}(\psi_i(t)) \to 0$ as $t\to\infty$. Since $U(\bar{q}_N)$ assumes the structure (7) and $\vec{W}_i$ is defined by (22), we have $\vec{W}_i = \nabla_{q_i} U(\bar{q}_N)$ and hence conclude that the closed-loop system satisfies condition iv) of Problem 2.1.

We now verify condition i) of Problem 2.1 also by a contradiction argument. Assume that the closed-loop system does not satisfy this condition, i.e., $U(\bar{q}_N(t)) \to \infty, t\to t_1$ for some $t_1 \geq 0$. Then, in view of (15) and (28), we must have

$$\lim_{t\to t_1} \sum_{i=1}^{N} \eta_i^2(t) = 0. \tag{36}$$

Thus, using the same argument as above, we have

$$\lim_{t\to t_1} \|\vec{W}_i(t)\| = 0 \tag{37}$$

which, in view of (22), indicates that both $U_0(q_i(t), q_C(t))$ and $U_1(q_i(t), q_j(t))$ remain finite as $t \to t_1$ and hence so does $U(\bar{q}_N(t))$. This is a contradiction. Thus, the closed-loop system satisfies condition i) of Problem 2.1.

The satisfaction of condition ii) of Problem 2.1 is obvious from (30). Furthermore, by the design (24) of $\alpha$, the satisfaction of condition i), and the definition (10) of $\zeta_i$, we have

$$\begin{aligned}\lim_{t\to\infty} &|\nu_i(t) - \nu_j(t)| \\ &\leq \lim_{t\to\infty} |\zeta_i(t) - \zeta_j(t)| \\ &\quad + \lim_{t\to\infty} |\kappa_i(t)\alpha_i(t) - \kappa_j(t)\alpha_j(t)| = 0\end{aligned} \tag{38}$$

which indicates that condition iii) is satisfied.

Finally, by the designed Property 2.1 and the satisfaction of condition i) of $U(\bar{q}_N)$, both collision avoidance and cohesion maintenance are guaranteed. ■

## V. Simulation Study

In this section, we present result of the simulation of the above theory. The desired performance is that the group of hovercrafts as a whole shrinks toward its center of mass $q_C = \sum_i q_i/N$ and, at the same time, forms an circular ring centered at $q_C$. The general design for the control $\tau_{u,i}$ and $\tau_{r,i}$ has been given by (12), (13), (22), (24), and (27) . Our job now is to specify the design functions $U_0, U_1, \psi_r$, and $\delta_\psi$.

To specify functions $U_0$ and $U_1$, let us specify the behavior of the $i$-th vehicle as follows. The $i$-th vehicle moves away $q_C$ if its distance to $q_C$, $r_i = \|q_i - q_C\|$, is smaller than a reference value $r_0$ and the vehicle move towards $q_C$ if $r_i > r_1$. Within the range $r_0 \leq r_i \leq r_1$, the vehicle rests on its circulation motion. Clearly, such behavior is ensured by the design

$$U_0(q_i, q_C) = \frac{h(r_i; r_0+\varepsilon, r_0)}{r_i - r_0} + h(r_i; r_1, r_1+\varepsilon)(r_i - r_1), \tag{39}$$

where $\varepsilon > 0$ is a small constant and $h(x; a, b)$ is the bump function defined by [13]

$$h(x; a, b) = g\Big(\frac{x-a}{b-a}\Big) \tag{40}$$

where

$$g(s) = \frac{f(s)}{f(s) + f(1-s)}, \quad f(s) = \begin{cases} e^{-1/s} & \text{if } s > 0 \\ 0 & \text{if } s \leq 0 \end{cases}. \tag{41}$$

Similarly, we have the following design for $U_1(q_i, q_j)$

$$\begin{aligned}U_1(q_i, q_j) = &\frac{h(d_{ij}, d_0+\varepsilon, d_0)}{(d_{ij} - d_0)^2} \\ &+ h(d_{ij}, d_1, d_1+\varepsilon)(d_{ij} - d_1),\end{aligned} \tag{42}$$

where $d_{ij} = \|q_i - q_j\|$ and $d_0, d_1 > 0$ are design parameters.

To generate circular flocking, it is relevant to have the hovercrafts move on circles all centered at the center of mass $q_C = [x_C, y_C]^T$. According, we specify the desired

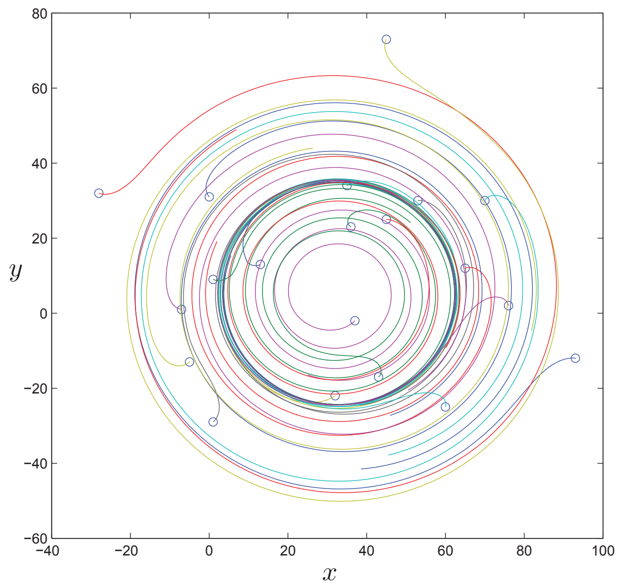


Fig. 2. Trajectories of 20 hovercrafts

orientation $\psi_r(q_i, q_C)$ for the $i$-th hovercraft visiting the location $q_i = [x_i, y_i]^T$ as follows. Consider the following equation of the the circle passing $q_i$ and centering at $q_C$

$$(x - x_C)^2 + (y - y_C)^2 = (x_i - x_C)^2 + (y_i - y_C)^2. \quad (43)$$

From the following expression of the unit vector tangent to the circle (43) at $q_i$

$$\vec{e}(q_i, q_C) = \frac{1}{(x_i - x_C)^2 + (y_i - y_C)^2} \begin{bmatrix} -(y_i - y_C) \\ x_i - x_C \end{bmatrix}, \quad (44)$$

we have the specification $\psi(q_i, q_C) = \arg\vec{e}(q_i, q_C)$, the argument of $\vec{e}(q_i, q_C)$.

Finally, we have the specification

$$\delta_\psi(\chi_i) = -k_\psi \arctan((\vec{U}'(q_i))^\perp \cdot \vec{e}(q_i, q_C)\|\nabla_{q_i} U_i\|), \quad (45)$$

where $k_\varphi > 0$ is a design parameter, $(\vec{U}'(\cdot))^\perp$ is the rotation by an angle of $\pi/2$ of $\vec{U}'(q_i) = \sum_j (\nabla_{q_C} U_0(q_j, q_C) + \nabla_{q_i}(U_0(q_i, q_C) + U_1(q_i, q_j)))$, and $U_i = U_0(q_i, q_C) + \sum_j U_1(q_i, q_j)$. Such design (45) is to ensure the performance that the $i$-th hovercraft turns away $q_C$ if it is close to $q_C$, i.e., $(\vec{U}'(q_i))^\perp \cdot \vec{e}(q_i, q_C) < 0$, turns towards $q_C$ if it is far from $q_C$, i.e., $(\vec{U}'(q_i))^\perp \cdot \vec{e}(q_i, q_C) > 0$, and rests on circular motion if $(\vec{U}'(q_i))^\perp \cdot \vec{e}(q_i, q_C) = 0$ which, by Theorem 4.1, is the ultimate state of the system.

The result of our simulation with 20 hovercrafts is shown in Fig. 2 and Fig. 3. We selected: $k_1 = 2, k_2 = 1, a = 3, b = 2, \varepsilon = 0.2, r_0 = 10, r_1 = 30, d_0 = 5, d_1 = 10$, and $k_\psi = 0.2$. Fig. 2 indicates that a ring had been established, and Fig. 3 indicates that the minimal distance among all pairs of vehicles $d_{\min}(t) = \min\{\|q_i(t) - q_j(t)\| : i \neq j\}$ is always greater than $d_0 = 5$, i.e., collision avoidance was guaranteed.

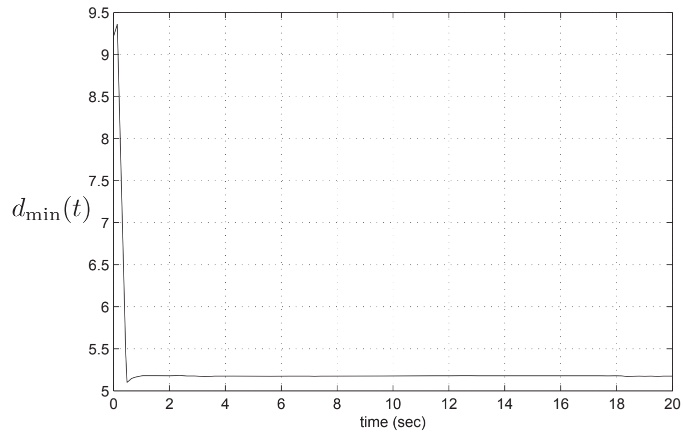


Fig. 3. Time diagram of the minimal distance among all pairs of hovercrafts

## VI. Conclusion

We presented a systematic design for synthesizing individual control for circular flocking with collision avoidance of underactuated hovercrafts. The control design take the curvature-dependent motion into account. Using linear velocity in surge and angular velocity in yaw as virtual controls, we obtained the desired actual control force in surge and actual control torque in yaw in a systematic way of Lyapunov-based design. In such a way, we have proved collision avoidance and cohesion maintenance with mathematical rigor.

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