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On global feedback stabilization of decentralized formation control

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***Abstract*— We address the problem of global stabilization in decentralized formation control. Formation control is concerned with problems in which autonomous agents are required to stabilize at a given distance of other agents. In this context, a graph associated to a formation encodes both the information flow in the system and the distance constraints, by fixing the lengths of the edges. While globally stabilizing control laws for the case of $n = 3$ agents in a cyclic formation have been proposed, the case of $n = 4$ agents has so far resisted attempts to obtain globally stabilizing control laws. We show that a large class of control laws, including all control laws shown to work in the three agents case, cannot satisfactorily stabilize a four agents formation. The proof relies on applying ideas from singularity theory and dynamical systems theory which can be used to address global stabilization of a broad class of decentralized control systems.**

I. Introduction

With the exception of some work on the relation between Lyapunov theory and Morse theory [1], ideas from topology and singularity theory have not played an important role in the analysis of globally stabilizing control laws. The main reason behind this fact is that singularities are, in general, easily avoided by considering a small perturbation of the system or the control law. In this paper, we will show that *the information flow constraints inherent to decentralized control can make such singularities generic* and that these singularities affect the global stabilization properties of the system. Using this approach, we prove that if one tries to stabilize a particular formation at a given configuration via a continuous feedback law, other equilibrium configurations appear and are stable, thus preventing global stability.

Consider the following problem, depicted in Figure 1a: agent 1 observes the position of agent 2, agent 2 the position of agent 3 and agent 3 the position of agent 1. Let the vector μ parametrize the configuration agents $1, 2$ and 3 are required to reach (i.e. a triangle in the plane) and x the state of the system. Is there a continuous feedback control $u(\mu; x)$ such that the system stabilizes to any prescribed configuration in the plane? Problems of this type have been a focal point of attention of control theory for the past decade or more, as they arise in a wide variety of natural (think schooling, herding, etc) and engineering situations. There are three key features to problems of this type:

- the objective (i.e. reaching a given formation in the plane) is in general *parametric*, and the feedback control law depends on the parameter describing the objective.
- the control law is decentralized both in its design (i.e. different agents have access to only "part" of the vector μ) and in its implementation (i.e. the agents have only access to "part" of the state vector x). For a detailed discussion of these points, see [2]
- the interactions are not symmetric: e.g. agent 1 knows the position of agent 2 whereas agent 2 is not aware of agent 1.

In this paper, we prove that there are no decentralized feedback control that globally stabilizes the so-called two-cycles formation, depicted in Figure 1b. For a survey of related results, see [3]

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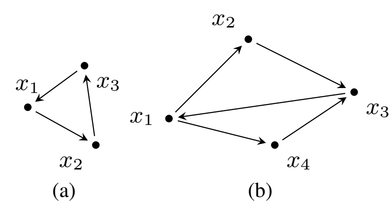

Fig. 1: (a). Three agent in a cyclic formation in the plane. Agent 1 observes agent 2, which observes agent 3 which in turn observes agent 1. (b). The two-cycles formation.

The relevance of the two-cycles formation stems from the following: it was shown in a series of papers ([4], [5], [6] and references therein) that for the case of triangular formations, global stabilization to almost any triangle is possible in a decentralized manner. Even more, a careful analysis exhibited a whole family of stabilizing laws and gave a unified framework to analyze their global convergence properties. It was then conjectured that the results would extend to the case of $n > 3$ agents—which we prove here is not the case.

The fact that the two-cycles formations is the "second simplest formation" is a consequence of a Theorem of Baillieul and Suri [14] that built upon earlier work of Brockett. The result states that when the information flow graph is directed, one can generically require of an agent to have two leaders at the most. From this point of view, we can understand the formation depicted in Figure 1b as being the formation right above the triangle in terms of complexity for finding a control law. Indeed, this formation was singled-out in [6] as an example of the difficulty to make progress in this area and underscored the need of new results to address these decentralized control problem.

II. Notions of stability

Consider the control system

$$\dot{x} = f(x, u(x)) \tag{1}$$

where $x \in M$, a smooth manifold, and all functions are assumed smooth.

We are interested in *global* stabilization properties. Because formation control is inherently nonlinear, we need to introduce some definitions to precisely state the results. From [7], we know that formation control problems evolve on a manifold M with non-trivial homology groups [8]. As a result of the Morse inequalities [8], these systems cannot be globally stable in the usual sense: under reasonable assumptions on the dynamics, there is no continuous u such that (1) has a *unique* equilibrium. Such situations happen frequently in nonlinear control, e.g. in steering control.

From a practical standpoint, however, if one could make one equilibrium stable, and all other equilibria either saddles or unstable, the system would behave as if it were globally stable. Indeed, a vanishingly small perturbation in the state of the system would ensure that, if at a saddle or unstable equilibrium, it evolves to the unique stable equilibrium. We formalize and elaborate on this observation here.

Let $\mathcal{E}_d$ be a finite subset of M containing configurations that we would like to stabilize via feedback. All configurations in $\mathcal{E}_d$ are equally appropriate for the stabilization purpose. We are thus interested in the design of a smooth feedback control $u(x)$ that will stabilize the system to any point $x_0 \in \mathcal{E}_d$. We call these points the *design targets* or *design equilibria*:

$$\mathcal{E}_d = \{x_0 \in M \text{ s.t. } x_0 \text{ is a design equilibrium}\}$$

Let

$$\mathcal{E} = \{x_0 \in M \text{ s.t. } f(x_0, u(x_0)) = 0\},$$

the set of equilibria of (1). We assume that $\mathcal{E}$ is finite.

As explained above, when the system evolves on a non-trivial manifold, the Morse inequalities make it unreasonable to expect that there exists a control $u(x)$ that makes the design equilibria the *only* equilibria of the system, i.e. such that $\mathcal{E}_d = \mathcal{E}$. We call the additional equilibria, whose existence is a consequence of the non-trivial topology of the space, *ancillary equilibria*:

$$\mathcal{E}_a = \mathcal{E} - \mathcal{E}_d.$$

Let us assume for the time being that the linearization of the system at an equilibrium has no eigenvalues with zero real part. We decompose the set $\mathcal{E}$ into *stable* equilibria, by which we mean equilibria such that *all the eigenvalues* of the linearized system have a negative real part, and *unstable equilibria*, where *at least one eigenvalue* of the linearization has a positive real part. Observe that under this definition, saddle points are considered unstable. In summary:

$$\mathcal{E} = \mathcal{E}_s \cup \mathcal{E}_u$$

where

$$\mathcal{E}_s = \{x_0 \in \mathcal{E} \mid x_0 \text{ is stable}\} \text{ and } \mathcal{E}_u = \{x_0 \in \mathcal{E} \mid x_0 \text{ is unstable}\}.$$

With these notions in mind, we introduce the following definition:

Definition 1. *Consider the smooth control systems $\dot{x} = f(x, u(x))$, where $u \in \mathcal{U}$, a set of admissible controls, $x \in M$ and the set $\mathcal{E}$ of equilibria of a system given an admissible u is finite. Let $\mathcal{E}_d \subset M$ be a finite set. We say that $\mathcal{E}_d$ is*

1) feasible *if we can choose a smooth $u(x)$ such that $\mathcal{E}_d \cap \mathcal{E} \neq \varnothing$.*
2) type-A stable *if we can choose a smooth $u(x)$ such that $\mathcal{E}_s \subset \mathcal{E}_d$.*

When the set $\mathcal{E}_d$ is clear from the context, we say that the system is feasible or type-A stable. If the system depends on a parameter μ, we say that it is feasible (resp. type-A stable) if we can choose $u(\mu; x)$ such that it is feasible (resp. type-A stable) for almost all *parameters.*

The set $\mathcal{E}_d$ is feasible if we can choose $u(x)$ such that *at least one* equilibrium of the system is a design target. It is said to be *type-A* stable if the system stabilizes to $\mathcal{E}_d$ almost surely for any initial condition on M.

The usual notion of global stability is a particular instance of type-A stability; indeed, it corresponds to having $u(x)$ such that $\mathcal{E}_d = \mathcal{E} = \mathcal{E}_s$. Looking at the contrapositive of this definition, a system is *not type-A stable* if there exists a set of initial conditions, which is of strictly positive Lebesgue measure, that lead to an ancillary equilibrium.

Example 1. *Consider a system*

$$\dot{x} = x(1 - kx^2)$$

where $k \in \mathbb{R}$ is a feedback parameter to be chosen by the user. We show that any $\mathcal{E}_d \subset (0, \infty)$ is not type-A stable. We first observe that the system has an equilibrium at 0 and two equilibria at $x = \pm\sqrt{1/k}$ if $k > 0$. The system is thus feasible for any $\mathcal{E}_d \subset \mathbb{R}$. The Jacobian of the system is 1 at $x = 0$ and -2 at $x = \pm\sqrt{1/k}$. For $k > 0$, we thus have

$$\mathcal{E} = \{0, \pm\sqrt{1/k}\} = \underbrace{\{\sqrt{1/k}\}}_{\mathcal{E}_d} \cup \underbrace{\{0, -\sqrt{1/k}\}}_{\mathcal{E}_a}.$$

From the linearization of the system, we have that $\mathcal{E}_s = \{\pm\sqrt{1/k}\}$ and $\mathcal{E}_u = \{0\}$. We conclude that the system is not type-A stable for $\mathcal{E}_d \subset (0, \infty)$ since $\mathcal{E}_s \nsubseteq \mathcal{E}_d$.

III. Genericity, robustness and jet spaces

Informally speaking, a property of elements of a topological space is said to be *generic* if it is satisfied by *almost all* elements of the set.

Definition 2. *A property $\mathcal{P}$ is* generic *for a topological space S if it is true on an everywhere dense intersection of open sets of S.*

Everywhere dense intersections of open sets are sometimes called *residual* sets [9]. In general, asking for a given property to be generic is a rather strong requirement, and oftentimes it is enough to show that a given property is true on an open set of parameters, initial conditions, etc. We define

Definition 3. *An element u of a topological space S satisfies the property $\mathcal{P}$* robustly *if there exists a neighborhood U of u in $\mathcal{S}$ such that $\mathcal{P}$ is true for all $u' \in U$. A property $\mathcal{P}$ is robust if there exists a robust u which satisfies the property.*

In practical terms, if a property satisfied only at *non-robust* u's, then it fails to be satisfied under the slightest error in modelling or measurement.

IV. Singularities and transcritical bifurcation

We now recall a few definitions from dynamical systems theory. Consider a dynamical system of the form

$$\dot{x} = f(\mu; x) \tag{2}$$

where $x \in M$, an $n-$dimensional manifold, and $\mu \in \mathbb{R}^k$ is a vector of parameters on which the system smoothly depends.

Definition 4 (Hyperbolic and singular equilibria, bifurcation)**.**

1) *An equilibrium x_0 of* (2) *is called* hyperbolic *if the eigenvalues of the linearization of* (2) *at x_0 have non-zero real-parts. It is called* singular *or* degenerate *otherwise.*
2) *A value μ_0 in the parameter space $\mathbb{R}^k$ for which the flow of* (2) *has a singular equilibrium is called a* bifurcation value*.*

Singularities of vector fields are, generally speaking, not generic since a small perturbation of the vector field f will make its Jacobian, or linearization, non-singular [9]. Our approach relies on showing that, by opposition to the general case, *the decentralized nature of the system makes the existence of such singularities generic*.

We show below that the 2-cycles behaves similarly to the logistic equation, which is presented here, in the sense that they both exhibit the same type of bifurcation. The logistic equation, which is often used to describe systems in which two competing effects—such as supply and demand or predator and prey— are at play, is the one-dimensional ODE given by

$$\dot{x} = x(\mu - x). \tag{3}$$

This equation displays what is called a *transcritical* or *transfer of stability* bifurcation at $\mu = 0$, which we explain here. Observe that the system has two equilibria, one at $x = 0$ and one at $x = \mu$. The linearization of the system at x is

$$df = (\mu - x) - x = \mu - 2x.$$

From this linearization, we see that for $\mu > 0$, the equilibrium $x = 0$ is unstable whereas the equilibrium $x = \mu$ is stable. The situation is reversed for $\mu < 0$. We conclude that at the *bifurcation value* $\mu = 0$, the two equilibria coalesce and *exchange their stability properties*.

Our approach relies on showing that the 2-cycles dynamics is in some sense equivalent to the one system (3). The most common approach to establish this equivalence near a non-hyperbolic equilibrium relies on the use of the *center manifold theorem* [10]. This theorem asserts the existence of a nonlinear change of coordinates, valid near the equilibrium, where the dynamics can be put in a so-called normal form. While very useful in general, such an approach is without much hope for success for our objective. Indeed, the change of variables involved in the analysis depends on the control u, and tracking the effect of this dependence through the whole procedure is not feasible if one considers a broad class of control laws.

In order to overcome this difficulty, we have recourse to the following result of Sotomayor [11], which characterizes the generic behavior of dynamical systems near non-hyperbolic fixed-points *without recourse* to the center manifold.

Theorem 1 (Sotomayor). *Let $\dot{x} = f(\mu; x)$ be a system of ODE in $\mathbb{R}^n$ depending on a scalar parameter μ. For $\mu = \mu_0$, assume that the system has an equilibrium x_0 satisfying the following conditions:*

1) $\frac{\partial f(\mu_0;x)}{\partial x}|_{x_0}$ *has a unique zero eigenvalue with left and right eigenvectors w and v respectively. The other eigenvalues are negative.*
2) $w^T \frac{\partial f(\mu;x)}{\partial \mu}|_{x_0,\mu_0} v = 0$
3) $w^T \frac{\partial^2 f(\mu_0;x)}{\partial x^2}|_{x_0}(v,v) \neq 0$ *and* $w^T \frac{\partial^2 f(\mu_0;x)}{\partial x \partial \mu}|_{x_0,\mu_0}(v,v) \neq 0$

Then the phase portrait is topologically equivalent to the phase portrait of the logistic equation, i.e. we have a transcritical bifurcation about x_0 for $\mu = \mu_0$. Thus around $\mu = \mu_0$, there are two arcs of equilibria whose stability properties are exchanged when passing through μ_0.

V. Formation Control

Let $G = (V, E)$ be a *directed graph* with n vertices — that is $V = \{x_1, x_2, \ldots, x_n\}$ is an ordered set of vertices and $E \subset V \times V$ is a set of edges. We let $|E| = m$ be the cardinality of E. We call the *outvalence* of a vertex the number of edges originating from this vertex and the *invalence* the number of incoming edges.

Directed graphs are used to encode the *information flow* in formation control problems. We follow the convention that an arrow leaving vertex i for vertex j means that agent i measures the relative—to its own location— position of agent j.

The mixed-adjacency matrix of a graph $G = (V, E)$ is a $|E| \times |V|$ matrix whose entry i, j is -1 if edge i originates from vertex j, 1 if edge i ends at vertex j and 0 otherwise:

Definition 5 (Mixed adjacency matrix). *Given a directed graph $G = (V, E)$, its mixed adjacency matrix $A_m \in \mathbb{R}^{n \times m}$ is defined by*

$$A_{m,ij} = \begin{cases} -1 & \text{if } e_i = (j,s), s \in V \\ +1 & \text{if } e_i = (k,j), k \in V \\ 0 & \text{otherwise.} \end{cases}$$

We will often have to consider the matrix $A_m \otimes I$ where $\otimes$ is the Kronecker product and I the two-by-two identity matrix. In order to keep the notation simple, we write $A_m^{(2)}$ for this Kronecker product.

A. Rigidity

We briefly cover the fundamentals of rigidity and establish the relevant notation. We refer the reader to [3], [12] for a more detailed presentation. We call a *framework* an embedding of a graph in $\mathbb{R}^2$ endowed with the usual Euclidean distance, i.e. given $G = (V, E)$, a framework p attached to a graph G is a mapping

$$p : V \to \mathbb{R}^2.$$

By abuse of notation, we write x_i for $p(x_i)$. We define the *distance function* δ of a framework with n vertices (represented as an element of $\mathbb{R}^{2n}$) as

$$\delta(p) : \mathbb{R}^{2n} \to \mathbb{R}_+^{n(n-1)/2} : (x_1, \ldots, x_n) \\ \to \frac{1}{2}\left[\|x_1 - x_2\|^2, \ldots, \|x_2 - x_3\|^2, \ldots, \|x_{n-1} - x_n\|^2\right],$$

where $\mathbb{R}_+ = [0, \infty)$. We denote by $\delta(p)|_E$ the restriction of the range of δ to edges in E.

For a graph G with m edges, we define

$$\mathcal{L} = \left\{ d = (d_1, \ldots, d_m) \in \mathbb{R}_+^m \text{ s.t. } \exists p \text{ with } \delta(p(V))|_E = \sqrt{d} \right\},$$

where the square root of d is taken entry-wise. Properties of this set and its relations to the number of ancillary equilibria are discussed in [7]. We have taken the square root of d for computational convenience, as will be seen below. We denote by $\mathcal{L}_0$ the interior of $\mathcal{L}$.

The *rigidity matrix* of the framework is the Jacobian $\frac{\partial \delta}{\partial x}$ restricted to the edges in E. We denote it by $\frac{\partial \delta}{\partial x}|_E$. A framework is said to be infinitesimally rigid if there are no vanishingly small motions of the vertices, except for rotations and translations, that keep the edge-length constraints on the framework satisfied. This translates into [13] $\text{rank}(\frac{\partial \delta}{\partial x}|_E) = 2n - 3$. A framework attached to a graph G is said to be *rigid* if there are no motions, save for rotations and translations of the plane, of the vertices that keep the edge length constraints satisfied and *minimally rigid* if all the edges of the graph are necessary for rigidity.

B. Directed formation control

We formalize in this section the type of control system considered. We are given a graph $G = (V, E)$ which is assumed to be *minimally rigid* with $|V| = n$ and $|E| = m$. Let $d \in \mathcal{L}$ be a feasible edge-length vector. The objective of the formation control problem is to find a decentralized control law, where the information flow is given by G, that will stabilize the system around a framework where the inter-agent distances are given by d.

In more detail, each agent with position $x_i \in \mathbb{R}^2$ is represented by a vertex i in V. The dynamics of x_i is allowed to depend only on the position of agents x_k for which there is an edge originating at x_i and ending at x_k:

$$\dot{x}_i = u_i(\mu; x_k, x_l, \ldots), \text{ where } (i,k), (i,l), \ldots \in E.$$

The objective is then to find u_i such that the system stabilizes around a framework with the prescribed edge lengths μ. Following [14], we assume from now on that G has a maximum outvalence of two.

We denote by the variables $z_i \in \mathbb{R}^2$ the relative position of the agents. Precisely, given (arbitrary) orderings of the edges and vertices of G, we define

$$z_i = x_k - x_l \text{ and } e_i = z_i^T z_i - d_i$$

where edge i links nodes x_k to x_l, and e_i is the corresponding error in edge length.

With the convention of Section II, we have

$$\mathcal{E}_d = \{x \in \mathbb{R}^{2n} | e_i(x) = 0, \text{for all edges in } E\}.$$

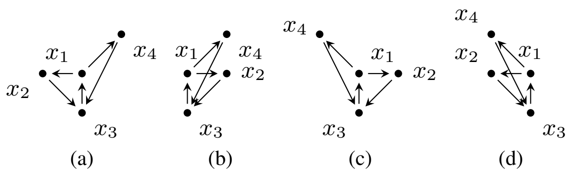

Fig. 2: Four formations in the plane that are not equivalent under rotations and translation and that have the same corresponding edge lengths. (a) is the mirror-symmetric of (c) and (b) is the mirror-symmetric of (d).

An important feature of the formation control problem is that it is defined up to a rigid transformation of the plane: if $x \in \mathbb{R}^{2n}$ is a framework of G, frameworks obtained by a rotation and translation of x—we write them as $A \cdot x$, for $A \in SE(2)$, the special Euclidean group [7]— are equivalent to x in formation control. As a consequence, we can assume without loss of generality that the agents measure only the *relative* position of other agents.

We emphasize that the feedback laws can depend explicitly on the design distances d, and not solely on them through the e_i's as is implicitly assumed in most related work on formation control. We write this dependence of the feedback on d, which is a parameter by opposition to a dynamical variable, by using a semi-colon as in previous sections. We have, for $x_i, z_i \in \mathbb{R}^2$, $u_i \in \mathbb{R}$:

$$\dot{x}_i = u_i(d_i; e_i) z_i$$

in case agent i follows a single agent and

$$\dot{x}_i = u_{1i}(d_k, d_l; e_k, e_l, z_k^T z_l) z_k + u_{2i}(d_k, d_l; e_k, e_l, z_k^T z_l) z_l$$

in case agent i follows two agents. This general form respects both the invariance under the SE(2) action and the decenralization constraints on d and x given by G.

We established in [7] a few conditions a feedback control law had to satisfy in order to yield a well-defined formation control system (Definition 7). We recall them here:

Definition 6. *A feedback control law u_i is* compatible *with a formation control problem if*

1) *$u_i(d_j; e_j)$ is such that $u_i(d_j; 0) = 0$ if agent i has one co-leader.*
2) *$u_i(d_j, d_k; e_j, e_k, z_j \cdot z_k)$ is such that $u_i(d_j, d_k; 0, 0, z) = 0$ for all z if agent i has two co-leaders.*

We accordingly define the class of controls $\mathcal{U}$ to be all smooth control laws such that $u_i(d_i; e_i) = 0$ and $u_j(d_i, d_j; e_i, e_j, \cdot) = 0$ for $e_i = e_j = 0$.

From the above discussion, we conclude that:

Proposition 1. *Let $\mathcal{E}_d$ be the set of design equilibria for a formation control problem with underlying information flow graph G. If the graph G is* rigid*, the set $\mathcal{E}_d/SE(2)$ is finite. In other words, the set of design equilibria is finite up to rigid transformations.*

C. The two-cycles formation

The two-cycles is the formation represented in Figure 1b. Let $x_i \in \mathbb{R}^2$, $i = 1 \ldots 4$ represent the position of 4 agents in the plane. We define the vectors

$$\begin{cases} z_1 = x_2 - x_1; & z_2 = x_3 - x_2; & z_3 = x_1 - x_3 \\ z_4 = x_3 - x_4; & z_5 = x_4 - x_1 \end{cases} \quad (4)$$

Hence a general control law for such a system is

$$\begin{cases} \dot{x}_1 &=& u_{11}(d_1, d_5; e_1, e_5, z_1^T z_5) z_1 + u_{12}(d_1, d_5; e_5, e_1, z_1^T z_5) z_5 \\ \dot{x}_2 &=& u_2(d_2; e_2) z_2 \\ \dot{x}_3 &=& u_3(d_3; e_3) z_3 \\ \dot{x}_4 &=& u_4(d_4; e_4) z_4 \end{cases} \quad (5)$$

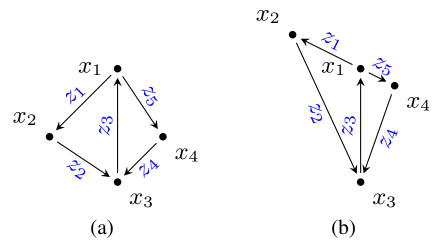

Fig. 3: The formation in (a) is such that $(\|z_1\|, \ldots, \|z_5\|) \notin \mathcal{S}$, whereas $(\|z_1\|, \ldots, \|z_5\|) \in \mathcal{S}$ for the formation depicted in (b)

The set of design equilibria $\mathcal{E}_d$ for the 2-cycles is of cardinality 4, up to rigid transformations, since there are four frameworks in the plane for which $e_i = 0$; they are depicted in Figure 2.

The set $\mathcal{E}_a$ of ancillary equilibria of (5) depends on the choice of feedbacks $u_i \in \mathcal{U}$. Due to the invariance and distributed nature of the system, some configurations belong to $\mathcal{E}_a$ for all elements of $\mathcal{U}$:

Proposition 2. *The set $\mathcal{E}$ of* (5) *contains, in addition to the equilibria in $\mathcal{E}_d$, the frameworks characterized by*

1) *$z_i = 0$ for all i, which corresponds to all the agents being at the same position.*
2) *all z_i are aligned, which corresponds to having all agents on the same one-dimensional subspace in $\mathbb{R}^2$. These frameworks form a three dimensional invariant subspace of the dynamics.*
3) *$e_2 = e_3 = e_4 = 0$, z_1 and z_5 are aligned and so that*

$$u_1(e_1, e_5, z_1 \cdot z_5)\|z_1\| = \pm u_5(e_1, e_5, z_1 \cdot z_5)\|z_5\|,$$

where the sign depends on whether z_1 and z_5 point in the same or opposite directions.

This result is straightforward from an inspection of Equation (5). Frameworks of type 2 above are non-infinitesimally rigid and they define an invariant submanifold of the dynamics.

The main result of this paper is to show that the 2-cycles formation is not robustly type-A stabilizable for an open set of parameters $d \in \mathcal{L}$ or, equivalently, that there does not exist robust feedback laws that will stabilize any of its four design equilibria without stabilizing an ancillary equilibria:

Theorem 2. *The two-cycles formation is not robustly type-A stable.*

Because of space constraints, we prove Theorem 2 in a particular case.

D. Singular formations for $n = 4$ agents.

Our work singles out a particular type of frameworks which, even though they are infinitesimally rigid, show a certain degree of degeneracy. We first observe that, in general, the angle between z_1 and z_5 is not uniquely determined by the edge lengths. We define $\mathcal{S}$ to be set of edge lengths such that, *at least* one of the four frameworks corresponding to d has z_1 parallel to z_5 with the notation of Figure 3:

$$\mathcal{S} = \{d \in \mathcal{L} \text{ s.t. } z_1 \text{ parallel } z_5 \text{ for one framework at least.}\}.$$

We define $\mathcal{S}_0 = \mathcal{S} \cap \mathcal{L}_0$.

We will need the following properties of this set:

Lemma 1. *The following properties of $\mathcal{S}$ hold:*

1) *$\mathcal{S}$ is of codimension one in $\mathcal{L}$.*
2) *The frameworks corresponding to edge lengths in $\mathcal{S}_0$ are infinitesimally rigid.*

Proof. For the first part, observe that we can parametrize $\mathcal{S}$ by first choosing a feasible d_1, d_2, d_3 yielding a triangle x_1, x_2, x_3

and one additional parameter giving the signed length of z_5, with the sign referring to z_5 going in the same direction as z_1 or the opposite direction. Hence 4 parameters are sufficient and necessary to describe a formation in $\mathcal{S}$; it is thus of codimension one in $\mathcal{L}$

For the second part, we have that the rigidity matrix of the two-cycles is given by R in Equation (6):

$$R = \begin{bmatrix} z_1^T & -z_1^T & 0 & 0 \\ 0 & z_2^T & -z_2^T & 0 \\ -z_3^T & 0 & z_3^T & 0 \\ 0 & 0 & z_4^T & -z_4^T \\ z_5^T & 0 & 0 & -z_5^T \end{bmatrix}, Z = \begin{bmatrix} z_1^T & 0 & \dots & 0 \\ 0 & z_2^T & \dots & 0 \\ & & \ddots & \vdots \\ 0 & \dots & & \vdots \\ 0 & 0 & \dots & z_5^T \end{bmatrix}. \tag{6}$$

Some simple algebra shows that one has

$$R = ZA_m^{(2)} \tag{7}$$

where we recall that A_m is the mixed adjacency matrix. In the case of the 2-cycles, the mixed adjacency matrix $A_m \in \mathbb{R}^{5\times 4}$ is of rank 3. The cokernel of A_m is spanned by $[0,0,1,1,1]^T$ and $[1,1,1,0,0]$. Hence, the cokernel of $A_m^{(2)}$ is four dimensional and spanned by the vectors $[0,0,1,1,1]^T \otimes [1,0]^T$, $[0,0,1,1,1]^T \otimes [0,1]^T$, $[1,1,1,0,0]^T \otimes [1,0]^T$ and $[1,1,1,0,0]^T \otimes [0,1]^T$

The matrix Z is of full rank unless $z_i = 0$ for some i, which corresponds to two agents superposed. We thus have that Z is of full rank for formations in $\mathcal{L}_0$. The kernel of Z is given by the relations $z_1 + z_2 + z_3 = 0$ and $z_3 + z_4 + z_5 = 0$. Because Z is of full row rank, R is of full (row) rank if $A_m^{(2)}$ maps *onto* the coimage of Z. It is readily verified to be the case from the above relations describing the cokernel of $A_m^{(2)}$ and the kernel of Z. ∎

VI. The two-cycles is not type-A stable

We now focus our attention on the system,

$$\begin{cases} \dot{x}_1 = u(e_1)z_1 + u(e_5)z_5; & \dot{x}_2 = u(e_2)z_2 \\ \dot{x}_3 = u(e_3)z_3; & \dot{x}_4 = u(e_4)z_4 \end{cases} \tag{8}$$

We show that the system of Equation 8 is equivalent to the logistic equation around $\mathcal{S}_0$. We denote by $F(x)$ the vector field on the right-hand side of Equation (8). In this particular case, the set of admissible control laws $\mathcal{U}$ contains twice differentiable u with $u(0) = 0$.

Theorem 3. *Consider the set of vector fields $\mathcal{F} = \{F(x) | u(x) \in \mathcal{U}\}$. For all robust $u \in \mathcal{U}$, the system undergoes a transcritical bifurcation at frameworks with $d \in \mathcal{S}_0$.*

We will prove Theorem 2 as a corollary of this result. We prove Theorem 3 in several steps:

Proposition 3. *For $d \in \mathcal{S}_0$ there is a non-zero vector $w \in \mathbb{R}^8$ such that $w^T \frac{\partial F}{\partial x}|_{e_i=0,d} = w^T \frac{\partial F}{\partial d}|_d = 0$ for at least one framework with edge lengths d.*

We will prove Proposition 3 by relying on some technical lemmas. We use the notation $Z_i = z_i z_i^T$.

Lemma 2. *The Jacobian of system 8 at a design equilibrium is given by*

$$\frac{\partial F}{\partial x}|_{e_i=0,d} = -2u'(0)\begin{bmatrix} -Z_1 - Z_5 & Z_1 & 0 & Z_5 \\ 0 & -Z_2 & Z_2 & 0 \\ Z_3 & 0 & -Z_3 & 0 \\ 0 & 0 & Z_4 & -Z_4 \end{bmatrix}.$$

Proof. We write $F_1(x) = u(e_1)z_1 + u(e_5)z_5$, $F_2(x) = u(e_2)z_2$, and so forth. Observe that

$$\frac{\partial u(e_1)z_1}{\partial x_1} = u(e_1)\begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix} + u_1'(e_1)z_1 z_1^T.$$

Since $u(0) = 0$ and design equilibria are such that $e_i = 0$, we have that $\frac{\partial F_1}{\partial x_1}|_{e_1=0} = 2u'(0)(z_1 z_1^T + z_5 z_5^T)$. We have similar relations for other derivatives $\frac{\partial F_i}{\partial x_j}$. Gathering the above relations, we obtain the result. ∎

Lemma 3. *The Jacobian of F with respect to the parameters d at a design equilibrium is given by*

$$\frac{\partial F}{\partial d}|_{e_i=0,d} = -u'(0)\begin{bmatrix} z_1 & 0 & 0 & 0 & z_5 \\ 0 & z_2 & 0 & 0 & 0 \\ 0 & 0 & z_3 & 0 & 0 \\ 0 & 0 & 0 & z_4 & 0 \end{bmatrix} \tag{9}$$

Proof. We have that $\frac{\partial F_1}{\partial d_1} = -u_1'(e_1)z_1$ and similar expressions for the other entries. ∎

Lemma 4. *Let $d \in \mathcal{S}_0$, then w is a left eigenvector of $\frac{\partial F}{\partial x}|_{e_i=0,d}$ with eigenvalue 0 if and only if $w^T \frac{\partial F}{\partial d}|_{e_i=0,d} = 0$*

Proof. Let $R \in \mathbb{R}^{5\times 8}$ be the rigidity matrix of the two-cycles formation. A direct computation yields

$$\frac{\partial F}{\partial d}R = \frac{\partial F}{\partial x}. \tag{10}$$

From Equation 10 and Lemma 1, we conclude that $w^T \frac{\partial F}{\partial x} = 0$ if and only if $w^T \frac{\partial F}{\partial d} = 0$. ∎

Because the control system is invariant under an action of the Euclidean group $SE(2)$ on $\mathbb{R}^2$, its Jacobian at an equilibrium has three zero eigenvalues:

Proposition 4. *The eigenvalues at a desired equilibrium of the linearized system of Equation 8 are $(0,0,0)$ and the eigenvalues of the matrix*

$$J = ZA_m^{(2)}\frac{\partial F}{\partial d}. \tag{11}$$

Proof. We know from Lemma 4 and Equation (7) that $\frac{\partial F}{\partial x} = \frac{\partial F}{\partial d}R = \frac{\partial F}{\partial d}ZA_m^{(2)}$, which has the same non-zero spectrum as $ZA_m^{(2)}\frac{\partial F}{\partial d}$ [15]. The result is a simple consequence of the dimensions of the operators involved. ∎

Corollary 1 (Singular formations)**.** *The Jacobian J of the 2-cycles formation is of corank 1 for at least one framework in $\mathcal{S}_0$.*

Proof. An explicit computation yields

$$J = \begin{bmatrix} z_1^T z_1 & -z_1^T z_2 & 0 & 0 & -z_1^T z_5 \\ 0 & z_2^T z_2 & -z_2^T z_3 & 0 & 0 \\ -z_3^T z_1 & 0 & z_3^T z_3 & 0 & -z_3^T z_5 \\ 0 & 0 & -z_3^T z_4 & z_4^T z_4 & 0 \\ -z_1^T z_5 & 0 & 0 & -z_4^T z_5 & z_5^T z_5 \end{bmatrix}$$

The first and last column are multiple of each other if z_1 is parallel to z_5, and quick computation shows that the first four columns, are linearly independent. The corank is higher if one of the z_i is in addition zero. ∎

Proof of Proposition 3. Consider a framework with $d \in \mathcal{S}_0$ and z_1 parallel to z_5. From Corollary1, we know that $\frac{\partial F}{\partial x}$ is generically of rank 4. Let w be an eigenvector corresponding to the zero eigenvalue. We conclude using Lemma 4 that $w^T \frac{\partial F}{\partial d} = 0$. ∎

Proof of Theorem 3. Write $d_0 = (d_1, d_2, d_3, d_4, d_5)$. We consider the *one parameter system* where only $\mu \in \mathbb{R}$ is allowed to vary:

$$\begin{cases} \dot{x}_1 &= u(e_1)z_1 + u(e_5)z_5 \\ \dot{x}_2 &= u(e_2)z_2 \\ \dot{x}_3 &= u(z_3^T z_3 - (d_3 + \mu))z_3 \\ \dot{x}_4 &= u(e_4)z_4 \end{cases}. \tag{12}$$

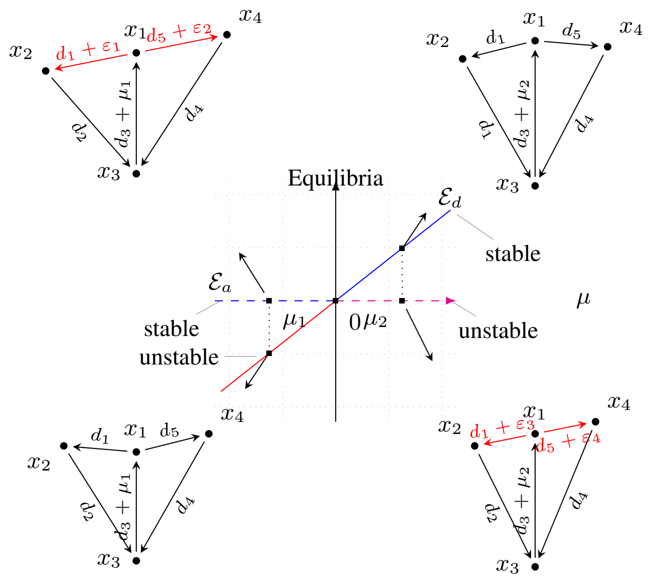

Fig. 4: We illustrate the stability properties of ancillary and design equilibria around $\mathcal{S}_0$. Let the vector $(d_1, d_2, d_3, d_4, d_5) \in \mathcal{S}_0$. The horizontal dashed line corresponds ancillary equilibria and the slanted line that intersects it to design equilibria. They coincide at $\mu = 0$, as seen in Proposition V-C; for $\mu \neq 0$ configurations in $\mathcal{S}_0$ are ancillary equilibria. For $\mu_1 < 0$, there is an ancillary equilibrium with $e_2, e_3, e_4 = 0$ but $e_1 = \varepsilon_1$ and $e_5 = \varepsilon_2$ and z_1 and z_5 aligned. It is illustrated in the top-left corner of the figure. This equilibrium is moreover stable. For $\mu_2 > 0$, there is a similar ancillary equilibria with $e_1 = \varepsilon_3$ and $e_5 = \varepsilon_4$, illustrated in the bottom-right corner, but this equilibrium is unstable. We see that around the bifurcation value μ_0, there is a *transfer of stability* from $\mathcal{E}_d$ to $\mathcal{E}_a$. The orientation may be reversed (i.e. $\mu_1 > 0, \mu_2 < 0$ and all else the same in the figure) depending on the sign of the second derivatives in Theorem 3. ■

We prove that conditions (1), (2) and (3) of Theorem 1 are satisfied at robust $u \in \mathcal{U}$. From Corollary 1 and the fact that $u' \neq 0$ generically at a zero of u [9], we know that the Jacobian of the two-cycles at $\mathcal{S}_0$ has a unique zero eigenvalue zero generically for $F \in \mathcal{F}$. Hence condition (1) is verified. Condition (2) follows from Proposition 3.

The second derivatives of condition (3) in Theorem 1 are, at a design equilibrium, the sum of two terms:

$$\frac{\partial^2 F}{\partial x^2} = u' \sum_l w_l v^T Q_l v + u'' \sum_l w_l v^T \tilde{Q}_l v$$

where $Q_l, \tilde{Q}_l \in \mathbb{R}^{8\times 8}$ are obtained by evaluating the Hessian of F_l. One can easily check that $\sum_l w_l v^T Q_l v$ and $\sum_l w_l v^T \tilde{Q}_l v$ are generically non-zero on $\mathcal{S}_0$. Set $a = \sum_l w_l v^T Q_l v$ and $b = \sum_l w_l v^T \tilde{Q}_l v$. L We thus have $\frac{\partial^2 F}{\partial x^2}$ is zero only if

$$u'a + u''b = 0$$

when u vanishes. Let $C \subset J^2(\mathbb{R}, \mathbb{R})$[1]be defined by the equations $au' + bu'' = 0$ and $u = 0$. Since C is of codimension 2 in J^2, the 2-jet extension of u is transversal [9] to C if and only if $u \neq 0$ or $au' + bu'' \neq 0$. We conclude using Thom's transversality Theorem [9] that $\frac{\partial^2 F}{\partial z^2}$ is generically non-zero.

Observe that $\frac{\partial F^2}{\partial x \partial d}$ is a constant matrix independent of the configuration. Using a similar reasoning as above, we can conclude that $w^T \frac{\partial^2 F}{\partial x \partial d} v$ is generically non-zero. ■

Proof of Theorem 2. We depict the setting in Figure 4. We will show that there is a set of positive measure in $\mathcal{L}$ which cannot be made robustly type-A stable. We do so by showing that for $\mathcal{E}_d$ corresponding to distances in that subset of $\mathcal{L}$, there is a stable ancillary equilibrium for all robust $u \in \mathcal{U}$.

Denote by $\mathcal{S}^\varepsilon$ a tubular neighborhood of $\mathcal{S}$:

$$\mathcal{S}^\varepsilon = \{d \in \mathcal{L} \text{ s.t. } \exists\, d_0 \in \mathcal{S}, \text{ with } \|d - d_0\| < \varepsilon\}$$

and $\mathcal{S}_0^\varepsilon = \mathcal{S}^\varepsilon \cap \mathcal{S}_0$. The set $\mathcal{S}^\varepsilon$ contains frameworks where z_1 and z_5 are close to parallel. These frameworks are infinitesimally rigid and non-singular. Let $d \in \mathcal{S}_0^\varepsilon$ and $d_0 \in \mathcal{S}_0$ be such that there is $-\varepsilon < \mu < \varepsilon$ with $d = d_0 + (0, 0, \mu, 0, 0)$. Such d_0 and μ exist by definition of $\mathcal{S}_0^\varepsilon$.

Because the system is invariant under mirror symmetry [7] along z_3, the stability property of the equilibria (a) and (c) and (b) and (d) in Figure 2 are the same. Assume without loss of generality that u is such that the design equilibria for the frameworks with x_1 and x_4 on the same side of z_3 are stable. Because the system undergoes a transcritical bifurcation when $\mu = 0$ by Theorem 3, and because $u' \neq 0$ generically when u vanishes, we have that for ε small enough, $\mathcal{E}_a$ contains the framework where z_1 is parallel to z_5 for all frameworks with $-\varepsilon < \mu < \varepsilon$. Furthermore, for either $\mu > 0$ or $\mu < 0$, we have that this framework is asymptotically stable, i.e. $\mathcal{E}_s \cap \mathcal{E}_a \neq \emptyset$. Hence, there is a set of positive measure of target frameworks in $\mathcal{S}_0^\varepsilon$ which contains a stable ancillary equilibrium and thus the system is not robustly type-A stable. ■

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