On feedback stabilization of stochastic nonlinear systems with discrete and distributed delays

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Abstract— In this paper, the problem of feedback stabilization of stochastic differential delay systems is considered. The systems under study are nonlinear, nonaffine and involve both discrete and distributed delays. By using a LaSalle-type theorem for stochastic systems, general conditions for stabilizing the closed-loop system with delays are obtained. In addition, stabilizing state feedback control laws are proposed.

I. INTRODUCTION

During the last decades, the problems of stabilization and controller design for linear systems with delays has been extensively studied and is still under investigation (see [6], [8], [11], [18], [20], [23]). In practice, many control processes involve delays (often due to transmission or transportation phenomenons). Delays may significantly affect the closed-loop performances or even be a source of instability.

In the case of nonlinear systems with delays, the problem of stabilization is more complex. This is mainly due to the infinite dimensionality of the system state combined with the nonlinear structure of the differential equations.

In ([1]-[4]) we investigated the problem of stabilization of nonlinear, nonaffine systems involving delays in both continuous and discrete-time cases. These works have been treated in a deterministic context.

However, the presence of delays and nonlinearities are not the only sources of complexity. Indeed, various disturbances that are not measurable may arise which, in turn, limit the application of classical control systems design. This motivates the study of the stabilizability and the control design in a stochastic framework, where the state equation is described by an Itô differential delay equation driven by Wiener noise.

The stability analysis of the equilibrium positions of stochastic differential equations with delays has been extensively studied (see for instance [12],[15],[19]). In the case of linear stochastic systems with delays, some results on stabilization have been proposed (in [7], [22], for instance). However, the stabilization of nonlinear stochastic systems with delays still remains an open problem.

In this paper, the state feedback stabilizability problem of equilibrium positions of stochastic nonlinear systems with delays is considered. The class of systems under study is nonaffine in control. Moreover, the system without drift - in other words, the related autonomous system - also involves

delays. By combining a suitable mathematical formalism and a LaSalle-type theorem dedicated to stochastic systems ([16], [17]), sufficient conditions guaranteeing the stability of the closed-loop system are developed and feedback controllers for these systems are proposed. The approach adopted in this paper allows considering a rather large class of nonlinear stochastic systems. The methodology is developed for systems involving both discrete and distributed delays. Moreover, the autonomous system ($u = 0$) as well as the controlled part are affected by a noise.

The organization of the paper is as follows. In Section 2 the class of systems under consideration is presented and some basic notions are recalled. Some notations are introduced and usual notions about the stability and LaSalle-type Theorem for stochastic differential systems with time delay are recalled. In Section 3, the main results are given and proved. Sufficient conditions of stabilizability and related feedback laws are proposed. Moreover, an illustrative example is presented before conclusions.

II. PROBLEM FORMULATION AND PRELIMINARIES

Consider the following class of systems:

$$
\begin{cases}
dx(t) = \Big(f(x(t), x(t-\tau_1)) + \int_0^{\tau_2} \tilde{f}(x(t), x(t-\theta), u(t))d\theta\Big)dt \\
\quad + \Big(g(x(t), x(t-\tau_3)) + \int_0^{\tau_4} \tilde{g}(x(t), x(t-\theta), u(t))d\theta\Big)d\xi(t) \\
x(t) = \phi(t), \quad t \in [-\tau, 0], \quad \tau = \max_{i=1..4}\{\tau_i\}
\end{cases}
$$

$$(1)$$

where f, g, $\tilde{f}$ and $\tilde{g}$ are smooth vector fields such that $f(0,0) = g(0,0) = \tilde{f}(0,0,0) = \tilde{g}(0,0,0) = 0$. In the following, $x(t) \in \mathbb{R}^n$ is the state vector and $u \in \mathbb{R}$ is the input vector. τ_i ($i = 1..4$) are positive scalars representing delays.

The function $\phi(t) \in \mathcal{C} = \mathcal{C}([-\tau, 0], \mathbb{R}^n)$ represents the initial condition. $\mathcal{C}([-\tau, 0], \mathbb{R}^n)$ is the banach space of continuous function mapping $[-\tau, 0]$ into $\mathbb{R}^n$, with the norm $\|\phi\| = \sup_{t \in [-\tau, 0]} |\phi(t)|$ where $|\phi(t)|$ stands for the Euclidean norm of $\phi(t) \in \mathbb{R}^n$.

$\{\xi(t), t \geq 0\}$ is a standard Wiener process defined on the usual complete probability space $(\Omega, \mathcal{F}, (\mathcal{F}_t)_{t\geq 0}, P)$ with $(\mathcal{F}_t)_{t\geq 0}$ being the complete right-continuous filtration generated by ξ and $\mathcal{F}_0$ contains all P-null sets.

Let $\mathcal{C}^b_{\mathcal{F}_0}([-\tau,0],\mathbb{R}^n)$ be the set of all $\mathcal{F}_0$-measurable bounded $\mathcal{C}([-\tau,0],\mathbb{R}^n)$-valued random variables ϕ. $\mathcal{W}([-\tau,0],[0,\infty))$ represents the family of all Borel measurable bounded nonnegative functions η such that
$$\int_{-\tau}^0 \eta(s)ds = 1$$

A. Preliminaries

In order to deal with the previously defined class of systems, some useful notations are introduced and usual notions about the stability of stochastic systems with time delay are first recalled as preliminaries for the further results. To that purpose, the differential stochastic system of the general form (2) is considered :

$$\begin{cases} dx(t) = F(t,x_t)dt + G(t,x_t)d\xi(t) \\ x(t) = \phi(t), \quad t \in [-\tau,0] \end{cases} \tag{2}$$

where $F, G : [0,\infty) \times \mathcal{C}([-\tau,0],\mathbb{R}^n) \mapsto \mathbb{R}^n$ is continuous with respect to the first argument, locally Lipschitz with respect to the second and satisfy $F(t,0) = G(t,0) = 0$ for all $t \geq 0$.

For $t \geq \sigma - \tau$, we denote by $x(\sigma,\phi)(t)$ its solution at time t with initial data ϕ, specified at time σ, i.e., $x(\sigma,\phi)(\sigma+\theta) = \phi(\theta), \forall\theta \in [-\tau,0]$. For $\theta \in [-\tau,0]$,

$$x_t(\theta) = x(t+\theta)$$

and represents the state of the delay system.

Under the previous assumptions on F and G, it is known (see e.g. [15], [19], [21]) that equation (2) has a unique solution $x(\sigma,\phi)(t)$ for $t \geq \sigma - \tau$.

The infinitesimal generator associated to (2), obtain by differentiating V in the sense of Itô, is given by

$$\mathcal{L}V(t,\phi) = \frac{\partial V(t,\phi(0))}{\partial t} + <F(t,\phi),\nabla V(t,\phi(0))>$$

$$+\frac{1}{2}\text{Tr}\Big([G(t,\phi)G^T(t,\phi)]\nabla^2 V(t,\phi(0))\Big) \tag{3}$$

where ∇ denotes the gradient and $\langle.,.\rangle$ designates the scalar product. The matrix $\nabla^2 V(t,x) = \dfrac{\partial^2 V(t,x)}{\partial x^2}$ is the Hessian matrix of the second order partial derivatives.

The notation Tr(.) designates the trace of a matrix. In the following, we will also use Ker(.) to designate the Kernel of a matrix or a function and $d(x,D)$ will represents the Haussdorf semi-distance between a point $x \in \mathbb{R}^n$ and a set D (i.e. $d(x,D) = \inf_{y\in D}|x-y|$.) For any matrix M, M^T denotes its transpose.

B. Notions of stability and LaSalle-type Theorem for stochastic differential delay systems

In the following, some notions of stability of equilibrium solution of stochastic differential delay equations are given. They generalize the notions initially dedicated to stochastic differential systems (cf. [5], [10], [13]).

DEFINITION 1:
The equilibrium solution, $x \equiv 0$ of the stochastic differential delay equation (2) is said to be
1) stable in probability, if for any $\sigma \in \mathbb{R}$, $\varepsilon > 0$, there is a $\beta = \beta(\varepsilon,\sigma)$ such that $\|\phi\| < \beta$ implies $P\big(\sup_{t\geq\sigma} \|x_t(\sigma,\phi)\| > \varepsilon\big) = 0$.
2) uniformly stable in probability, if the number β is independent of σ.

For all $\beta > 0$, let us denote by $\mathcal{B}(0,\beta)$, the ball
$$\mathcal{B}(0,\beta) = \{\phi \in \mathcal{C}([-\tau,0],\mathbb{R}^n) : \|\phi\| < \beta\}.$$

DEFINITION 2:
The equilibrium solution, $x \equiv 0$ of the stochastic differential delay equation (2) is said to be asymptotically stable in probability, if it is stable and there exists $b_0 = b_0(\sigma) > 0$ such that $\phi \in \mathcal{B}(0,b_0)$ implies $P\big(\lim_{t\to+\infty} \|x_t(\sigma,\phi)\| = 0\big) = 1$.

In practice, we can use the following LaSalle-type Theorem for stochastic differential delay system (cf. [16], [17]). This is a stochastic version of the well-known LaSalle Theorem (cf. [9], [14]).

THEOREM 1:
Assume that there are functions $V \in \mathcal{C}^{1,2}([0,\infty) \times \mathbb{R}^n, [0,\infty))$, $\gamma \in L^1([0,\infty),[0,\infty))$, $\eta \in \mathcal{W}([-\tau,0],[0,\infty))$ and $w_1, w_2 \in \mathcal{C}(\mathbb{R}^n,[0,\infty))$ such that

$$\mathcal{L}V(t,\phi) \leq \gamma(t) - w_1(\phi(0)) + \int_{-\tau}^0 \eta(\theta)w_2(\phi(\theta))d\theta,$$

$$\forall\phi \in \mathcal{C}([-\tau,0],\mathbb{R}^n), t \geq 0,$$

$$w_1(x) \geq w_2(x) \quad x \in \mathbb{R}^n$$

and $\lim_{|x|\to\infty} \inf_{0\leq t<\infty} V(t,x) = \infty$

$$\tag{4}$$

Then, $Ker(w_1 - w_2) \neq \emptyset$,

$$\int_0^\infty E(w_1(x(t,\phi))-w_2(x(t,\phi))) < \infty \quad \text{a.s (almost surely)},$$

$$\limsup_{t\to} V(t,x(t,\phi)) \quad \text{a.s},$$

and

$$\lim_{t\to\infty} d(x(t,\phi),Ker(w_1-w_2)) = 0 \quad \text{a.s}$$

for every $\phi \in \mathcal{C}^b_{\mathcal{F}_0}([-\tau,0],\mathbb{R}^n)$.

Note that $L^1([0, \infty), [0, \infty))$ are the functions $\gamma : [0, \infty) \to [0, \infty)$, firstly integrable i.e. such that $\int_0^\infty \gamma(t)dt < \infty$. Moreover, E denotes the expectation operator with respect to the given probability measure P.

We shall now state and prove our main results.

III. MAIN RESULTS

We consider system (1) where we assume that $\tilde{f}$ can be developed in the form :

$$\tilde{f}(x(t), x(t - \theta), u) = \tilde{f}_0(x(t), x(t - \theta))$$

$$+ u\, \tilde{f}_1(x(t), x(t - \theta)) + u^2\, \tilde{f}_2(x(t), x(t - \theta), u)$$

for all $\theta \in [0, \tau_2]$ and

$$\tilde{g}(x(t), x(t - \theta), u) = \tilde{g}_0(x(t), x(t - \theta))$$

$$+ u\, \tilde{g}_1(x(t), x(t - \theta)) + u^2\, \tilde{g}_2(x(t), x(t - \theta), u)$$

for all $\theta \in [0, \tau_4]$.
$\tilde{f}_0$ and $\tilde{g}_0$ are functions defined by :

$$\tilde{f}_0(x(t), x(t - \theta)) = \tilde{f}(x(t), x(t - \theta), 0)$$

and

$$\tilde{g}_0(x(t), x(t - \theta)) = \tilde{g}(x(t), x(t - \theta), 0).$$

$\tilde{f}_i$ and $\tilde{g}_i$ $(i = 1, 2)$ are suitable smooth functions.

Note that since $\tilde{f}$ and $\tilde{g}$ are smooth, such expansion are possible where

$$\tilde{f}_1(x(t), x(t - \theta)) = \frac{\partial \tilde{f}}{\partial u}(x(t), x(t - \theta), 0)$$

and $\tilde{f}_2(x(t), x(t - \theta), u)$ correspond to the rest of the Taylor expansion. A similar remark stands for $\tilde{g}$ as well.

For further simplification, we will use, for any function $h : \mathbb{R}^n \times \mathbb{R}^n \to \mathbb{R}^n$, the following type of notation indexed by $s \in [0, \tau]$,

$$h_s(x(t)) = h(x(t), x(t - s)), \quad \forall s \in [0, \tau]. \qquad (5)$$

Let us denote by $L_{0,\delta}$ and $L_{1,\delta}$ the second order differential operators defined for all $\Xi \in \mathcal{C}^2(\mathbb{R}^n)$ by :

$$L_{0,\delta}\Xi(\phi(0)) = < f_{\tau_1}(\phi(0)), \nabla\Xi(\phi(0)) >$$

$$+ \int_0^{\tau_2} < \tilde{f}_{0,\theta}(\phi(0)), \nabla\Xi(\phi(0)) > d\theta$$

$$+ \frac{1}{2}\mathrm{Tr}\Big(\big(\int_0^{\tau_4} \tilde{g}_{0,\theta}(\phi(0))d\theta\big)\big(\int_0^{\tau_4} \tilde{g}_{0,\theta}^T(\phi(0))d\theta\big).\nabla^2\Xi(\phi(0))\Big)$$

and

$$L_{1,\delta}\Xi(\phi(0)) = \int_0^{\tau_2} < \tilde{f}_{1,\theta}(\phi(0)), \nabla\Xi(\phi(0)) > d\theta$$

$$+ \frac{1}{2}\mathrm{Tr}\Big([\big(\int_0^{\tau_4} \tilde{g}_{0,\theta}(\phi(0))d\theta\big)\big(\int_0^{\tau_4} \tilde{g}_{1,\theta}^T(\phi(0))d\theta\big)$$

$$+ \big(\int_0^{\tau_4} \tilde{g}_{1,\theta}(\phi(0))d\theta\big)\big(\int_0^{\tau_4} \tilde{g}_{0,\theta}^T(\phi(0))d\theta\big)]\nabla^2\Xi(\phi(0))\Big)$$

where $\delta = \{\tau_1, \tau_2, \tau_3, \tau_4\}$.
We introduce $H_{\Xi,\delta}(\phi(0), u)$ defined by

$$H_{\Xi,\delta}(\phi(0), u) =$$

$$\int_0^{\tau_2} < \tilde{f}_{2,\theta}(\phi(0), u), \nabla\Xi(\phi(0))d\theta >$$

$$+ \frac{1}{2}\mathrm{Tr}\Big(\{\big(g_{\tau_3}(\phi(0)) + \int_0^{\tau_4} \tilde{g}_{0,\theta}(\phi(0))d\theta\big).$$

$$\cdot \big(\int_0^{\tau_4} \tilde{g}_{2,\theta}^T(\phi(0), u)d\theta\big)$$

$$+ \big(\int_0^{\tau_4} \tilde{g}_{2,\theta}(\phi(0), u)d\theta\big).\big(g_{\tau_3}^T(\phi(0)) + \int_0^{\tau_4} \tilde{g}_{0,\theta}^T(\phi(0))d\theta\big)$$

$$+ u\big(\int_0^{\tau_4} \tilde{g}_{1,\theta}(\phi(0))d\theta\big)\big(\int_0^{\tau_4} \tilde{g}_{2,\theta}^T(\phi(0), u)d\theta\big)$$

$$+ u\big(\int_0^{\tau_4} \tilde{g}_{2,\theta}(\phi(0), u)d\theta\big)\big(\int_0^{\tau_4} \tilde{g}_{1,\theta}^T(\phi(0))d\theta\big)$$

$$+ \big(\int_0^{\tau_4} \tilde{g}_{1,\theta}(\phi(0))d\theta\big)\big(\int_0^{\tau_4} \tilde{g}_{1,\theta}^T(\phi(0))d\theta\big)$$

$$+ u^2\big(\int_0^{\tau_4} \tilde{g}_{2,\theta}(\phi(0), u)d\theta\big)\big(\int_0^{\tau_4} \tilde{g}_{2,\theta}^T(\phi(0), u)d\theta\big)$$

$$\}\nabla^2\Xi(\phi(0))\Big).$$

$$(6)$$

where we use the notation (5) and set

$$\tilde{f}_{2,\theta}(\phi(0), u) = \tilde{f}_2(\phi(0), \phi(\theta), u)$$

and

$$\tilde{g}_{2,\theta}(\phi(0), u) = \tilde{g}_2(\phi(0), \phi(\theta), u),$$

in order to get more compact expressions.

We suppose that there exists a Lyapunov function $V \in \mathcal{C}^2(\mathbb{R}^n, [0, \infty))$ and functions $\eta \in \mathcal{W}([-\tau, 0], [0, \infty))$, α and $\alpha_d \in \mathcal{C}(\mathbb{R}^n, [0, \infty))$ such that

$$L_{0,\delta}V(\phi(0)) \leq -\alpha(\phi(0)) + \int_{-\tau}^0 \eta(\theta)\alpha_d(\phi(\theta))d\theta \qquad (7)$$

where

$$\alpha_d(x) \leq \alpha(x) \quad \forall x \in \mathbb{R}^n.$$

Let us denote by M the set :

$$\mathcal{M} = Ker(\alpha - \alpha_d) \cap Ker(L_{1,\delta}V) \qquad (8)$$

We then have the following result :

THEOREM 2: Let

$$u(\phi(0)) = -\psi(\phi(0)) \, L_{1,\delta}V(\phi(0)) \qquad (9)$$

where $\psi \in \mathcal{C}^\infty(R^n;]0, \infty))$ is a function satisfying :

$$\psi(\phi(0)) \leq \frac{1}{\sup\limits_{|u| \leq 1} |H_{\Xi,\delta}(\phi(0), u)|^2 + |L_{1,\delta}V(\phi(0))|^2 + 2} . \qquad (10)$$

If the set $\mathcal{M}$ defined by (8) is reduced to the origin, then the trivial solution of the system (1) with (9) is almost surely asymptotically stable.

Proof :

With the control law (9), the closed-loop system is of the form :

$$dx(t) = f(x(t), x(t - \tau_1)) \, dt$$

$$+ \left(\int_0^{\tau_2} \tilde{f}_0(x(t), x(t - \theta)) \, d\theta \right) dt$$

$$+ u \left(\int_0^{\tau_2} \tilde{f}_1(x(t), x(t - \theta)) \, d\theta \right) dt$$

$$+ u^2 \left(\int_0^{\tau_2} \tilde{f}_2(x(t), x(t - \theta), u) \, d\theta \right) dt$$

$$+ g(x(t), x(t - \tau_3)) \, d\xi(t) \qquad (11)$$

$$+ \left(\int_0^{\tau_4} \tilde{g}_0(x(t), x(t - \theta)) \, d\theta \right) d\xi(t)$$

$$+ u \left(\int_0^{\tau_4} \tilde{g}_1(x(t), x(t - \theta)) \, d\theta \right) d\xi(t)$$

$$+ u^2 \left(\int_0^{\tau_4} \tilde{g}_2(x(t), x(t - \theta), u) \, d\theta \right) d\xi(t).$$

The infinitesimal generator associated to this system (11) is given by :

$$\mathcal{L}_\delta V(\phi(0)) = < f_{\tau_1}(\phi(0)), \nabla V(\phi(0)) >$$

$$+ \int_0^{\tau_2} < \tilde{f}_{0,\theta}(\phi(0)), \nabla V(\phi(0)) > d\theta$$

$$+ u \int_0^{\tau_2} < \tilde{f}_{1,\theta}(\phi(0)), \nabla V(\phi(0)) > d\theta$$

$$+ u^2 \int_0^{\tau_2} < \tilde{f}_2(\phi(0), \phi(\theta), u), \nabla V(\phi(0)) > d\theta$$

$$+ \frac{1}{2} \mathrm{Tr} \Big([g_{\tau_3}(\phi(0)) + \int_0^{\tau_4} \tilde{g}_{0,\theta}(\phi(0)) d\theta$$

$$+ u \int_0^{\tau_4} \tilde{g}_{1,\theta}(\phi(0)) d\theta \qquad (12)$$

$$+ u^2 \int_0^{\tau_4} \tilde{g}_2(\phi(0), \phi(\theta), u) d\theta].$$

$$[g_{\tau_3}(\phi(0)) + \int_0^{\tau_4} \tilde{g}_{0,\theta}(\phi(0)) d\theta$$

$$+ u \int_0^{\tau_4} \tilde{g}_{1,\theta}(\phi(0)) d\theta$$

$$+ u^2 \int_0^{\tau_4} \tilde{g}_2(\phi(0), \phi(\theta), u) d\theta]^T \, \nabla^2 V(\phi(0)) \Big)$$

Using (9)(10) and the definition of H given by (6), we get

$$\mathcal{L}_\delta V(\phi(0)) = L_{0,\delta}V(\phi(0)) + u(\phi(0)) \, L_{1,\delta}V(\phi(0))$$

$$+ u^2(\phi(0)) \, H_{\Xi,\delta}(\phi(0), u) \qquad (13)$$

It is easy to check that u satisfies the following inequalities

$$u(\phi(0)) L_{1,\delta}V(\phi(0)) \leq 0,$$

$$|u(\phi(0))| \leq \frac{1}{2} \qquad (14)$$

$$\text{and } |\psi(\phi(0)) \, H_{\Xi,\delta}(\phi(0), u)| \leq \frac{1}{2},$$

Then

$$\mathcal{L}_\delta V(\phi(0)) = L_{0,\delta}V(\phi(0)) - \psi(\phi(0))(L_{1,\delta}V(\phi(0)))^2$$

$$+ \psi^2(\phi(0))(L_{1,\delta}V(\phi(0)))^2.$$

$$H_{\Xi,\delta}(\phi(0), u(\phi(0)))(L_{1,\delta}V(\phi(0)))) \qquad (15)$$

By definition of ψ with (10) and (14),

$$\left[1 - \psi(\phi(0)) \, H_{\Xi,\delta}(\phi(0), u) \right] \geq \frac{1}{2} > 0$$

and

$$\mathcal{L}_\delta V(\phi(0)){=}L_{0,\delta}V(\phi(0))- \psi(\phi(0))\big(L_{1,\delta}V(\phi(0))\big)^2$$

$$.\big[\,1 - \psi(\phi(0))\,H_{\Xi,\delta}(\phi(0),u)(\phi(0))\,\big]$$

$$\le L_{0,\delta}V(\phi(0))-\frac{\Psi(\phi(0))}{2}\big(L_{1,\delta}V(\phi(0))\big)^2$$

$$\le 0.$$

(16)

Therefore, we can deduce that the system (1)(9) is stable in probability.

By applying the stochastic version of LaSalle Theorem (cf. [16], [17]), we can deduce that the solution $x(t)$ tend to the set asymptotically $\mathcal{N} = \{x \in \ker(\mathcal{L}_\delta V)\}$ with probability one.

If x is an element of $\mathcal{N}$ then with (16), and the fact that

$$\big[\,1 - \psi(\phi(0))\,H_{\Xi,\delta}(\phi(0),u)(\phi(0))\,\big] > 0$$

we can deduce that

$$L_{0,\delta}V(x) = 0 \quad \text{and} \quad L_{1,\delta}V(x) = 0.$$

From the condition (7) on $L_{0,\delta}V(x)$, it follows that :

$$\alpha(x) - \alpha_d(x) = 0 \quad \text{and} \quad L_{1,\delta}V(x) = 0.$$

Therefore, x is an element of $\mathcal{M}$. Since $\mathcal{M} = \{0\}$, the almost sure attractivity of the origin is proved. Consequently, the closed-loop system is almost surely asymptotically stable. This completes the proof of Theorem 2.

REMARK 1: For sake of simplicity our main result is given for $u \in \mathbb{R}$. A similar result for $u \in \mathbb{R}^p\ (p > 1)$ can be analogously established.

REMARK 2: When the system is affine in control, i.e., of the form :

$$dx(t) = \Big(f(x(t),x(t-\tau_1))+\int_0^{\tau_2}\tilde{f}_0(x(t),x(t-\theta))d\theta$$

$$+u\int_0^{\tau_2}\tilde{f}_1(x(t),x(t-\theta))d\theta\Big)dt$$

(17)

$$+\Big(g(x(t),x(t-\tau_3)) + \int_0^{\tau_4}\tilde{g}_0(x(t),x(t-\theta))d\theta$$

$$+u\int_0^{\tau_4}\tilde{g}_1(x(t),x(t-\theta))d\theta\Big)d\xi(t)$$

we have the following result :

COROLLARY 1: Suppose there exists a Lyapunov function $V \in \mathcal{C}^2\,(\mathbb{R}^n, [0,\infty))$ and functions α and $\alpha_d \in \mathcal{C}(\mathbb{R}^n, [0,\infty))$ such that condition (7) is satisfied. If the set

$$\mathcal{M} = Ker(\alpha - \alpha_d) \cap Ker(L_{1,\delta}V)$$

is reduced to $\{0\}$, then the system (17) is almost surely stabilizable by means of the feedback law

$$u_\delta(x(t)) = -\kappa\,L_{1,\delta}V(x(t)) \tag{18}$$

with $\kappa > 0$.

In order to illustrate the main result developed, an example is proposed.

Example

Consider the following example

$$\begin{cases} dx(t){=}-2x^3(t)dt + ux(t)u\,dt + \dfrac{1}{\tau}\Big(\int_0^\tau x^2(t-\theta)d\theta\Big)d\xi(t) \\ x(t){=}\phi(t),\quad t\in[-\tau,0] \end{cases}$$

(19)

We consider the Lyapunov function $V(x) = x^2$. This function satisfy the condition $\lim\limits_{|x|\to\infty} V(x) = \infty$.

The infinitesimal generator associated to this system is given by

$$\mathcal{L}_\delta V(x(t)) \;=\; \big(-4x^4(t) + 2x^2(t)u\big)$$

$$+\; \frac{1}{\tau^2}\Big(\int_0^\tau x^2(t-\theta)d\theta\Big)^2 d\xi(t)$$

(20)

Here

$$H_{V,\delta}\big(x(t),u\big) = 0$$

$$L_{1,\delta}V(x(t)) = 2x^2(t),$$

and

$$L_{0,\delta}V(x(t)) = -4x^4(t) + \Big(\int_0^\tau x^2(t-\theta)d\theta\Big)^2.$$

Since

$$\frac{1}{\tau^2}\Big(\int_0^\tau x^2(t-\theta)d\theta\Big)^2 \le \frac{1}{\tau}\int_0^\tau x^4(t-\theta)d\theta$$

$$L_{0,\delta}V(x(t)) \le -4x^4(t) + \frac{1}{\tau}\int_0^\tau x^4(t-\theta)d\theta.$$

Note that condition (7) is satisfied with $\eta(\theta) = \dfrac{1}{\tau}, \forall\theta \in [0,\tau]$.

By Theorem 2, the equilibrium position $x \equiv 0$ of the closed-loop system (19) to which we apply the control law of the form

$$u(x(t) \;=\; -\Theta(x(t))L_{1,\delta}V(x(t))$$

$$=\; -2\Theta(x(t))x^2(t).$$

(21)

is almost surely asymptotically stable if the set

$$\mathcal{M} = \text{Ker}(\alpha - \alpha_d) \cap Ker(L_{1,\delta}V)$$

is reduced to $\{0\}$.

We can remark that $\text{Ker}(\alpha-\alpha_d) = \mathbb{R}^n$. Thus, concerning the open-loop system, we can only conclude that it is stable in probability but not asymptotically stable in probability. We can also note that $\mathcal{M} = \{0\}$. Thus, the closed-loop system (19)(21) is almost surely asymptotically stable.

IV. CONCLUSIONS

In this paper, we have considered the problem of feedback stabilization of a class of nonlinear stochastic time-delayed systems. We considered the case where the systems involve both discrete and distributed delays. We have used the Stochastic version of the Invariance Principle of LaSalle for stochastic differential delayed systems in order to tackle this problem. In addition, we have obtained sufficient conditions for guaranteeing the asymptotic stability of the closed-loop system and derived stabilizing state feedback control laws.

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