2011 50th IEEE Conference on Decision and Control and
European Control Conference (CDC-ECC)
Orlando, FL, USA, December 12-15, 2011

Rejection of sinusoids from nonlinearly perturbed uncertain regular linear systems

Vivek Natarajan and Joseph Bentsman, *Member, IEEE*

***Abstract* — An application-motivated system class - nonlinearly perturbed regular linear systems (NPRLS) – is considered, and the response of the latter to periodic inputs is characterized. A recently proposed robust control scheme for tracking bandlimited periodic signals by uncertain exponentially stable regular linear systems (RLS) is then applied to an uncertain system belonging to the NPRLS class, whose linearization is an exponentially stable RLS, to reject internally generated sinusoids from the system output. Assuming the uncertain NPRLS to be unknown, but its gain at the frequency of interest known and bounded away from zero, and using the aforementioned characterization, the stability and disturbance rejection of the resulting topology are shown to be guaranteed for sufficiently small nonlinearity.**

***Index Terms*—Internal model principle, regular linear systems, nonlinear perturbation, periodic response.**

I. INTRODUCTION

INTERNAL model principle [1] has been used to track and reject periodic signals in finite dimensional linear [2]-[6] and nonlinear systems [7]-[9] when the plant model is known. When the plant model is unknown, this principle has been applied using only the transfer function (TF) gains at the frequencies of interest, to track bandlimited periodic signals in case of stable linear finite [10], linear infinite [11]-[13] and nonlinear finite [14] dimensional systems. The control schemes in [13] and [14] are similar and are motivated by a motion distortion problem in steel casting mold oscillation systems that include a servo-beam structure. This application can be characterized as a sinusoidally excited linear infinite dimensional structure with a small finite dimensional nonlinear perturbation causing the generation of unwanted harmonics which excite a structural resonance. The control objective is to eliminate the harmonic at the resonance frequency from the output. In the absence of the nonlinear perturbation, [13] addressed the tracking of bandlimited periodic signals by exponentially stable regular linear systems (RLS), a class which encompasses a large number of infinite-dimensional physical models. In the presence of nonlinear perturbation [14] addresses rejection of unwanted harmonics only in a stable finite dimensional linear system.

The present work considers the class of systems - nonlinearly perturbed exponentially stable regular linear system (NPRLS) – that directly matches the phenomenology of the application of interest, and extends the applicability of the control scheme in [13] to this class of systems. Namely, definition of this system class is given and conditions for closed loop stability and rejection of unwanted harmonics for systems in this class are rigorously ascertained. This is accomplished by first considering the closed loop without the nonlinear perturbation and then analyzing the effect of the perturbation on stability and performance using a detailed characterization of the response of NPRLS to periodic excitation.

II. NOTATION AND BACKGROUND ON RLS AND NPRLS

The paper uses the following notation:

$\mathcal{L}(X,Y)$ - Space of bounded linear operators from X to Y. Let $\mathcal{L}(X,X)=\mathcal{L}(X)$,

$L^2([t,T),X)$ / $L^\infty([t,T),X)$ / $L^2_{loc}([0,\infty),X)$ - Space of square integrable / essentially bounded functions from $[t,T)$ to X with the usual norm / Space of locally square integrable functions from $[0,\infty)$ to X,

$D(A)$ / $\rho(A)$ - Domain / resolvent set of an operator A,

$\|\bullet\|_X$ - Norm in space X,

$\mathbb{R}/\mathbb{C}$ - Space of real/complex numbers,

$\|x\|_{L^2[0,T)}$ - Same as $\|x(t)\|_{L^2([0,T),X)}$, when X is clear,

$\|x\|_{X,\infty}$ - Same as $\|x(t)\|_{L^\infty([0,\infty),X)}$ and

$C_B([0,\infty),X)$ - Banach space of continuous bounded functions from $[0,\infty)$ to X with $\|\bullet\|_{X,\infty}$ as norm.

Regular Linear Systems (RLS). The following definitions and background on RLS can be found in [15]-[17] and references therein and are reproduced below to enhance the clarity of presentation. Let U, X and Y be Hilbert spaces and $\Omega=L^2([0,\infty),U)$ and $\Gamma=L^2([0,\infty),Y)$ be input and output spaces, respectively. An abstract linear system on Ω, X and Γ is a quadruple $\Sigma=(\mathbb{T},\Phi,\mathbb{L},\mathbb{F})$ where $\mathbb{T}=(\mathbb{T}_t)_{t\geq 0}$

V. Natarajan (email: vnatara2@illinois.edu) and J. Bentsman (corresponding author: Ph: 217-244-1076, Fax: 217-244-6534, email: jbentsma@illinois.edu) are with the Department of Mechanical Science and Engineering, University of Illinois at Urbana-Champaign, IL 61801, USA. This work was partly supported by NSF grant CMMI-0900138.

is a C_0-semigroup on X, $\Phi = (\Phi_t)_{t\geq 0}$, $\mathbb{L} = (\mathbb{L}_t)_{t\geq 0}$, $\mathbb{F} = (\mathbb{F}_t)_{t\geq 0}$ are families of bounded linear operators from Ω to X (input to state), X to Γ (initial state to output) and Ω to Γ (input to output), respectively. The operators Φ, $\mathbb{L}$ and $\mathbb{F}$ can be extended naturally to the Frechet space of inputs $\tilde{\Omega} = L^2_{loc}([0,\infty),U)$ and outputs $\tilde{\Gamma} = L^2_{loc}([0,\infty),Y)$, with the extensions denoted by Φ_∞, $\mathbb{L}_\infty$ and $\mathbb{F}_\infty$, respectively.

Let A be the infinitesimal generator of $\mathbb{T}$. The Hilbert spaces X_1 and X_{-1} are defined as follows: X_1 is $D(A)$ with the norm $\|x\|_{X_1} = \|(\beta I - A)x\|_X$, where $\beta \in \rho(A)$ is fixed and X_{-1} is the completion of X with respect to the norm $\|x\|_{X_{-1}} = \left\|(\beta I - A)^{-1} x\right\|_X$. The semigroup $\mathbb{T}$ can be extended to a semigroup on X_{-1} isomorphic to $\mathbb{T}$ and will be denoted by the same symbol. For any abstract linear system, there exists an unique $B \in \mathcal{L}(U, X_{-1})$ called the control operator such that $\Phi_t u = \int_0^t \mathbb{T}_{t-\tau} B u(\tau)\, d\tau$, for all inputs $u \in \tilde{\Omega}$ and all $t \geq 0$. Then, for any initial state x_0 the state x of Σ at time $t \geq 0$ is expressed as

$$x(t) = \mathbb{T}_t x_0 + \Phi_t u = \mathbb{T}_t x_0 + \int_0^t \mathbb{T}_{t-\tau} B u(\tau)\, d\tau. \qquad (1)$$

The state $x : [0,\infty) \to X$, as defined, is a continuous function. In general, $\hat{B} \in \mathcal{L}(U, X_{-1})$ is called a $\mathbb{T}$-admissible control operator if the map $\int_0^t \mathbb{T}_{t-\tau} \hat{B} u(\tau)\, d\tau \in \mathcal{L}(\Omega, X)$ for each $t \geq 0$.

Associated with every abstract linear system is a unique $C \in \mathcal{L}(X_1, Y)$ called the observation operator such that $\forall x \in X_1$, $(\mathbb{L}_\infty x)(t) = C\mathbb{T}_t x$. $\mathbb{L}_\infty$ is completely determined by this equation due to the density of X_1 in X. The Λ-extension of C is defined by

$$C_\Lambda x' = \lim_{\lambda\to\infty} C\lambda(\lambda I - A)^{-1} x'$$

with λ real and for all x' in the domain

$$D(C_\Lambda) = \{x' \in X : \text{the limit above exists}\}.$$

The input-to-output operator of any abstract linear system can be described by a TF which is an operator valued analytic function defined and bounded on some right half complex plane. Let $\boldsymbol{G}$ denote the TF of Σ. $\boldsymbol{G}$ is called regular if the following limit exists $\forall v \in U$,

$$Dv = \lim_{\lambda\to\infty} G(\lambda) v, \ \lambda \text{ real}.$$

Then $D \in \mathcal{L}(U,Y)$ is called the feedthrough operator of $\boldsymbol{G}$. In this case, the state $x(t)$ given in (1) and the output of Σ, $y = \mathbb{L}_\infty x_0 + \mathbb{F}_\infty u$, where x_0 is the initial state and $u \in \tilde{\Omega}$ is the input, satisfy pointwise almost everywhere (a.e.) in time, the equation

$$y(t) = C_\Lambda x(t) + Du(t). \qquad (2)$$

If $\boldsymbol{G}$ is regular, Σ is called RLS. (A,B,C,D) as introduced are the generating operators (GOs) of Σ. A RLS Σ is called exponentially stable if the associated semigroup $\mathbb{T}$ satisfies $\|\mathbb{T}_t\|_{\mathcal{L}(X)} \leq Me^{-at}$, $\forall t \geq 0$, for some $M \geq 1$ and $a > 0$.

In the rest of the paper, an exponentially stable RLS $\Sigma_P = (\mathbb{T}, \Phi, \mathbb{L}, \mathbb{F})$, with GOs (A_P, B_P, C_P, D_P) and TF $G_P(s)$ is considered.

Nonlinearly Perturbed Regular Linear Systems (NPRLS). The NPRLS, $\Sigma_{P,N}$, corresponding to the above RLS Σ_P, is defined as being represented by the differential equation

$$\begin{aligned} \dot{x}_P &= A_P x_P + B_P u + \varepsilon g(x_P), \quad x_P(0) = x_0 \\ y_P &= C_{P,\Lambda} x_P + D_P u \end{aligned} \qquad (3)$$

where $C_{P,\Lambda}$ is the Λ-extension of C_P, $\varepsilon > 0$ is a small constant, g is a nonlinear term, and (3) satisfies the following: :

Assumption 1: $\mathbb{T}$, the C_0-semigroup on X generated by A_P, is exponentially stable and satisfies $\|\mathbb{T}_t\|_{\mathcal{L}(X)} < Me^{-at}$ with $a > 0$ and $M \geq 1$.

Assumption 2: $g(x) : X \to X$ satisfies the Lipschitz condition that, for any bounded domain $D \subset X$, $\exists L(D)$: $\|g(x) - g(y)\|_X < L(D)\|x - y\|_X$, $\forall\ x, y \in D$ and $g(0) = 0$.

Assumption 3: $(A_P, I_X, C_P, 0)$ are the GOs of an RLS $(\mathbb{T}, \Phi^g, \mathbb{L}, \mathbb{F}^g)$ with state X and input and output taking values in X and Y respectively. I_X is the identity operator on X.

Under Assumptions 1-2, Lemma 3 shows that for a given $u \in L^\infty([0,\infty),U)$ and $x_0 \in X$, $\exists \bar{\varepsilon} : \forall \varepsilon < \bar{\varepsilon}$, the state equation in (3) has a unique continuous solution, bounded in X, and satisfying a variation of parameters formula (like (1)). The existence proof is along the lines of the proof in chapter 6 of [18] and is sketched here since, though $B_P \in \mathcal{L}(U, X_{-1})$ it may not belong to $\mathcal{L}(U, X)$. Assumption 3 enables treating the nonlinearity as an input and renders the output equation in (3) meaningful.

III. PROBLEM STATEMENT

Consider an NPRLS satisfying Assumptions 1-3 that tracks the reference sinusoid of frequency ω satisfactorily, but for the presence of an internally generated small magnitude higher harmonic at ω_r, an r-integer multiple of ω.

Objective. *Design a control scheme that eliminates the harmonic at ω_r from the NPRLS output without affecting system stability and tracking performance at frequency ω.*

This problem statement is motivated by an industrial application represented in Fig. 1, where an actuator modeled as a linear finite dimensional system perturbed by a smooth nonlinearity (Section V.A [14]) is coupled to an infinite dimensional beam. In this application, the input is a sinusoid of frequency ω and the output contains small amplitude higher harmonics, usually negligible, due to the nonlinearity. But, though small, the sinusoid at ω_r in the output is particularly undesirable and must be eliminated since ω_r is a resonance frequency of the beam. This was accomplished in [14] assuming a finite dimensional plant. Given the infinite dimensional dynamics of the beam, in the present work the plant model is more appropriately assumed to be a NPRLS $\Sigma_{P,N}$ with $U = Y = \mathbb{R}$.

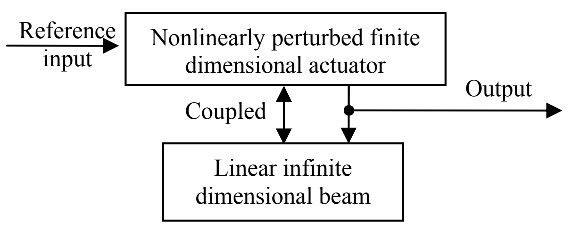

Fig. 1. Schematic representation of motivating application

IV. RESPONSE OF NPRLS TO PERIODIC EXCITATION

This section contains a series of lemmas that characterize the response of NPRLS to periodic excitation. Consider Σ_P and its GOs as defined in Section II. Since $\mathbb{T}$ is exponentially stable (Assumption 1), $\Phi_t : \Omega \to X$ is continuous for all t and $\|\Phi_t(u)\|_X \le \tilde{N}\|u\|_{L^2[0,t]}$, $\forall u \in \tilde{\Omega}$ [19]. For bounded input u, a bound on $\Phi_t(u)$ depending on $\|u\|_{U,\infty}$ is desired and can be given by the following lemma.

Lemma 1: If $u \in L^\infty([0,\infty),U)$, then $\|\Phi_t u\|_X \le \bar{N}\|u\|_{U,\infty}$ for some fixed $\bar{N} > 0$ independent of t.

Proof: Consider the exponentially stable C_0-semigroup $\tilde{\mathbb{T}}_t = \mathbb{T}_t e^{bt}$ generated by $A_P + bI$ with $0 < b < a$. Fix any $t \ge 0$ and $v \in \Omega$. Then,

$$\tilde{\Phi}_t v = \int_0^t \tilde{\mathbb{T}}_{t-\tau} B_P v(\tau)\, d\tau = \int_0^t \mathbb{T}_{t-\tau} B_P \left(e^{b(t-\tau)} v(\tau)\right) d\tau = \int_0^t \mathbb{T}_{t-\tau} B_P \tilde{v}(\tau)\, d\tau = \Phi_t \tilde{v},$$

where $\tilde{v}(\tau) = e^{b(t-\tau)} v(\tau) \in L^2([0,t],U)$. $\tilde{\Phi}_t \in \mathcal{L}(\Omega, X)$, since $\|\tilde{\Phi}_t v\|_X = \|\Phi_t \tilde{v}\|_X \le \tilde{N}\|\tilde{v}\|_{L^2[0,t]} \le \tilde{N} e^{bt}\|v\|_{L^2[0,t]}$ for any $v \in \Omega$. Since this holds for all t, B_P is a $\tilde{\mathbb{T}}$-admissible control operator and $(\tilde{\mathbb{T}}, \tilde{\Phi})$ is an abstract linear control system [19] where $\tilde{\Phi} = (\tilde{\Phi}_t)_{t\ge 0}$. Moreover, since $\tilde{\mathbb{T}}$ is exponentially stable, $\tilde{\Phi}$ is uniformly bounded with $\|\tilde{\Phi}_t\|_{\mathcal{L}(\Omega,X)} \le \tilde{\tilde{N}}$. Therefore,

$$\begin{aligned}\|\Phi_t u\|_X &= \left\|\tilde{\Phi}_t \left(e^{-b(t-\tau)} u(\tau)\right)\right\|_X \le \tilde{\tilde{N}} \left\|e^{-b(t-\tau)} u\right\|_{L^2[0,t]} \\ &\le \tilde{\tilde{N}} \|u\|_{U,\infty} \left\|e^{-b(t-\tau)}\right\|_{L^2[0,t]} \le \bar{N}\|u\|_{U,\infty}\end{aligned}$$

and the lemma is proven. □

Lemma 2: Let $u : [0,\infty) \to U$ and $v : [0,\infty) \to X$ be continuous T-periodic functions. Then under Assumption 1,

$$w_u(t) = \int_0^t \mathbb{T}_{t-\tau} B_P u(\tau)\, d\tau \qquad \text{and} \qquad w_v(t) = \int_0^t \mathbb{T}_{t-\tau} v(\tau)\, d\tau$$

converge in $\|\bullet\|_X$, asymptotically, to continuous T-periodic functions. The rate of convergence depends on $\|u\|_{U,\infty}$ and $\|v\|_{X,\infty}$ but not on the particular functions.

Proof: Since $B_P \in \mathcal{L}(U, X_{-1})$, the integration for w_u is in X_{-1}. That $w_u(t) \in X$, $\forall t$, and the continuity of w_u in X follows from the properties of RLS. Using Lemma 1 and a change of variables, it follows that for $n > m$, $p = n - m$ and $\forall s \in [0,T]$

$$\begin{aligned}&\left\| \int_0^{nT+s} \mathbb{T}_{nT+s-\tau} B_P u(\tau)\, d\tau - \int_0^{mT+s} \mathbb{T}_{mT+s-\tau} B_P u(\tau)\, d\tau \right\|_X \\ &= \left\| \int_0^{pT} \mathbb{T}_{nT+s-\tau} B_P u(\tau)\, d\tau \right\|_X = \left\| \mathbb{T}_{mT+s} \int_0^{pT} \mathbb{T}_{pT-\tau} B_P u(\tau)\, d\tau \right\|_X . \\ &= \left\| \mathbb{T}_{mT+s} \Phi_{pT} u \right\|_X \le M e^{-aTm} \bar{N} \|u\|_{U,\infty} .\end{aligned}$$

Hence the sequence of continuous functions f_n defined as $f_n(s) = w_u(nT+s)$, $\forall s \in [0,T]$, is Cauchy in the supremum norm. Hence $f_n(s)$ converges to a function $f(s)$ uniformly $\forall s \in [0,T]$, which implies that $w_u(t)$ converges to a continuous *T*-periodic function asymptotically. A similar argument establishes the result for w_v. The claim about the rate of convergence follows from the proof. □

Lemma 3, concludes that the state of the periodically excited NPRLS in (3) is periodic, asymptotically, with the same period.

Lemma 3: Consider the RLS Σ_P, the corresponding NPRLS $\Sigma_{P,N}$ given in (3) and the associated nonlinear integral map $q = \mathbb{I}(p)$ defined $\forall t \in [0,\infty)$ as:

$$q(t) = p(t) - \mathbb{T}_t x_0 - \int_0^t \mathbb{T}_{t-\tau} B_P u(\tau)\, d\tau - \varepsilon \int_0^t \mathbb{T}_{t-\tau} g\big(p(\tau)\big)\, d\tau .$$

Let $\|x_0\|_X < C_0$ and $u:[0,\infty) \to U$ be a continuous T-periodic function. Let Assumptions 1-2 hold. Then, since Σ_P is exponentially stable, $\|\Phi_t u\|_X \le \tilde{N} \|u\|_{L^2[0,t)}$ and from Lemma 1 $\|\Phi_t u\|_X \le \bar{N} \|u\|_{U,\infty}$, for some $\tilde{N}, \bar{N} > 0$ and $\forall t$. Hence the map $\mathbb{I}$ is well defined from $C_B([0,\infty), X)$ to itself. Assume that ε is small so that for some $0 < \alpha < 1$ and $D_0 = \left\{ x \in X : \|x\|_X \le \left(C_0 + \left(MC_0 + \bar{N}\|u\|_\infty\right)\right) / (1-\alpha) = d_0 \right\}$, $(\varepsilon M L_0)/a < \alpha$. Here L_0 is the Lipschitz constant of $g(x)$ for the domain D_0. Then, $\exists x_P, \hat{x}_P \in C_B([0,\infty), X)$ with $I(x_P) = 0$ and $\hat{x}_P$ a T-periodic function such that for any given $\lambda > 0$, $\exists\ \hat{t} > 0$: $\left\| x_P[\hat{t},\infty) - \hat{x}_P[\hat{t},\infty) \right\|_{X,\infty} < \lambda$.

Proof: The main ideas of the proof are sketched below. Step 1 is to establish the existence of a unique solution to $\mathbb{I}(p) = 0$, which would then be the variation of parameters solution to (3). This is done by first showing that if the initial condition satisfies $\|x_0\|_X < d_0$, a unique solution $x_P(t)$ exists for $t \in [0,\sigma]$. Next, arguing that $\|x_P(t)\|_X < d_0$, $\forall t \in [0,\sigma]$, enables extending the solution to $[\sigma, 2\sigma]$. Repeating this argument shows that $x_P(t)$ exists $\forall t \in [0,\infty)$. Step 2 proves asymptotic convergence of $x_P(t)$ to a *T*-periodic function, $\hat{x}_P(t)$. Let $\int_0^t \mathbb{T}_{t-\tau} B_P u(\tau)\, d\tau$ and $\int_0^t \mathbb{T}_{t-\tau} \varepsilon g\big(v_0(\tau)\big)\, d\tau$ converge asymptotically to T-periodic functions $v_0(t)$ and $v_1(t)$, respectively. For $k > 1$, let v_k be the T-periodic function to which $\int_0^t \mathbb{T}_{t-\tau} \varepsilon g\big(v_0(\tau) + v_{k-1}(\tau)\big)\, d\tau$ converges asymptotically. The sequence of T-periodic functions $\{v_0 + v_k\}$ is shown to be Cauchy in $C_B([0,\infty), X)$ and its limit $\hat{x}_P$ is the function to which $x_P(t)$ converges asymptotically. □

Remark 1: With $x_P(t)$ and $\hat{x}_P(t)$ as in Lemma 3, it can be shown that for a fixed $0 < \delta < 1$, a $\bar{t}$ can be found such that $\forall t > n^2 \bar{t}$, $\|\hat{x}_P(t) - x_P(t)\|_X < \delta^n$ for every integer $n > 0$. Hence $\hat{x}_P(t) - x_P(t) \in L^2((0,\infty], X)$.

If $C_P \in \mathcal{L}(X,Y)$ in (3), it follows from Lemma 3 that $y(t)$ is continuous and tends to a *T*-periodic function asymptotically. But when $C_P \in \mathcal{L}(X_1,Y)$, as is the case for a general RLS, Lemma 4 clarifies the behavior of $y(t)$.

Lemma 4: Let the assumptions in Lemma 3 hold. Then the output $y_P(t)$ of (3) can be written as $y_{l,P}(t) + y_{nl,P}(t) + y_{0,P}(t)$ where $y_{l,P}$ and $y_{l,P} + y_{nl,P}$ are the T-periodic components of the output in the absence and presence of the nonlinearity, respectively and $y_{0,P} \in L^2((0,\infty], Y)$.

Proof: From Remark 1 and Assumptions 1-3, since $x_P(t)$ is known and bounded, the NPRLS (3) can be regarded as a RLS with 2 inputs, u and $\varepsilon g(x_P)$. Let n satisfy $\|\mathbb{T}(nT)\|_{\mathcal{L}(X)} < 1$. Consider the system

$$\begin{aligned} &\dot{\bar{x}}_P = A_P \bar{x}_P + B_P u + \varepsilon g(\hat{x}_P), \quad \bar{y}_P = C_{P,\Lambda} \bar{x}_P + D_P u, \\ &\bar{x}_P(0) = \left(I - \mathbb{T}(nT)\right)^{-1} \left(\Phi_{nT} u + \Phi_{nT}^g \varepsilon g(\hat{x}_P)\right). \end{aligned} \tag{4}$$

Since the inputs to this system are *T*-periodic, with the chosen initial conditions (as in finite dimensions), $\bar{x}_P$ is a *T*-periodic continuous function. From the output equation, which is valid a.e. it follows that $\bar{y}_P$ is periodic a.e. and by linearity can be written as $\bar{y}_P = y_{l,P} + y_{nl,P}$ to distinguish the contribution of the 2 inputs: u and $\varepsilon g(\hat{x}_P)$. Equations (3) and (4) differ in the initial conditions and the second input. But the difference in the second input is in $L^2((0,\infty], X)$ since $\|g(\hat{x}_P) - g(x_P)\|_X \le L_0 \|\hat{x}_P - x_P\|_X$. (use Remark 1, $\|\hat{x}_P\|_{X,\infty} \le d_0$, $\|x_P\|_{X,\infty} \le d_0$). From the properties of exponentially stable RLS the difference in the outputs of (3) and (4), $y_{0,P} = y_P - \bar{y}_P$, must satisfy $y_{0,P} \in L^2((0,\infty], Y)$. Clearly, the L^2-norm over one period of $\varepsilon g(\hat{x}_P)$ and therefore of $y_{nl,P}$ is proportional to ε. □

V. Controller Design

In Theorem 1, the controller designed in [13] for tracking of finite number of sinusoids by RLS is applied to the NPRLS in (3) to address the problem statement of Section III. In the present section, the input and outputs take values in $\mathbb{R}$, i.e. $U = Y = \mathbb{R}$. For all TFs $G(s)$ and $\forall \tau \in \mathbb{R}$, it is assumed that $G(-j\tau)$ is the complex conjugate of $G(j\tau)$. In Lemmas 3 and 4, if u is a sinusoid of period *T*, then using the theory of TFs for RLS [17], $y_{l,P}$, the *T*-periodic component of the output in the absence of the nonlinearity, can be shown to be a sinusoid.

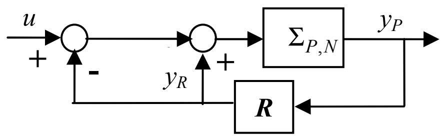

Fig. 2. Unaugmented system

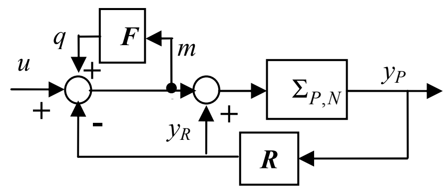

Fig. 3. Augmented system

*Theorem 1: Consider the RLS Σ_P and the NPRLS $\Sigma_{P,N}$ ((3)) as in Section II, satisfying Assumptions 1-3. The unaugmented system shown in Fig. 2 is a cascade interconnection (the feedbacks cancel out) of $\Sigma_{P,N}$ and a stable s-dimensional SISO filter **R**:*

$$\dot{x}_R = A_R x_R + B_R u_R, \qquad y_R = C_R x_R .$$

Also assume the following:

B1. The input u to $\Sigma_{P,N}$ is a sinusoid of frequency $\omega = 2\pi/T$ and amplitude C_u. Let the initial condition x_0 satisfy $\|x_0\|_X < C_0$. By Lemma 4, for sufficiently small ε, the output y_P of (3) is a sum of a L^2-function and a T-periodic function.

B2. Consistent with the problem statement, assume that $\Sigma_{P,N}$ tracks the input sinusoid of frequency ω satisfactorily, but for the output y_P containing small amplitude higher harmonics of ω, including ω_r, due to nonlinear effects.

B3. If the input u to $\Sigma_{P,N}$ is u_{ω_r}, a small amplitude sinusoid of frequency ω_r, the periodic component of the output y_P contains a sinusoid at ω_r (Lemma 4); let it be $S_{y_P,\omega_r}(t)$. Let the complex gain from u_{ω_r} to S_{y_P,ω_r}, $G_{P,N}(j\omega_r)$, be bounded away from zero.

*Choose **R** so that its strictly proper TF, G_R, satisfies $\left|1 - G_{P,N}(j\omega_r) G_R(j\omega_r)\right| = 0$. In the augmented system (Fig. 3), **F** is the linear stable SISO system with TF*

$$G_F(s) = \frac{2\zeta\omega_r^2}{s^2 + 2\zeta\omega_r s + \omega_r^2} \frac{s - \omega_r}{s + \omega_r} \tag{5}$$

*with $0 < \zeta \ll 1$, a parameter to be chosen. Let the GOs of **F** be $\left(A_F(\zeta), \zeta B_F, C_F, 0\right)$ where $A_F(\zeta)$ is a linear function of ζ, and B_F and C_F are independent of ζ. Then, for small ε, $\exists \zeta^* : \forall \zeta \in (0, \zeta^*]$ the augmented system in Fig. 3 with $\Sigma_{P,N}$ replaced by Σ_P is an exponentially stable RLS. Assuming that $(A_P, B_P, I, 0)$ are the GOs of an RLS, $\exists \varepsilon^*$ such that if $\varepsilon < \varepsilon^*$ the augmented system in Fig. 3, now with $\Sigma_{P,N}$, is stable for each $\zeta \in (0, \zeta^*]$ and tracks the input sinusoid of frequency ω akin to the unaugmented system and possibly contains small magnitude higher harmonics of ω generated by the nonlinear perturbation, but the harmonic at frequency ω_r is eliminated from its output.*

Proof: Let $G_P(j\omega_r)$ be the TF gain of Σ_P at ω_r. From Lemma 4 $\left|G_P(j\omega_r) - G_{P,N}(j\omega_r)\right|$ is small for small ε, and hence $\left|1 - G_P(j\omega_r) G_R(j\omega_r)\right| < 1$. From Theorem 1 in [13] (**R** is the same as 1-**Q** in [13]; The additional assumption in this paper that **R** be strictly proper does not alter the results in [13]) it follows that $\exists \zeta^* : \forall \zeta \in (0, \zeta^*]$, the augmented system in Fig. 3 with $\Sigma_{P,N}$ replaced by Σ_P, denoted as $\Sigma_P^c(\zeta)$, is an exponentially stable RLS; the superscript c indicates 'closed loop'. The state space and GOs for $\Sigma_P^c(\zeta)$ are $X^c = X \times \mathbb{R}^s \times \mathbb{R}^3$ and $\left(A_P^c(\zeta), B_P^c(\zeta), C_P^c, D_P^c\right)$ respectively, with input u and output y_P where:

$$A_P^c = \begin{bmatrix} A_P & 0 & B_P C_F \\ B_R C_{P,\Lambda} & A_R & B_R D_P C_F \\ 0 & -\zeta B_F C_R & A_F(\zeta) + \zeta B_F C_F \end{bmatrix}, B_P^c = \begin{bmatrix} B_P \\ B_R D_P \\ \zeta B_F \end{bmatrix}$$

$$C_P^c = \begin{bmatrix} C_{P,\Lambda} & 0 & D_P C_F \end{bmatrix}, \quad D_P^c = [D_P].$$

Let the exponentially stable semigroup generated by A_P^c be $\mathbb{T}^c(\zeta)$. Denote the augmented system in Fig. 3 (now, with $\Sigma_{P,N}$) by $\Sigma_{P,N}^c$. Its differential equation representation is:

$$\begin{aligned} \dot{x} &= A_P^c x + B_P^c u + \varepsilon g^c(x), \quad x(0) = \hat{x}_0 \\ y_P^c &= C_{P,\Lambda}^c x + D_P^c u, \end{aligned} \tag{6}$$

where $x = [x_P, x_R, x_F]^T$, $\hat{x}_0 = [x_0, 0, 0]^T$, $C_{P,\Lambda}^c$ is the λ-extension of C_P^c and $g^c(x) = \left[g(x_P), 0, 0\right]^T$. Using Assumption 3, it follows from [16] that $\left(A_P^c(\zeta), [I_X, 0, 0]^T, C_P^c, 0\right)$ are GOs of a RLS. From Lemmas 3 and 4, for each $\zeta \in (0, \zeta^*]$, $\exists \varepsilon(\zeta) : \forall \varepsilon < \varepsilon(\zeta)$, the NPRLS $\Sigma_{P,N}^c$ is stable and has a unique output that can be written as $y_P^c = y_{l,P}^c + y_{nl,P}^c + y_{0,P}^c$ where $y_{l,P}^c$, a sinusoid of frequency ω, and $y_{l,P}^c + y_{nl,P}^c$, a *T*-periodic function, are periodic components of the output of $\Sigma_{P,N}^c$ in the absence and presence of the nonlinearity, respectively and $y_{0,P}^c \in L^2([0,\infty), \mathbb{R})$. The L^2-norm over a period for $y_{nl,P}^c$

is small (proportional to ε). Comparing the TFs, from u to y_P, of $\Sigma_P(\zeta)$ and $\Sigma_P^c(\zeta)$ and noting that the gain of $\boldsymbol{F}$ at all frequencies away from ω_r (like ω) is small (proportional to ζ), it can be shown that $y_{l,P}^c - y_{l,P}$ is small, implying that the augmented system tracks the input sinusoid of frequency ω akin to the unaugmented system. The loop containing $\boldsymbol{F}$ in Fig. 3 can be replaced by 1/(1-$\boldsymbol{F}$), which has poles at $\pm j\omega_r$, with output $m = u - C_R x_R + C_F x_F$ and input $u - y_R$. Hence the presence of the harmonic at ω_r in $y_{nl,P}^c$ and so in y_R would lead to m becoming unbounded. Since this contradicts the fact that the states of the stable NPRLS $\Sigma_{P,N}^c$ are bounded, $y_{nl,P}^c$ must have no harmonic at ω_r.

In Lemma 3, the characterization of stability and periodic response depended on the nonlinearity (ε and L) and the growth bounds on the semigroup (M and a). Let $M(\zeta) = 2\sup_{t\ge 0}\left\|\mathbb{T}_t^c(\zeta)\right\|_{\mathcal{L}(X^c)}$ and $a(\zeta)$ the largest positive number such that $\left\|\mathbb{T}_t^c(\zeta)\right\|_{\mathcal{L}(X^c)} \le M(\zeta)e^{-a(\zeta)t}$. When $\zeta \ne 0$, one can use the perturbation theory of C_0-semigroups [18] to show that the growth bounds for $\mathbb{T}_t^c(\zeta)$ change continuously in the interval $(0, \zeta^*]$. Consequently, there exists a continuous function of ζ, $\varepsilon_m(\zeta) > 0$, such that for each $\zeta \in (0, \zeta^*]$ if $\varepsilon < \varepsilon_m(\zeta)$, then the augmented system is stable. But at $\zeta = 0$, the perturbation theory cannot be used. Instead a transfer function based approach can be used to show that in a neighborhood of $\zeta = 0$, $\varepsilon_m(\zeta) \ge \bar{\varepsilon} > 0$. Hence $\exists \varepsilon^* > 0$ such that if $\varepsilon < \varepsilon^*$, then for each $\zeta \in (0, \zeta^*]$ the augmented system in Fig. 3 (with $\Sigma_{P,N}$) is stable. □

VI. CONCLUSION

Nonlinear perturbations are considered in the framework of regular linear systems (RLS) and the stability and response of the resulting NPRLS to periodic excitation are characterized. On this basis, analysis of the stability and disturbance rejection performance of a control scheme developed for RLS when applied to a NPRLS is carried out. It is shown that for sufficiently small perturbations, the feedback loop is stable and eliminates the internally generated harmonic, while preserving the plant response at the input frequency.

REFERENCES

[1] B. A. Francis and W. M. Wonham, "The internal model principle for linear multivariable regulators," *Appl. Math. Optim.*, vol. 2, no. 2, pp. 170-194, 1975.

[2] S. Hara, Y. Yamamoto, T. Omata and M. Nakano, "Repetitive control system: A new type servo system for periodic exogenous signals," *IEEE Transactions on Automatic control*, Vol. 33, No. 7, pp. 659-668, 1988.

[3] M. Tomizuka, T-C. Tsao, and K-K. Chew, "Analysis and synthesis of discrete-time repetitive controller," *ASME Journal of Dynamic Systems, Measurement, and Control*, Vol. 111, pp. 353-358, 1989.

[4] L. J. Brown and Q. Zhang, "Periodic disturbance cancellation with uncertain frequency," *Automatica*, vol. 40, pp. 631-637, 2004.

[5] T. J. Manayathara, T-C. Tsao, J. Bentsman and D. Ross, "Rejection of unknown periodic load disturbances in continuous steel casting process using learning repetitive control approach", *IEEE Transactions on Control Systems Technology*, vol. 4, no. 3, pp. 259-265, 1996

[6] M. R. Bai and T. Wu, "Simulations of an internal model-based active noise control system for suppressing periodic disturbances," *Journal of Vibration and Acoustics*, vol. 120, pp. 111-116, 1998.

[7] Omata, T., Hara, S., and Nakano, M., 1987, "Nonlinear Repetitive Control with Application to Trajectory Control of Manipulators," Journal of Robotic systems, vol. 4, No. 5, pp. 631-652.

[8] Ghosh, J., and Paden, B., 2000, "Nonlinear Repetitive Control," IEEE Trans. on Automatic control, vol. 45, no. 5, pp. 949-954.

[9] Lucibello, P., 2003, "Comments on "Nonlinear Repetitive Control"," IEEE Trans. on Automatic control, vol. 48, No. 8, pp. 1470-1471.

[10] E. J. Davison, "Multivariable tuning regulators: The feedforward and robust control of a general servomechanism problem," *IEEE Transactions on Automatic control*, Vol. 21, pp. 35-47, 1976.

[11] T. Hamalainen and S. Pohjlainen, "A finite-dimensional robust controller for systems in the CD-algebra," *IEEE Transactions on Automatic control*, Vol. 45, No. 3, pp. 421-431, 2000.

[12] R. Rebarber and G. Weiss, "Internal model based tracking and disturbance rejection for stable well-posed systems," *Automatica*, Vol. 39, pp. 1555-1569, 2003.

[13] V. Natarajan and J. Bentsman, "Robust periodic reference tracking by stable uncertain infinite-dimensional linear systems," *Proceeding of the American Control Conference, San Francisco,* 2011, pp. 1777-1782.

[14] V. Natarajan and J. Bentsman, "Robust rejection of sinusoids in stable nonlinearly perturbed unmodelled linear systems: theory and application to servo," *Proceeding of the American Control Conference, San Francisco,* 2011, pp. 3289-3294.

[15] G. Weiss, "The representation of regular linear systems on Hilbert spaces," *Control and Estimation of Distributed Parameter Systems*, F. Kappel, K. Kunisch, and W. Schappacher, Eds., Basel: Birkhauser Verlag, vol. 91, pp. 401–416, 1989.

[16] G. Weiss and R. Curtain, "Dynamic stabilization of regular linear systems," *IEEE Transactions on Automatic control*, Vol. 42, No. 1, pp. 4-21, 1997.

[17] G. Weiss, "Transfer functions of regular linear systems. Part I: Characterization of regularity," *Transactions of the American Mathematical Society*, Vol. 342, No. 2, pp. 827-854, 1994.

[18] A. Pazy, *Semigroups of linear operators and applications to partial differential equations,* Applied Mathematical Sciences, vol. 44, Springer, 1983.

[19] G. Weiss, "Admissibility of unbounded control operators," *SIAM J. Control and Optimization,* Vol. 27, No. 3, pp. 527-545, 1989.