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# Impulsive solutions, inadmissible initial conditions and pole/zero structure at infinity

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***Abstract*— In this paper we study solutions to linear ordinary constant coefficient differential equations on the half-line and relate impulsive solutions to the pole/zero structure at infinity of an associated polynomial matrix. While this relation has been thoroughly studied for first order systems, and through first order analysis also for higher order systems partially, the use of the 'state map', in particular the shift and cut map, makes it very straightforward to characterize the inadmissible initial condition space and the smooth solution space. This paper contains results about the space of initial conditions that have impulsive solutions and those having smooth solutions, and the relation with the zero structure at infinity. We show, amongst other results, that the rank over the reals of the shift and cut map is precisely the dimension of the space of smooth and impulsive solutions for a linear differential system.**

**Keywords: States, initial conditions, zeros at infinity, impulsive solutions, inadmissible initial conditions**

## I. INTRODUCTION

We study solutions to differential equations on the half-line $\mathbb{R}^+$. It is well-known that for a first order system of equations, existence of impulsive solutions for certain conditions is related to presence of zeros at infinity of a related polynomial matrix. We use the behavioral approach to generalize this result to the case of higher order linear differential equations.

The problem studied in this paper is understood easily through the following example:

$$(\frac{d}{dt}-1)w_1 - \frac{d}{dt}w_2 = 0, \quad (\frac{d}{dt}+4)w_2 = 0.$$

Consider the associated polynomial matrix $R(s) = \begin{bmatrix} s-1 & -s \\ 0 & s+4 \end{bmatrix}$. The dimension of smooth solutions for this system of differential equations is easily seen to be 2 since the degree of the determinant of $R$ is 2. Consider the state map $X(\frac{d}{dt})$ (defined more precisely below in Subsection II-E) which acts on the variable $w$ and gives a variable, called $x$, satisfying the property of 'state': concatenability at $t = 0$ of two trajectories is guaranteed if the state at $t = 0$ of the two trajectories is equal. The state map $X(\frac{d}{dt})$ also has the property that the dimension of the row-span over $\mathbb{R}$, modulo the set of the system of differential equations, equals the dimension of the state space in a minimal input/state/output (I/S/O) representation of the system (see Rapisarda and Willems [11]). If the polynomial matrix $R(s)$ inducing the differential equation is row-reduced, this dimension also equals the number of poles of $R(s)$ at infinity.

Consider the following system of differential equations.

$$w_1 - \frac{d}{dt}w_2 = 0, \quad w_2 = 0. \qquad (1)$$

The polynomial matrix $R(s)$ in this case is $\begin{bmatrix} 1 & -s \\ 0 & 1 \end{bmatrix}$. While the only solution on $\mathbb{R}$ is the zero trajectory (with the solutions sought in either the space of all smooth solutions, or the space of all distributions), this system of differential equation admits nontrivial solutions on the *half-line*, i.e. $\mathbb{R}^+ := [0, \infty)$. One can check that $(w_1, w_2) = (b\delta, 0)$, with $w_2(0^-) = b$, for any $b \in \mathbb{R}$ satisfies the differential equation in the distributional sense.

Thus we have a one dimensional subspace of solutions (impulsive solutions on $\mathbb{R}^+$) though the polynomial matrix $R$ is unimodular; solutions on $\mathbb{R}$ (smooth or distributional) or solutions on $\mathbb{R}^+$ (smooth) for a unimodular matrix is just zero since elementary row operations can bring a unimodular matrix to the identity matrix, thus resulting in $w = 0$ as the differential equation.

In other words, the assumption that 'elementary row operations' do not change the set of solutions, that underlies the statement "one may assume, without loss of generality, that $R$ is row reduced", is questionable now. (See [12] for a different treatment about elementary row operations by which impulsive solutions, if any, are retained by the operations.)

However, we will show in more generality that the state map does give us the initial conditions that result in solutions, both smooth and impulsive.

The paper is organized as follows. The next section contains various preliminaries required for this paper. Section III formulates the problem we study, and Section IV considers the situation of unimodular matrices. This is the situation when there are only impulsive solutions: an elaborate example from Vardulakis [13] is considered here. The case that we have both impulsive and smooth solutions are studied next in Section V. Two circuit examples are studied in Section VI. The paper is concluded in Section VII.

## II. NOTATION AND PRELIMINARIES

This section contains the requried preliminaries and some new definitions for the context of this paper. Readers familiar

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with (some of) these preliminaries can skip to the concerned subsection/section directly: the next subsection contains the key function spaces we will need in this paper. Polynomial matrices, their properties and related definitions play a central role in this paper; this is reviewed in Subsection II-B. The pole/zero structure at infinity of a polynomial matrix is elaborated in Subsection II-C. This is followed with Subsection II-D which contains basics of behavioral theory of dynamical systems. Since the initial conditions are studied with respect to impulsive solutions, Subsections II-E and II-F contain definitions of state maps and inadmissible initial conditions respectively.

## A. Solution spaces

The solution space in which we seek solutions to the differential equations is the focus of this paper. The trajectories $w$ are functions on either $\mathbb{R}$ (the field of real numbers) or on $\mathbb{R}^{+}$ (the half-line $[0,\infty)$) and $w$ takes its values in $\mathbb{R}^q$, a finite dimensional vector space. Since the dimension of $\mathbb{R}^q$ is often clear from the context, we often suppress that. Define

1) $\mathcal{L}^{C}_{\mathbb{R}} = \mathfrak{C}^{\infty}(\mathbb{R},\mathbb{R}^q)$ : the space of smooth functions on $\mathbb{R}$.
2) $\mathcal{L}^{C}_{\mathbb{R}^+} = \mathfrak{C}^{\infty}(\mathbb{R}^+,\mathbb{R}^q)$ : the space of smooth functions on $\mathbb{R}^+$.
3) $\mathcal{L}^{I}_{\mathbb{R}} \subset \mathfrak{D}'(\mathbb{R},\mathbb{R}^q)$ : the space of distributions on $\mathbb{R}$.
4) $\mathcal{L}^{I}_{\mathbb{R}^+} \subset \mathfrak{D}'(\mathbb{R}^+,\mathbb{R}^q)$ : the space of distributions on $\mathbb{R}^+$.

The last two spaces will include distributions with support in 0, and with support at other values of time; this is elaborated later below.

The basic object of interest will be the set of solutions, *the behavior* of a system of linear, constant coefficient, ordinary differential equations

$$R_0 w + R_1 \dot{w} + \cdots + R_N \frac{d^N}{dt^N} w = 0. \tag{2}$$

with $R_i \in \mathbb{R}^{m\times q}$ and where the solution $w$ is sought in pre-specified function space $\mathcal{L}$. The above system of differential equations can be rewritten in the notation $R(\frac{d}{dt})w = 0$, with the polynomial matrix $R(s) := R_0 + sR_1 \cdots + s^N R_N$. Thus $R \in \mathbb{R}^{m\times q}[s]$ helps describe $m$ equations in $q$ unknowns.

The reason for listing four function spaces above is as follows. For the differential equation mentioned in the second example above (see Equation (1)), the four solution spaces have different dimension of solutions. More specifically, when solutions for the differential equation (1) are sought in $\mathcal{L}^{C}_{\mathbb{R}}$, then zero is the only solution. So is the case with $\mathcal{L}^{I}_{\mathbb{R}}$, and with $\mathcal{L}^{C}_{\mathbb{R}^+}$. However, when seeking solutions in $\mathcal{L}^{I}_{\mathbb{R}^+}$, the set of solutions is one-dimensional. This is investigated in the following sections.

To specify $\mathcal{L}^{I}_{\mathbb{R}^+}$, note that for this paper, at fist sight it looks like we only need $\mathcal{L}^{C}_{\mathbb{R}^+}$ together with distributions supported at zero. The solutions to the differential equations turn out to be just 'impulsive-smooth': the impulsive part as $\delta$ and its derivatives, all supported at zero, and the smooth part as analytic solutions. However, in this context, the concept of 'initial condition' $w(0^-)$ requires precise formulation. For this paper, we proceed as follows: define $\mathfrak{C}^{\infty}$ functions on $(-\epsilon,\infty)$ taking their values in $\mathbb{R}^q$, with $\epsilon > 0$. Further, we include the unit step $H$, defined as 1 on $(-\epsilon,0)$ and 0 on $\mathbb{R}^+$, and all distributions supported at zero. In addition to these inclusions, we also include all their integrals, together with all linear combinations over $\mathbb{R}$. The left-hand-side limit at $t=0$ is defined for each of these functions $w(t)$: we call this limit $w(0^-)$; the distributions supported at zero do not play a role in this. The intersection of these smooth function spaces (i.e. $\mathfrak{C}^{\infty}$ on $(-\epsilon,\infty)$ for all values of $\epsilon > 0$) and together with $H$ and its integrals, distributions at 0 and all their linear combinations is what we define as $\mathcal{L}^{I}_{\mathbb{R}^+}$. For the rest of this paper, whenever $w \in \mathcal{L}^{I}_{\mathbb{R}^+}$, we use $w(0^-)$ in this sense. Higher order derivatives $w^{(n)}(0^-)$ are defined in the same way[1].

## B. Polynomial matrices

We consider the rank of a polynomial matrix $R \in \mathbb{R}^{m\times q}[s]$: here the rank is over the ring of polynomials $\mathbb{R}[s]$ (equivalently, $\mathbb{R}(s)$, the field of rationals over $\mathbb{R}[s]$). We denote this rank by $\mathrm{rank}_{\mathbb{R}[s]}(R)$ in order to distinguish it from the rank over $\mathbb{R}$ of a polynomial matrix, say $X \in \mathbb{R}^{m\times q}[s]$, denoted by $\mathrm{rank}_{\mathbb{R}}(X)$ and defined as the maximum number of rows of $X$ that are linearly independent over $\mathbb{R}$.

We often stack matrices (of same column dimension) or vectors over each other: 'col' takes two or more matrices and gives a 'taller' matrix, i.e. $\mathrm{col}(R_1,R_2)$ denotes $[R_1^T \;\; R_2^T]^T$.

For a polynomial matrix $R \in \mathbb{R}^{m\times q}[s]$, we need its row-degrees $d_i$, for $i = \{1,\ldots,m\}$, the highest degree of the $i$-th row of $R$. The leading row coefficient matrix of $R$ is the constant matrix $\Gamma^r(R)$ whose $i$-th row is the coefficient of $s^{d_i}$ in the $i$-th row of $R$.

*Definition 1:* The polynomial matrix $R = \mathrm{col}(r_1,\ldots,r_m) \in \mathbb{R}^{m\times q}[s]$ is called *row proper*[2] if its leading row coefficient matrix $\Gamma^r(R)$ has full row rank.

A matrix $R \in \mathbb{R}^{q\times q}[s]$ is said to be unimodular if $\det(R)$ is a nonzero constant. These are, in a sense, opposite of row-reduced matrices: unimodular matrices cannot be row-reduced, unless the matrix is a constant matrix. This is easier to see using the notion of poles/zeros at infinity, which we pursue in the following section.

## C. Pole zero structure at infinity

We now define the poles and zeros of a matrix at infinity.

For any $\lambda \in \mathbb{C}$ and $P \in \mathbb{R}^{q\times q}(s)$ there exist square rational matrices $U$ and $V$ such that none of $U$ and $V$ have any poles or zeros at $\lambda$, and $UPV = \mathrm{diag}\ ((s-\lambda)^{n_i(\lambda)})$, the integers $n_i(\lambda)$ nondecreasing in $i$. It turns out that the integers $n_i(\lambda)$

[1]There are other ways of specifying this elements of interest, for example each element as a triple: $(f,p,a)$ with $f \in \mathfrak{C}^{\infty}\ (\mathbb{R}^+,\mathbb{R}^q)$, $p$ a distribution supported at zero, and $a$ a vector comprising a list of values of $w$ and a finite number of its derivatives at zero: we thank an anonymous reviewer for pointing this. Also see [7] and [6] for other alternative methods of dealing with the 'initial value' problem with distributional solutions.

[2]The term row-reduced, also often used in the literature, means the same.

depend only on $P$ and not on the $U$ and $V$ matrices. If $n_1 < 0$ we say that $P$ has (one or more) poles at $\lambda$, the negative $n_i(\lambda)$'s are called the structural pole indices at $\lambda$. If $n_q > 0$ we say that $P$ has (one or more) zeros at $\lambda$, and the positive $n_i(\lambda)$'s are called the structural zero indices at $\lambda$. The zeros/poles and their structural indices of $P$ at infinity are defined as those of $Q(s)$ at $s = 0$ with $Q(s) := P(\lambda)$ and $\lambda = 1/s$.

The total number of poles and zeros (counted with multiplicity) of $P$ at any $\lambda \in \mathbb{C} \cup \infty$ are respectively denoted by $z_P(\lambda)$ and $p_P(\lambda)$ and defined by

$$z_P(\lambda) := \sum_{n_i>0} n_i(\lambda) \text{ and } p_P(\lambda) = -\sum_{n_i<0} n_i(\lambda) \quad .$$

A more direct count of the zeros and poles at infinity can be obtained by counting the valuations at $\infty$ for a rational matrix as elaborated in Kailath [8]. For a rational $p(s) \in \mathbb{R}(s)$, with $p = n/d$ where $n$ and $d$ are polynomials, define $\nu^\infty(p) :=$ degree $d -$ degree $n$. This is used for defining the valuations of a rational matrix $R \in \mathbb{R}^{q\times q}(s)$. Define $\sigma_1^\infty(R)$ as the minimum of the valuations of all $1 \times 1$ minors of $R$. Let $\sigma_2^\infty(R)$ be the minimum of the valuations of determinants of all nonsingular $2 \times 2$ minors of $R$. This procedure allows defining upto $\sigma_q^\infty(R)$, for a nonsingular polynomial/rational $q \times q$ sized $R$. The structural indices at infinity of the rational matrix $R$ are defined as $\nu_1^\infty(R) := \sigma_1^\infty(R)$, $\nu_j^\infty(R) := \sigma_j^\infty(R) - \sigma_{j-1}^\infty(R)$ for $j$ equal to 2 upto $q$. While $\nu_i$ satisfy $\nu_1 \leqslant \nu_2 \cdots \leqslant \nu_p$, one or more $\nu_i$ can be negative or positive. The absolute values of the negative ones are summed to give the poles at infinity of $R$ (with multiplicity), while the positive ones are summed to give the zeros at $\infty$ of $R$ with multiplicity:

$$z_R(\infty) = \sum_{\nu_i>0} \nu_i(\infty) \quad \text{ and } \quad p_R(\infty) = -\sum_{\nu_i<0} \nu_i(\infty) \ .$$

Further, the indices $n_i(\infty)$ defined above are just respectively the integers $\nu_i^\infty$.

We briefly review this concept in the special context of polynomial and unimodular matrices. A polynomial matrix has no finite poles, but has poles at infinity (unless the matrix is a constant matrix). A polynomial matrix in general has finite zeros. A unimodular matrix has no finite zeros, since the determinant is a nonzero constant. However, unless again the matrix is a constant matrix, there are poles at infinity, which are equalled by zeros at infinity, due to which the determinant is a nonzero constant. Thus unimodular matrices are special in the sense that they have neither finite zeros nor finite poles, but they have both poles and zeros at infinity, and moreover, with the same multiplicity. Further, a unimodular matrix can be neither row-reduced nor column reduced, unless the matrix is a constant matrix: this is because row-reduced/column-reduced implies no zeros at infinity.

### D. Behavioral theory of systems

We defined above that the behavior of a system is the set of solutions (in a pre-specified signal space) to the governing equations of the system. In behavioral theory we generally consider *weak* solutions: elements of the solution space that satisfy the differential equation in a distributional sense. We will adopt this approach here also. In this paper, we study those systems whose equations are of the form $R(\frac{d}{dt})w = 0$ (see equation (2)). Thus the behavior $\mathfrak{B}$ is

$$\mathfrak{B} := \{w \in \mathfrak{L} \mid R(\frac{d}{dt})w = 0\}$$

with $\mathfrak{L}$ prespecified as one of the four function spaces defined in Subsection II-A above. For obvious reasons, $R(\frac{d}{dt})w = 0$ is called a kernel representation of $\mathfrak{B}$. A behavior $\mathfrak{B}$ is called autonomous if the following implication holds:

$$\left( \begin{array}{c} w_1 \text{ and } w_2 \in \mathfrak{B} \text{ and } T \in \mathbb{R}^+ \\ \text{satisfy } w_1(t) = w_2(t) \text{ for all } t < T \end{array} \right) \Rightarrow (w_1 = w_2) \ .$$

Of course, when dealing with $\mathfrak{L}_{\mathbb{R}}^I$ and $\mathfrak{L}_{\mathbb{R}^+}^I$, both equalities of trajectories above is also to be understood in a distributional sense. A behavior is autonomous if and only if the polynomial matrix $R$ defining the behavior above has full column rank. For simplicity we assume that the autonomous behavior is given by a kernel representation $R(\frac{d}{dt})w = 0$ with $R \in \mathbb{R}^{q\times q}[s]$, square and nonsingular.

### E. States and state maps

The property of state in the familiar input/state/output[3] system captures both the concept of initial condition and that of concatenability.

Consider first $\mathfrak{L} = \mathfrak{L}_{\mathbb{R}}^C$. A variable $x$ is said to satisfy the property of concatenability if whenever $(w_1, x_1)$ and $(w_2, x_2)$ are in $\mathfrak{B}_{\text{full}}$, and such that $x_1(T) = x_2(T)$, then the trajectory $w_1 \underset{T}{\wedge} w_2$ also satisfies the differential equations of $\mathfrak{B}$ in a distributional sense. (The function $w_1 \underset{T}{\wedge} w_2$ is defined to be equal to $w_1$ for $t \leqslant T$ and equal to $w_2$ for $t > T$.) The variable $x$ is then called a state of the system.

The construction of a state variable can be done through the 'state map' $X(\frac{d}{dt})$: a map that acts on the variable $w$ and gives a state variable $x$, i.e. $x := X(\frac{d}{dt})w$. We focus in this paper on the state map constructed using the 'shift-and-cut-map' on a polynomial matrix $R$ defining the behavior.

The shift and cut operator $\sigma : \mathbb{R}^{w\times w}[s] \to \mathbb{R}^{w\times w}[s]$ for a polynomial matrix $R$ is defined by

$$\sigma(R) := s^{-1}(R(s) - R(0)) \ .$$

Higher order actions of $\sigma$ are defined in the obvious way: $\sigma^2(R) = \sigma(\sigma(R))$, etc.. Let $N$ be the highest degree amongst the entries in $R \in \mathbb{R}^{w\times w}[s]$. Then a state map $X(s) \in \mathbb{R}^{Nw\times w}[s]$ is constructed by $X(s) := \text{col}(\sigma(R), \sigma^2(R), \cdots, \sigma^N(R))$. One can remove the zero rows from this matrix $X(s)$ and retain only the nonzero rows. The state map $X$ obtained by the above procedure (shift and cut, and then remove zero rows) on a polynomial matrix $R$ is denoted as $X_R$ and is called the canonical state map. Clearly

[3]This refers to the system of equations $\dot{x} = Ax + Bu$ and $y = Cx + Du$.

$X_R \in \mathbb{R}^{n_X \times q}[s]$, where $n_X := \sum_i d_i(R)$; the sum of the row degrees of $R$.

Consider the map $X_\tau : \mathcal{L}^C_{\mathbb{R}} \to \mathbb{R}^{n_X}$ defined as $X_\tau(f) := (X(\frac{d}{dt})f)_{|t=\tau}$ and consider the subspace $S \subseteq \mathbb{R}^{n_X}$ defined as $S := X_0(\mathcal{L}^C_{\mathbb{R}})$. It is well-known (see Rapisarda & Willems [11], for example) that if (and only if) $R$ is row-reduced, then for every $a \in S$, there exists a trajectory $w \in \mathfrak{B}$, with $\mathfrak{B}$ defined by kernel representation $R(\frac{d}{dt})w = 0$ and $\mathcal{L} = \mathcal{L}^C_{\mathbb{R}}$ such that $X_0(w) = a$. In this sense, $S$ also serves the purpose of 'space of initial conditions'. We now turn to the case of a different function space $\mathcal{L}$.

We first note that, when studying distributional solutions, the equality aspect in the definition of concatenability ($x_1(T) = x_2(T)$) is not directly applicable. While careful modification of this definition by ruling out 'step discontinuities' is possible, we focus on the *initial condition aspect* of the state since in the context of autonomous systems and when dealing with $\mathcal{L}^C_{\mathbb{R}}$ and $\mathcal{L}^C_{\mathbb{R}^+}$, there is a one-to-one correspondence between initial conditions and solutions. This paper investigates the one-to-one relation between initial conditions and solutions for $\mathcal{L}^I_{\mathbb{R}^+}$, and the 'state' is to be understood in that sense.

It turns out that $X_\tau(\mathcal{L}^C_{\mathbb{R}})$, $X_0(\mathcal{L}^I_{\mathbb{R}^+})$ and $X_0(\mathfrak{B})$ (with $\mathfrak{B} \subseteq \mathcal{L}^C_{\mathbb{R}}$) are different in general, and further, elements in $S$ may not correspond to trajectories in $\mathfrak{B}$ when $\mathfrak{B} \subseteq \mathcal{L}^C_{\mathbb{R}}$ (nor when $\mathfrak{B} \subseteq \mathcal{L}^I_{\mathbb{R}}$), but there are trajectories in $\mathfrak{B} \subseteq \mathcal{L}^I_{\mathbb{R}^+}$ that correspond to these elements.

We call an initial condition $a$ redundant if there does not exist any function $f \in \mathcal{L}^C_{\mathbb{R}}$ such that $X_{0^-}(f) = a$.

For autonomous behavior, where the solution space consists of only smooth functions: $\mathcal{L}^C_{\mathbb{R}}$ or $\mathcal{L}^C_{\mathbb{R}^+}$, concatenability is the same as equality of two trajectories. The smooth behavior for a kernel representation $R(\frac{d}{dt})w = 0$ does not change with elementary row operations on $R$, but it is easily verified that the state map does change when elementary row operations are performed on $R$.

In the context of a state map $X(\frac{d}{dt})$, we frequently use the constant matrix $Z$ associated to the polynomial matrix $X(s)$. For $X \in \mathbb{R}^{n_X \times q}[s]$, with $N$ the highest of the degrees of the polynomials in $X$, define $Z \in \mathbb{R}^{n_X \times Nq}$ as the matrix such that

$$X(s) = Z\mathrm{col}(I_q, sI_q, \cdots, s^N I_q).$$

The constant matrix $Z$ helps in stating the following straightforward relation:

$$X_\tau(f) = Z\mathrm{col}(f, \frac{d}{dt}f, \cdots \frac{d^N}{dt^N}f)_{|t=\tau} \quad .$$

### F. Inadmissible initial conditions

Next we define an inadmissible initial condition vector for an autonomous system $R(\frac{d}{dt})w = 0$, with $R(s) \in \mathbb{R}^{q\times q}[s]$ nonsingular. We say that $a \in S$ is admissible if there exists a $w \in \mathfrak{B} \cap \mathcal{L}^C_{\mathbb{R}}$ such that $a = X_0(w)$. Otherwise, $a$ is called inadmissible.

A state map $X$ is said to be admissible if the initial condition space obtained by $X_\tau$ consists of only admissible initial conditions for every $\tau \geqslant 0$.

In the context of state maps, polynomial matrices and initial conditions, three integers play a key role. We define them now. Let $R \in \mathbb{R}^{q\times q}[s]$ be nonsingular. Define

- $n_X := \sum d_i(R)$, equalling the number of rows of $Z$ and $X$ (defined above in Subsection II-E),
- $n_{\text{full}}$ := the rank of $X$ over $\mathbb{R}$ (which also equals the rank of $Z$ over $\mathbb{R}$)
- $n_{\text{slow}} := \sum d_i(UR)$, where $U$ is unimodular and $UR$ is row proper.

We will show in this paper that these three integers are respectively the dimensions of the space of all (redundant and non-redundant) initial conditions, of non-redundant (admissible and inadmissible) initial conditions and of admissible initial conditions.

## III. Problem formulation

We summarize the main questions addressed in this paper below. We deal with autonomous LTI systems, and seek solutions on the half-line $\mathbb{R}^+$. The state map built using the shift-and-cut map on the polynomial matrix $R$ defining a dynamical system gives initial conditions. This state space/initial-condition space is based on the number of nonzero rows of the shift and cut map, the corresponding canonical state map is denoted by $X_R$. We pose the following questions: what elements of the state space can be considered as initial conditions; which of these result in smooth solutions and which ones result in impulses? We investigate the relation between these questions, and with the pole/zero structure at infinity, and then with the rank of $X$ over $\mathbb{R}$ and over $\mathbb{R}[s]$.

We start this study first addressing the case of the kernel representation matrix $R$ being unimodular (Section IV). In Section V we study the non-unimodular case.

## IV. Unimodular systems

This section considers systems described by unimodular matrices:

$$U(\frac{d}{dt})w = 0, \quad U \in \mathbb{R}^{q\times q}[s], \quad \det(U) \in \mathbb{R}^*(= \mathbb{R} - \{0\}) \ .$$

As is well known, on the real line there are no nonzero solutions, even in the distributional sense, but on the half line $\mathbb{R}_+$ there might be might be solutions for certain initial conditions, see for instance Vardulakis [13]. In fact, all non-redundant initial conditions are inadmissible and hence there are nontrivial solutions in $\mathcal{L}^I_{\mathbb{R}^+}$ but none in $\mathcal{L}^C_{\mathbb{R}}$ nor in $\mathcal{L}^I_{\mathbb{R}}$. For this case, the state map gives us all the initial conditions that result in impulsive solutions. Below is one of the main results of this paper.

*Theorem 2:* Let $U$ be a unimodular matrix, and let $X$ be its canonical state map. Let $r = \mathrm{rank}_{\mathbb{R}}(X)$. Then,

1) There exist precisely $r$ linearly independent impulsive solutions for the systems $U(\frac{d}{dt})w = 0$ on $\mathbb{R}_+$.

2) The integer $r$ equals the number of zeros of the polynomial matrix $U$ at infinity.
3) For each $a \in X_0(\mathcal{L}^C_{\mathbb{R}^+})$, there exists a $w \in \mathcal{L}^I_{\mathbb{R}^+}$ such that $U(\frac{\mathrm{d}}{\mathrm{d}t})w = 0$ and $a = Z\mathrm{col}(w, \frac{d}{dt}w, \cdots \frac{d^N}{dt^N}w)_{|t=0^-}$.

Note that the meaning of $w(0^-)$ for $w \in \mathcal{L}^I_{\mathbb{R}^+}$ is to be understood in the sense defined at the end of Subsection II-A above.

We skip the proof of this result and the other results below due to shortage of space.

### A. An illustrative example

The canonical state map $X_U$ associated to $U = \begin{bmatrix} 1 & -s \\ 0 & 1 \end{bmatrix}$, as defined in equation (1) above is $X_U = [0 \quad -1]$, which exactly defines the initial condition that triggers the impulsive solutions. We consider the following slightly-larger example from Vardulakis [13, Example 4.48, Page 191].

*Example 3:* Consider the differential equation $U(\frac{d}{dt})w = 0$ where $U(s)$ and the obtained state map $X$ are

$$U(s) = \begin{bmatrix} 1 & s^3 & 0 \\ 0 & 1 & s \\ 0 & 0 & 1 \end{bmatrix} \qquad X(s) = \begin{bmatrix} 0 & s^2 & 0 \\ 0 & s & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}.$$

Clearly, $\mathrm{rank}_{\mathbb{R}}(X) = 4$ and also $U$ has four poles at $\infty$. Since $U$ is unimodular, all the zeros are at infinity, and hence $X$ gives initial conditions all resulting in impulsive solutions. When $X(\frac{d}{dt})$ acts on $w$, we see that the initial conditions on $w$ that play a role are $w_2(0^-)$, $w_2'(0^-)$, $w_2''(0^-)$ and $w_3(0^-)$. Assuming these four values to respectively be equal to $a$, $b$, $c$ and $d \in \mathbb{R}$, and substituting these into the differential equation $U(\frac{d}{dt})w = 0$ we obtain the general solution

$$w(t) = \begin{bmatrix} -c\delta - b\delta' - a\delta'' + d\delta^{(3)} \\ -d\delta \\ 0 \end{bmatrix}.$$

The significance of using the state map is that the relevant initial conditions can be obtained far more easily than by the development in [13]. □

## V. Non-unimodular autonomous systems

In this section we consider the case of dynamical systems $R(\frac{d}{dt})w = 0$ for the case that $\det(R)$ is a nonconstant polynomial. In this case, the polynomial matrix $R$ has finite zeros; thus the behavior over $\mathcal{L}^C_{\mathbb{R}}$ is nonzero, and in fact, the dimension of this solution set is precisely $\deg(\det(R))$ (see Polderman & Willems [9]).

*Theorem 4:* Let $R \in \mathbb{R}^{q\times q}[s]$. The state map $X$ is an admissible state map if and only if $R$ is row proper.

The following lemma is relevant when $R$ is not row-proper.

*Lemma 5:* Consider a nonsingular polynomial matrix $R$ and the three integers defined above at the end of Subsection II-F. Then,

$$n_{\mathrm{slow}} \leqslant n_{\mathrm{full}} \leqslant n_X.$$

Lemma 5 can be improved for the case of a row-reduced matrix $R$: the inequalities in Lemma 5 become equalities.

*Theorem 6:* Consider $R \in \mathbb{R}^{q\times q}[s]$. The following statements are equivalent:

1) $R$ is row reduced
2) $n_X = n_{\mathrm{full}}$
3) $n_X = n_{\mathrm{slow}}$

In fact this theorem also follows from a result in Antslakis [1], see also Antsaklis and Gao [2, theorem A1] and Zúñiga and Henrion [14], where this result is obtained in a different context.

The next theorem shows that the number of linearly independent rows over $\mathbb{R}$ of the polynomial matrix $X$ is exactly the number of poles of $R$ at infinity.

*Theorem 7:* Consider $R \in \mathbb{R}^{q\times q}[s]$ and its canonical state map $X$. Then, $n_{\mathrm{full}} = p_R(\infty)$.

Using the fact that $n_{\mathrm{slow}} = p_R(\infty) - z_R(\infty)$, this theorem gives immediately the relation we are after: the relation between the dimension of the minimal 'smooth' state space $n_{\mathrm{slow}}$ and the dimension of the canonical state space $X$ constructed above: the difference is the number of zeros of $R$ at infinity.

*Theorem 8:* Let $R \in \mathbb{R}^{q\times q}[s]$ be nonsingular. Then

$$n_{\mathrm{full}} = n_{\mathrm{slow}} + z_R(\infty) \ .$$

We end this section with a generalization of Theorem 2 to the nonunimodular case. In this case the state map gives all non-redundant initial conditions: both admissible and inadmissible, with the dimension of smooth and impulsive solutions related to the rank of the canonical state map over $\mathbb{R}$.

*Theorem 9:* Let $R \in \mathbb{R}^{q\times q}[s]$ be square and nonsingular. There exist

- $n_{\mathrm{slow}}$ linearly independent smooth solutions, and
- $(n_{\mathrm{full}} - n_{\mathrm{slow}})$ linearly independent pure impulsive solutions.

Further, for any $a \in X_R(\mathcal{L}^C_{\mathbb{R}})$ there exists a unique solution $w \in \mathcal{L}^I_{\mathbb{R}^+}$ such that $X_0(w) = a$.

This theorem shows that our definition of inadmissible initial values in Section II-F concurs with the definition given in Dai [4] and Vardulakis [13]: an initial condition vector is said to be *inadmissible* if the corresponding solution $w(t)$ contains the Dirac impulse $\delta(t)$ and/or its higher order distributional derivatives. Compare our theorem with the following classical result, see for example [13, Theorem 4.32]:

*Proposition 10:* Consider the autonomous system defined by $R(\frac{d}{dt})w = 0$ where $R \in \mathbb{R}^{q\times q}[s]$ is nonsingular, and suppose $N$ is the degree of the highest degree entry in $R(s)$. Then, for every initial condition vector $\bar{w}(0) = (w(0), w^{(1)}(0), ..., w^{(N-1)}(0)) \in \mathbb{R}^{Nq}$, the corresponding solution $w(t)$ has no Dirac impulses nor its distributional derivatives if and only if $R(s)$ has no zeros at infinity. In other words, there exist no inadmissible initial conditions for $R(\frac{d}{dt})w = 0$ if and only if $R$ has no zeros at $\infty$.

## VI. Two electrical circuit examples

In this section we consider two circuit[4] examples, the first one has just one zero: at infinity, while the second example has three zeros: two finite and one at infinity. In both cases the state map is studied to see the initial conditions that define the solutions.

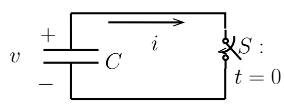


Fig. 1. Charged capacitor with one switch

*Example 11:* Consider the circuit shown in Figure 1. The switch $S$ is closed at $t = 0$, due to which we have the equation $R(\frac{d}{dt})w = 0$ for $t \geqslant 0$ with

$$R(s) = \begin{bmatrix} 1 & -Cs \\ 0 & 1 \end{bmatrix}, \qquad w = \begin{bmatrix} i \\ v \end{bmatrix}.$$

Of course, when the switch is closed, the charge in the capacitor discharges immediately resulting in the current becoming an impulse. Due to the voltage across the capacitor suddenly jumping to zero, we obtain that the current is an impulse at the origin. The state map gives the initial condition as the voltage across the capacitor, which upon being nonzero only, the current is an impulse. $\square$

*Example 12:* Consider another circuit shown in Figure 2. The switch $S$ is closed at $t = 0$, and now the system of

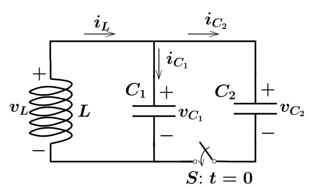


Fig. 2. An inductor and two capacitors with a switch

equations for $t \geqslant 0$ are $R(\frac{d}{dt})w = 0$ with

$$R(s) = \begin{bmatrix} 1 & 1 & 0 & 0 & 0 \\ 0 & 1 & -1 & 0 & 0 \\ 1 & 0 & 0 & -Ls & 0 \\ 0 & C_1 s & 0 & 0 & -1 \\ 0 & 0 & C_2 s & -1 & 1 \end{bmatrix} \text{ and } w = \begin{bmatrix} v_L \\ v_{C_1} \\ v_{C_2} \\ i_L \\ i_{C_1} \end{bmatrix}.$$

Notice that $\det(R) = 1 + L(C_1 + C_2)s^2$ and thus there are two finite zeros, and $n_{\text{slow}} = 2$. Construct $X$ using the shift and cut map to get

$$X(s) = \begin{bmatrix} 0 & 0 & 0 & -L & 0 \\ 0 & C_1 & 0 & 0 & 0 \\ 0 & 0 & C_2 & 0 & 0 \end{bmatrix}$$

which clearly has linearly independent rows. This implies that $R$ has three poles at infinity, suggesting one zero at $\infty$ and hence one impulsive mode. A look at the circuit tells that when the voltages across the capacitors are not equal, we indeed get an impulse in the current $i_{C_1}$ (and hence $i_{C_2}$) when the switch is closed. Further, the three relevant initial conditions are $v_{C_1}$, $v_{C_2}$ and $i_L$. $\square$

## VII. Conclusive remarks

In this paper, we considered autonomous systems and noted that while unimodular matrices have just the zero behavior when the solutions are sought over the full line (either continuous functions or distributional solutions), the half-line aspect of the solutions allows nonzero impulsive solutions. The dimension of the space of impulsive solutions is exactly the number of zeros at infinity. This relation turns out to be the case also when the polynomial matrix defining the system of equations is not unimodular. The state map provides the space of initial conditions which gives rise to nonzero solutions, both impulsive and smooth. We finally related the dimension of impulsive and smooth solutions to the number of finite and infinite zeros of the polynomial matrix. While some of these results are deducible from the Kronecker canonical form of the pair $(E, A)$ of the singular system obtained from the higher order polynomial matrix $R(s)$, our approach is very direct due to the use of the state map for obtaining explicitly the variables whose initial values play a role in the solutions: smooth or impulsive.

It remains to extend these results to the non-autonomous case, for example like was done in [3] for first order systems.

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