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Near-real-time GPS Integer Ambiguity Resolution

Anning Chen, Dongfang Zheng, Arvind Ramanandan and Jay A. Farrell

***Abstract*— Real-time decimeter accuracy GPS positioning can be achieved using carrier phase measurements. This requires fast and reliable on-the-fly integer ambiguity resolution. However, in GPS challenged areas (e.g. Urban canyons, tunnels, thick canopy etc.) the GPS receiver may not be able to track a sufficient number of satellites to resolve the integer ambiguities within one epoch. In this paper, we would like to find the optimal solution by combining the measurements from several epochs.**

In this paper, we present the theoretical derivation for a fast and efficient method for GPS integer ambiguity resolution with multiple GPS epoch measurements. Simulation results show the effectiveness of the proposed approach.

I. INTRODUCTON

Integrated GPS/INS (Inertial Navigation System) is a popular tool for localization [3], [2]. Localization accuracies of a few centimeters can be achieved using carrier phase processing assuming rapid and accurate on-the-fly integer ambiguity resolution.

However, in GPS challenged areas (e.g. Urban canyons, tunnels, thick canopy etc.) the GPS receiver may not be able to track a sufficient number of satellites to resolve the integer ambiguities within one epoch. In this paper, we would like to find the optimal solution by combining the measurements from several epochs.

For example, if there are few satellites in view, it will be difficult to solve the integer ambiguity with single epoch data. As we will discuss later, the single epoch GPS ambiguity problem has 4 degrees of freedom, assuming we have K $(K > 4)$ satellites in view each second, combining each additional epoch will add another 4 degrees of freedom but will also add K extra measurements. Therefore, combining GPS measurements from multiple epochs will help with GPS integer ambiguity resolution.

In this paper, we extend the approach in [1] with GPS measurements on multiple epochs. We present the theoretical derivation for a fast and efficient method for GPS integer ambiguity resolution. Two sets of simulation results show the effectiveness of the proposed approach.

II. MEASUREMENTS MODEL

A. DGPS Measurements

Through this paper, we consider single difference GPS (DGPS) measurements [5]. For simplicity of notation, we assume that the DGPS approach completely removes all common-mode errors (e.g., ionosphere, troposphere, satellite clock and ephemeris). The DGPS code and carrier phase measurements for satellite i can be modeled as

$$\rho^{(i)} = R^{(i)} + c\delta t_r + \varepsilon^{(i)} \quad (1)$$
$$\lambda\phi^{(i)} = R^{(i)} + c\delta t_r + \lambda N^{(i)} + \eta^{(i)} \quad (2)$$

where $R^{(i)} = \|X^{(i)} - X_a\|$ is the geometric distance between the satellite i position $X^{(i)}$ and receiver antenna position X_a. The symbol $c\delta t_r$ is the receiver clock bias. The symbol λ represents the signal wavelength. The symbols $\varepsilon^{(i)}$ and $\eta^{(i)}$ represent the measurement noises in code and phase measurements. The symbol $N^{(i)}$ represents the unknown integer ambiguity that is to be determined. The index $i = 1,\cdots,K$, where K is the number of available satellites.

B. Residual Measurements

This document will work with residual measurement computed relative to a position X_0 assumed to be sufficiently accurate so that the $h.o.t.$'s can be neglected in the linearization process. The residual measurements are

$$\delta\rho^{(i)} = \rho^{(i)} - \|X^{(i)} - X_0\| \quad (3)$$
$$\lambda\delta\phi^{(i)} = \lambda\phi^{(i)} - \|X^{(i)} - X_0\|. \quad (4)$$

The linearized residual measurements are modeled as

$$\delta\rho^{(i)} = h^{(i)}\mathbf{x}_{a0} + c\delta t_r + \varepsilon^{(i)} \quad (5)$$
$$\lambda\delta\phi^{(i)} = h^{(i)}\mathbf{x}_{a0} + c\delta t_r + \lambda N^{(i)} + \eta^{(i)} \quad (6)$$

where $\mathbf{x}_{a0} = (X_a - X_0)^\top \in \mathbb{R}^3$ and $h^{(i)} = \frac{X_a - X_0}{\|X_a - X_0\|} \in \mathbb{R}^3$. We assume that $\varepsilon^{(i)} \sim \mathcal{N}(0,\sigma_\rho^2)$ and $\eta^{(i)} \sim \mathcal{N}(0,\sigma_\Phi^2)$. In typical GPS applications, with σ_ρ is at the meter level and σ_Φ is at the centimeter level. All the noise terms are uncorrelated with each other.

The phase residual measurements can be put in matrix form as

$$\lambda\delta\boldsymbol{\phi} = \mathbf{H}\mathbf{x} + \lambda\mathbf{N} + \boldsymbol{\eta} \quad (7)$$

where $\boldsymbol{\phi} = \begin{bmatrix} \phi^{(1)} & \cdots & \phi^{(K)} \end{bmatrix}^\top$, $\mathbf{x} = \begin{bmatrix}\mathbf{x}_{a0}^\top \; c\delta t_r\end{bmatrix}^\top$, $\mathbf{H} = \begin{bmatrix} h^{(1)} & 1 \\ \vdots & \vdots \\ h^{(K)} & 1 \end{bmatrix}$, $\boldsymbol{\eta} = \begin{bmatrix} \eta^{(1)} & \cdots & \eta^{(K)} \end{bmatrix}^\top$, and $\mathbf{N} = \begin{bmatrix} N^{(1)} & \cdots & N^{(K)} \end{bmatrix}^\top \in \mathbb{Z}^K$ is the integer ambiguity vector that is to be determined.

C. Measurements over multiple epoches

For the simplicity of notation in the integer ambiguity problem, we rewrite (7) at time t_j as:

$$\mathbf{y}_j = \mathbf{G}_j\mathbf{x}_j + \mathbf{N}_j + \mathbf{v}_j \quad (8)$$

Anning Chen, Dongfang Zheng, Arvind Ramanandan and Jay A. Farrell are with Department of Electrical Engineering, University of California Riverside, Riverside, CA, 92521, U.S.A. Email: {achen;dzheng;aramanandan;farrell}@ee.ucr.edu.

where $\mathbf{y}_j = \delta\boldsymbol{\phi} \in \mathbb{R}^K$ represents the DGPS phase measurements at time t_j, $\mathbf{x}_j \in \mathbb{R}^n$ and $\mathbf{N}_j \in \mathbb{Z}^K$ are the parameters to be estimated, and $n = 4$. $\mathbf{G}_j = \lambda^{-1}\mathbf{H}_j \in \mathbb{R}^{K\times n}$ is the observation matrix characterizing the satellite-user geometry, the noise term $\mathbf{v}_j = \boldsymbol{\eta}_j/\lambda \in \mathbb{R}^K$ and $\mathbf{v}_j \sim \mathcal{N}(\mathbf{0}, \boldsymbol{\Sigma}_{vv})$, the covariance matrix of $\mathbf{v}_j$ is $\boldsymbol{\Sigma}_{vv} = \frac{\sigma_\Phi^2}{\lambda^2}\mathbf{I}$.

Assume from time t_1 to time t_M, the receiver maintains lock to the satellites, i.e., $\mathbf{N} = \mathbf{N}_1 = \cdots = \mathbf{N}_M$, then the measurements from t_1 to t_M can be grouped as:

$$\bar{\mathbf{y}} = \bar{\mathbf{G}}\mathbf{X} + \mathbf{L}\mathbf{N} + \bar{\mathbf{v}} \tag{9}$$

where the measurement $\bar{\mathbf{y}}^\top = \begin{bmatrix} \mathbf{y}_1^\top & \cdots & \mathbf{y}_M^\top \end{bmatrix} \in \mathbb{R}^{MK}$; the state vector $\mathbf{X}^\top = \begin{bmatrix} \mathbf{x}_1^\top & \cdots & \mathbf{x}_M^\top \end{bmatrix} \in \mathbb{R}^{MK}$; the measurement noise $\bar{\mathbf{v}}^\top = \begin{bmatrix} \mathbf{v}_1^\top & \cdots & \mathbf{v}_M^\top \end{bmatrix} \in \mathbb{R}^{MK}$ with $cov(\mathbf{v}_j) = \boldsymbol{\Sigma}_{vv_j}$,

$$\bar{\mathbf{G}} = \begin{bmatrix} \mathbf{G}_1 & & \mathbf{0} \\ & \ddots & \\ \mathbf{0} & & \mathbf{G}_M \end{bmatrix}, \tag{10}$$

$$\mathbf{L}^\top = \begin{bmatrix} \mathbf{I} & \cdots & \mathbf{I} \end{bmatrix}, \tag{11}$$

where $\mathbf{I}$ represents K-by-K identity matrix, and $\bar{\mathbf{G}} \in \mathbb{R}^{MK\times Mn}$ and $\mathbf{L} \in \mathbb{R}^{MK\times K}$.

Assuming that the GPS carrier phase measurement noise is uncorrelated over time, the covariance matrix of the noise vector $\bar{\mathbf{v}}$ is

$$\boldsymbol{\Sigma} = \begin{bmatrix} \boldsymbol{\Sigma}_{vv_1} & & \mathbf{0} \\ & \ddots & \\ \mathbf{0} & & \boldsymbol{\Sigma}_{vv_M} \end{bmatrix}.$$

By forming the problem as (9), we would solve the GPS integer ambiguity using measurements over the interval from $t_1, \cdots, t_M$. In this problem definition, we attempt to solve the integer ambiguity problem at $t = t_1$, if we succeed, then we have a real time solution. If we fail, then at $t = t_2$, we attempt to solve the problem using $\mathbf{y}_1$ and $\mathbf{y}_2$, etc. If the problem is solved at time $t > t_1$, the answer is attained in near real time, etc.

III. Integer Ambiguity Problem

A. Problem Statement

GPS integer ambiguity problem in near real time can be solved as a Maximum Likelihood (ML) estimation problem: given the GPS measurements, we would like to find the estimate of $\mathbf{N}$ and $\mathbf{x}$ that maximize the conditional probability $f(\mathbf{y}|\mathbf{N},\mathbf{x})$.

$$\begin{aligned}(\hat{\mathbf{N}},\hat{\mathbf{X}}) &= \arg\max_{\mathbf{N}\in\mathbb{Z}^K,\mathbf{X}\in\mathbb{R}^n} f(\mathbf{y}|\mathbf{N},\mathbf{x}) \\ &= \arg\max_{\mathbf{N}\in\mathbb{Z}^K,\mathbf{X}\in\mathbb{R}^n} \ln f(\mathbf{y}|\mathbf{N},\mathbf{x}) \\ &= \arg\max_{\mathbf{N}\in\mathbb{Z}^K,\mathbf{X}\in\mathbb{R}^n} \ln\left(\frac{e^{-\frac{1}{2}(\bar{\mathbf{y}}-\bar{\mathbf{G}}\mathbf{X}-\mathbf{N})^T\boldsymbol{\Sigma}^{-1}(\bar{\mathbf{y}}-\bar{\mathbf{G}}\mathbf{X}-\mathbf{N})}}{(2\pi)^{m/2}|\boldsymbol{\Sigma}|^{1/2}}\right) \\ &= \arg\min_{\mathbf{N}\in\mathbb{Z}^K,\mathbf{X}\in\mathbb{R}^n} (\bar{\mathbf{y}}-\bar{\mathbf{G}}\mathbf{X}-\mathbf{N})^T\boldsymbol{\Sigma}^{-1}(\bar{\mathbf{y}}-\bar{\mathbf{G}}\mathbf{X}-\mathbf{N})\end{aligned} \tag{12}$$

Therefore, the near real time GPS integer ambiguity problem can be solved as an Integer Weighted Least-Square (IWLS) problem [7]. Our objective is to find $\mathbf{N} \in \mathbb{Z}^K$, $\mathbf{X} \in \mathbb{R}^{Mn}$ that minimize the cost function

$$c(\mathbf{X},\mathbf{N}) = (\bar{\mathbf{y}}-\bar{\mathbf{G}}\mathbf{X}-\mathbf{N})^T\boldsymbol{\Sigma}^{-1}(\bar{\mathbf{y}}-\bar{\mathbf{G}}\mathbf{X}-\mathbf{N}). \tag{13}$$

B. Generating Searching Candidates

One of the leading algorithms in integer ambiguity problem is LMS [6]. Following the very useful insight from [4], that inspired LMS, although (9) contains $(K+Mn)$ unknown variables, there are only n degrees of freedom. Given any $\mathbf{x}_j$, all the integer ambiguities can be resolved. On the other hand, given any n integers, the states $\mathbf{X}$ can be computed accurately. These remarks show that not all combinations of integers are admissible and the challenge is to reformulate (9) properly to find admissible integer vectors efficiently. The basic idea of LMS is to search only over the admissible combinations of integer candidates so that the search space can be decreased. The original LMS procedure is presented in [6]. Alternative implementations are presented in [2], [5], [11].

Divide the integer vector $\mathbf{N}$ into two subvectors $\mathbf{N}_C$ and $\mathbf{N}_D$, where $\mathbf{N}_D$ contains n integers and $\mathbf{N}_C$ contains the remaining $(K-n)$ integers. The integers in $\mathbf{N}_D$ are searched exhaustively over some range of d integers, the remaining integers are computed as real value estimates and are rounded to an optimally selected integer (described below) to get the estimate of $\mathbf{N}_C$. This yields d^n integer vectors in $\mathbb{Z}^K$. We can evaluate each integer vector candidate to find the one with least cost. As in LMS, this decreases the search dimension from K to n, which decrease the integer vector candidates from d^K to d^n.

Starting from (9), for a given integer ambiguity $\mathbf{N}$, the weighted least square estimate of $\mathbf{x}$ would be:

$$\hat{\mathbf{X}} = \left(\bar{\mathbf{G}}^T\boldsymbol{\Sigma}^{-1}\bar{\mathbf{G}}\right)^{-1}\bar{\mathbf{G}}^T\boldsymbol{\Sigma}^{-1}(\bar{\mathbf{y}}-\mathbf{L}\mathbf{N}) \tag{14}$$

and the residual vector is

$$\begin{aligned}\hat{\boldsymbol{\varepsilon}} &= \bar{\mathbf{y}} - \bar{\mathbf{G}}\cdot\hat{\mathbf{X}} - \mathbf{L}\mathbf{N} \\ &= \left(\mathbf{I} - \bar{\mathbf{G}}\left(\bar{\mathbf{G}}^T\boldsymbol{\Sigma}^{-1}\bar{\mathbf{G}}\right)^{-1}\bar{\mathbf{G}}^T\boldsymbol{\Sigma}^{-1}\right)(\bar{\mathbf{y}}-\mathbf{L}\mathbf{N}) \\ &= \bar{\mathbf{Q}}_{\boldsymbol{\Sigma}}(\bar{\mathbf{y}}-\mathbf{L}\mathbf{N}).\end{aligned} \tag{15}$$

where

$$\bar{\mathbf{Q}}_{\boldsymbol{\Sigma}} = \mathbf{I} - \bar{\mathbf{P}}_{\boldsymbol{\Sigma}}, \tag{16}$$

$$\bar{\mathbf{P}}_{\boldsymbol{\Sigma}} = \bar{\mathbf{G}}\left(\bar{\mathbf{G}}^T\boldsymbol{\Sigma}^{-1}\bar{\mathbf{G}}\right)^{-1}\bar{\mathbf{G}}^T\boldsymbol{\Sigma}^{-1}. \tag{17}$$

Note that both $\bar{\mathbf{P}}_{\boldsymbol{\Sigma}}$ and $\bar{\mathbf{Q}}_{\boldsymbol{\Sigma}}$ are idempotent and that $Rank(\bar{\mathbf{P}}) = Mn$ and $Rank(\bar{\mathbf{Q}}) = M(K-n)$.

Proof: Because $\boldsymbol{\Sigma} > 0$ is a covariance matrix, $\boldsymbol{\Sigma}^{-1}$ is symmetric and positive definite. Thus it can be factored as

$$\boldsymbol{\Sigma}^{-1} = \mathbf{W}^\top\mathbf{M}^\top\mathbf{M}\mathbf{W}, \tag{18}$$

where $\mathbf{W} \in \mathbb{R}^{MK\times MK}$ is a unitary matrix (i.e., $\mathbf{W}\mathbf{W}^\top = \mathbf{W}^\top\mathbf{W} = \mathbf{I}$) and $\mathbf{M}^\top = \mathbf{M}$ is a diagonal matrix with positive elements on the diagonal. Substituting Eqn. (18) into Eqn. (17), we have

$$\begin{aligned}\bar{\mathbf{P}}_{\mathbf{\Sigma}} &= \mathbf{G}(\bar{\mathbf{G}}^\top\mathbf{\Sigma}^{-1}\bar{\mathbf{G}})^{-1}\bar{\mathbf{G}}^\top\mathbf{\Sigma}^{-1}\\ &= \bar{\mathbf{G}}(\bar{\mathbf{G}}^\top\mathbf{W}^\top\mathbf{M}^\top\mathbf{M}\mathbf{W}\bar{\mathbf{G}})^{-1}\bar{\mathbf{G}}^\top\mathbf{W}^\top\mathbf{M}^\top\mathbf{M}\mathbf{W}\\ &= \bar{\mathbf{G}}(\mathbf{A}^\top\mathbf{A})^{-1}\mathbf{A}^\top\mathbf{M}\mathbf{W}\\ &= \mathbf{W}^{-1}\mathbf{M}^{-1}\mathbf{A}(\mathbf{A}^\top\mathbf{A})^{-1}\mathbf{A}^\top\mathbf{M}\mathbf{W}\\ &= \mathbf{W}^{-1}\mathbf{M}^{-1}\mathbf{P}\mathbf{M}\mathbf{W}, \end{aligned} \quad (19)$$

where $\mathbf{P}=\mathbf{A}(\mathbf{A}^\top\mathbf{A})^{-1}\mathbf{A}^\top$ and $\mathbf{A}=\mathbf{M}\mathbf{W}\bar{\mathbf{G}}$. Eqn. (19) shows that $\bar{\mathbf{P}}_{\mathbf{\Sigma}}$ is similar to $\mathbf{P}$ where $\mathbf{P}$ is a projection matrix onto the range of $\mathbf{A}$; therefore, $rank(\mathbf{P})=rank(\mathbf{A})$. Because $\mathbf{M}$ and $\mathbf{W}$ are both nonsingular, $rank(\mathbf{A})=rank(\bar{\mathbf{G}})=Mn$; hence, $rank(\mathbf{P})=Mn$ yields $rank(\bar{\mathbf{P}}_{\mathbf{\Sigma}})=Mn$.

Let $\mathbf{Q}=\mathbf{I}-\mathbf{P}$, then $\mathbf{Q}$ is a projection matrix onto the subspace orthogonal to the range space of $\mathbf{A}$, and has rank $M(K-n)$. The following analysis shows that $\bar{\mathbf{Q}}_{\mathbf{\Sigma}}$ is similar to $\mathbf{Q}$:

$$\begin{aligned}\bar{\mathbf{Q}}_{\mathbf{\Sigma}} &= \mathbf{I}-\bar{\mathbf{P}}_{\mathbf{\Sigma}}\\ &= \mathbf{I}-\mathbf{W}^\top\mathbf{M}^{-1}\mathbf{P}\mathbf{M}\mathbf{W}\\ &= \mathbf{W}^\top\mathbf{M}^{-1}(\mathbf{I}-\mathbf{P})\mathbf{M}\mathbf{W}\\ &= \mathbf{W}^{-1}\mathbf{M}^{-1}\mathbf{Q}\mathbf{M}\mathbf{W}. \end{aligned} \quad (20)$$

Therefore, $\bar{\mathbf{Q}}_{\mathbf{\Sigma}}$ also has rank $M(K-n)$. ■

From (13), the cost function evaluated from candidate $\mathbf{N}$ is

$$\begin{aligned}c(\mathbf{N}) &= \|\bar{\mathbf{y}}-\bar{\mathbf{G}}\cdot\mathbf{x}-\mathbf{L}\mathbf{N}\|^2_{\mathbf{\Sigma}}\\ &= \|\bar{\mathbf{Q}}_{\mathbf{\Sigma}}(\bar{\mathbf{y}}-\mathbf{L}\mathbf{N})\|^2_{\mathbf{\Sigma}}\\ &= (\bar{\mathbf{y}}-\mathbf{L}\mathbf{N})^\top\bar{\mathbf{Q}}_{\mathbf{\Sigma}}^\top\mathbf{\Sigma}^{-1}\bar{\mathbf{Q}}_{\mathbf{\Sigma}}(\bar{\mathbf{y}}-\mathbf{L}\mathbf{N})\\ &= (\bar{\mathbf{y}}-\mathbf{L}\mathbf{N})^\top\bar{\mathbf{Q}}_0(\bar{\mathbf{y}}-\mathbf{L}\mathbf{N}) \end{aligned} \quad (21)$$

where

$$\begin{aligned}\bar{\mathbf{Q}}_0 &= \bar{\mathbf{Q}}_{\mathbf{\Sigma}}^\top\mathbf{\Sigma}^{-1}\bar{\mathbf{Q}}_{\mathbf{\Sigma}}\\ &= \left(\mathbf{I}-\bar{\mathbf{P}}_{\mathbf{\Sigma}}\right)^\top\mathbf{\Sigma}^{-1}\left(\mathbf{I}-\bar{\mathbf{P}}_{\mathbf{\Sigma}}\right) \quad (22)\\ &= \mathbf{\Sigma}^{-1}-\mathbf{\Sigma}^{-1}\bar{\mathbf{P}}_{\mathbf{\Sigma}}-\bar{\mathbf{P}}_{\mathbf{\Sigma}}^\top\mathbf{\Sigma}^{-1}+\bar{\mathbf{P}}_{\mathbf{\Sigma}}^\top\mathbf{\Sigma}^{-1}\bar{\mathbf{P}}_{\mathbf{\Sigma}}\\ &= \mathbf{\Sigma}^{-1}-2\mathbf{\Sigma}^{-1}\bar{\mathbf{G}}(\bar{\mathbf{G}}^\top\mathbf{\Sigma}^{-1}\bar{\mathbf{G}})^{-1}\bar{\mathbf{G}}^\top\mathbf{\Sigma}^{-1}\\ &\quad+\mathbf{\Sigma}^{-1}\bar{\mathbf{G}}(\bar{\mathbf{G}}^\top\mathbf{\Sigma}^{-1}\bar{\mathbf{G}})^{-1}\bar{\mathbf{G}}^\top\mathbf{\Sigma}^{-1}\bar{\mathbf{G}}(\bar{\mathbf{G}}^\top\mathbf{\Sigma}_{vv}^{-1}\bar{\mathbf{G}})^{-1}\bar{\mathbf{G}}^\top\mathbf{\Sigma}^{-1}\\ &= \mathbf{\Sigma}^{-1}-\mathbf{\Sigma}^{-1}\bar{\mathbf{G}}(\bar{\mathbf{G}}^\top\mathbf{\Sigma}^{-1}\bar{\mathbf{G}})^{-1}\bar{\mathbf{G}}^\top\mathbf{\Sigma}^{-1}\\ &= \mathbf{\Sigma}^{-1}\left(\mathbf{I}-\bar{\mathbf{P}}_{\mathbf{\Sigma}}\right)\\ &= \mathbf{\Sigma}^{-1}\bar{\mathbf{Q}}_{\mathbf{\Sigma}} \quad (23)\end{aligned}$$

Proposition 3.1: $Rank(\bar{\mathbf{Q}}_0)=M(K-n)$.

Proof: First we should notice that by use of (18) and (20), $\mathbf{Q}_0$ can be written as:

$$\begin{aligned}\bar{\mathbf{Q}}_0 &= \bar{\mathbf{Q}}_{\mathbf{\Sigma}}^\top\mathbf{\Sigma}^{-1}\bar{\mathbf{Q}}_{\mathbf{\Sigma}}\\ &= (\mathbf{W}^{-1}\mathbf{M}^{-1}\mathbf{Q}\mathbf{M}\mathbf{W})^\top\\ &\quad(\mathbf{W}^{-1}\mathbf{M}^\top\mathbf{M}\mathbf{W})(\mathbf{W}^{-1}\mathbf{M}^{-1}\mathbf{Q}\mathbf{M}\mathbf{W})\\ &= \mathbf{W}^{-1}\mathbf{M}^\top\mathbf{Q}\mathbf{M}^{-T}\mathbf{W}\mathbf{W}^{-1}\mathbf{M}^\top\mathbf{M}\mathbf{W}\mathbf{W}^{-1}\mathbf{M}^{-1}\mathbf{Q}\mathbf{M}\mathbf{W}\\ &= \mathbf{W}^{-1}\mathbf{M}^\top\mathbf{Q}\mathbf{M}\mathbf{W}. \end{aligned} \quad (24)$$

Following (19), we stated that $\mathbf{Q}$ is a projection matrix with rank $M(K-n)$. Because $\mathbf{M}$ and $\mathbf{W}$ are nonsingular, $\bar{\mathbf{Q}}_0$ is similar to $\mathbf{Q}$; therefore, $rank(\mathbf{Q}_0)=M(K-n)$. ■

Let the SVD (single value decomposition) of $\bar{\mathbf{Q}}_0$ be

$$\bar{\mathbf{Q}}_0=\bar{\mathbf{U}}\bar{\mathbf{S}}^2\bar{\mathbf{U}}^\top,$$

where $\bar{\mathbf{U}}$ is unitary and $\bar{\mathbf{S}}$ is diagonal with $diag(\bar{\mathbf{S}})=[\bar{s}_1,\ldots,\bar{s}_{M(K-n)},0,\ldots,0]$ with all $\bar{s}_i>0$ for $i=1,\ldots,M(K-n)$. Define $\bar{\mathbf{B}}=\bar{\mathbf{S}}\bar{\mathbf{U}}^\top$ such that

$$\bar{\mathbf{Q}}_0 = \bar{\mathbf{B}}^\top\bar{\mathbf{B}} \quad (25)$$

where the last Mn rows of $\bar{\mathbf{B}}$ are zero.

Given the above analysis, the cost function of (21) can be rewritten as

$$\begin{aligned}c(\mathbf{N}) &= (\bar{\mathbf{y}}-\mathbf{L}\mathbf{N})^\top\bar{\mathbf{B}}^\top\bar{\mathbf{B}}(\bar{\mathbf{y}}-\mathbf{L}\mathbf{N})\\ &= \|\bar{\mathbf{B}}\,(\bar{\mathbf{y}}-\mathbf{L}\mathbf{N})\,\|^2. \end{aligned} \quad (26)$$

Because $\bar{\mathbf{B}}$ does not have full rank, the null space of $\mathbf{B}$ is not empty. Therefore, there exists (non-unique) $\hat{\mathbf{N}}\in\mathbb{R}^m$ such that $(\bar{\mathbf{y}}-\mathbf{L}\hat{\mathbf{N}})$ is in the null space of $\bar{\mathbf{B}}$:

$$\bar{\mathbf{B}}(\bar{\mathbf{y}}-\mathbf{L}\hat{\mathbf{N}}) = \mathbf{0}. \quad (27)$$

$$\bar{\mathbf{B}}\bar{\mathbf{y}} = \bar{\mathbf{B}}\mathbf{L}\hat{\mathbf{N}} \quad (28)$$

The matrix $\bar{\mathbf{B}}$ can be represent as

$$\bar{\mathbf{B}}=\begin{bmatrix}\bar{\mathbf{A}}\\ \mathbf{0}\end{bmatrix},$$

where $\bar{\mathbf{A}}\in\mathbb{R}^{M(K-n)\times MK}$.

Our goal is to find an integer vector $\hat{\mathbf{N}}$ such that

$$\begin{aligned}\begin{bmatrix}\bar{\mathbf{A}}\\ \mathbf{0}\end{bmatrix}\bar{\mathbf{y}} &= \begin{bmatrix}\bar{\mathbf{A}}\\ \mathbf{0}\end{bmatrix}\mathbf{L}\hat{\mathbf{N}}\\ \bar{\mathbf{A}}\bar{\mathbf{y}} &= \bar{\mathbf{A}}\mathbf{L}\hat{\mathbf{N}} \end{aligned} \quad (29)$$

As $\bar{\mathbf{A}}\mathbf{L}\in\mathbb{R}^{M(K-n)\times K}$, let $\bar{\mathbf{A}}\mathbf{L}=\begin{bmatrix}\bar{\mathbf{C}} & \bar{\mathbf{D}}\end{bmatrix}$, where $\bar{\mathbf{C}}\in\mathbb{R}^{M(K-n)\times(K-n)}$ and $\bar{\mathbf{D}}\in\mathbb{R}^{M(K-n)\times n}$.

Let the last n elements of $\hat{\mathbf{N}}$ to be integers. We denote this subvector as $\hat{\mathbf{N}}_D$ and the first $K-n$ elements of $\hat{\mathbf{N}}$ are denoted as $\hat{\mathbf{N}}_C$. Then (29) can be rewritten as

$$\bar{\mathbf{A}}\bar{\mathbf{y}} = \begin{bmatrix}\bar{\mathbf{C}} & \bar{\mathbf{D}}\end{bmatrix}\begin{bmatrix}\hat{\mathbf{N}}_C\\ \hat{\mathbf{N}}_D\end{bmatrix}$$

Therefore, if we decompose $\mathbf{y}$ as $\mathbf{y}=\begin{bmatrix}\mathbf{y}_C^\top & \mathbf{y}_D^\top\end{bmatrix}^\top$ with $\mathbf{y}_C\in\mathbb{R}^{(K-n)}$ and $\mathbf{y}_D\in\mathbb{R}^n$ and given a hypothesized vector $\hat{\mathbf{N}}_D\in\mathbb{Z}^n$, then the real-value estimate of $\hat{\mathbf{N}}_C$ is:

$$\hat{\mathbf{N}}_C = \left(\bar{\mathbf{C}}^\top\bar{\mathbf{C}}\right)^{-1}\bar{\mathbf{C}}^\top\left(\bar{\mathbf{A}}\bar{\mathbf{y}}-\bar{\mathbf{D}}\hat{\mathbf{N}}_D\right) \quad (30)$$

The integer candidates $\mathbf{N}_D$ can be searched exhaustively over some finite range of integers using n "for" loops as shown in Fig. 1. For each integer vector $\mathbf{N}_D$, Eqn. (30) provides a float estimate $\hat{\mathbf{N}}_C$.

$\mathbf{A} = \mathbf{C}^{-1}\mathbf{D}$
for $i = -d : d$
  for $j = -d : d$
    for $k = -d : d$
      $\mathbf{N}_D = [i, j, k, 0]^\top$
      $\hat{\mathbf{N}}_C = \left(\bar{\mathbf{C}}^\top \bar{\mathbf{C}}\right)^{-1} \bar{\mathbf{C}}^\top \left(\bar{\mathbf{A}}\bar{\mathbf{y}} - \bar{\mathbf{D}}\hat{\mathbf{N}}_D\right)$
      … use $\hat{\mathbf{N}}_C$ to compute $\check{\mathbf{N}}_C$ minimizing $J(\mathbf{N}_C)$
      $\mathbf{N} = \left[\begin{array}{cc} \check{\mathbf{N}}_C & \mathbf{N}_D \end{array}\right]$
      if $c(\mathbf{N}) <$ current minimum
        Save $\mathbf{N}$
        current minimum $= c(\mathbf{N})$
      ⋮

Fig. 1: Triple 'for' loop to compute $\hat{\mathbf{N}}_C$ and $\mathbf{N}$ for the case where $n = 4$.

C. Rounding $\hat{\mathbf{N}}_C$

Having $\hat{\mathbf{N}}_C$, to get the optimal integer estimate of $\mathbf{N}_C$, we would like to find an integer vector $\check{\mathbf{N}}_C$ which is close to $\hat{\mathbf{N}}_C$ in an appropriate sense. As discussed in [8], [9], as the integer estimation error vector can be highly correlated, visualized by the level curve of the cost function being a tilted and elongated ellipse, directly rounding $\hat{\mathbf{N}}_C$ to $\check{\mathbf{N}}_C$ may yield incorrect integer estimates and cause a significant cost increase.

Proposition 3.2: Consider the cost function

$$\begin{aligned} J(\mathbf{N}_C) &= \|\mathbf{N}_C - \hat{\mathbf{N}}_C\|^2_{\mathbf{\Sigma}_{\hat{\mathbf{N}}_C}} \\ &= (\mathbf{N}_C - \hat{\mathbf{N}}_C)^\top \mathbf{\Sigma}^{-1}_{\hat{\mathbf{N}}_C} (\mathbf{N}_C - \hat{\mathbf{N}}_C), \end{aligned} \quad (31)$$

where from (30)

$$\mathbf{\Sigma}_{\hat{\mathbf{N}}_C} = \mathbf{C}^{-1}\mathbf{A}\mathbf{\Sigma}\mathbf{A}^\top\mathbf{C}^{-\top}. \quad (32)$$

Then the cost function $c(\mathbf{N})$ defined in (13) will be minimized by the same integer estimate that minimize $J(\mathbf{N}_C)$.

Proof: From Eqn. (26), for any $\mathbf{N} \in \mathbb{Z}^m$,

$$\begin{aligned} c(\mathbf{N}) &= (\bar{\mathbf{y}} - \mathbf{L}\mathbf{N})^\top \mathbf{B}^\top \mathbf{B} (\bar{\mathbf{y}} - \mathbf{L}\mathbf{N}) \\ &= (\bar{\mathbf{y}} - \mathbf{L}\hat{\mathbf{N}} - \mathbf{L}\mathbf{N} + \mathbf{L}\hat{\mathbf{N}})^\top \mathbf{B}^\top \mathbf{B} (\bar{\mathbf{y}} - \mathbf{L}\hat{\mathbf{N}} - \mathbf{L}\mathbf{N} + \mathbf{L}\hat{\mathbf{N}}) \\ &= \|\mathbf{B}(\bar{\mathbf{y}} - \mathbf{L}\hat{\mathbf{N}})\|^2 + \|\mathbf{B}\mathbf{L}(\mathbf{N} - \hat{\mathbf{N}})\|^2 \\ &\quad -2(\mathbf{N} - \hat{\mathbf{N}})^\top \mathbf{L}^\top \mathbf{B}^\top \mathbf{B} (\bar{\mathbf{y}} - \mathbf{L}\hat{\mathbf{N}}) \end{aligned} \quad (33)$$

where $\hat{\mathbf{N}}$ is the optimal real-valued estimate of $\mathbf{N}$, which by (27) satisfies $\mathbf{B}(\bar{\mathbf{y}} - \mathbf{L}\hat{\mathbf{N}})) = 0$ and therefore

$$c(\mathbf{N}) = \|\mathbf{B}\mathbf{L}(\mathbf{N} - \hat{\mathbf{N}})\|^2. \quad (34)$$

From (10) and (16), we know

$$\bar{\mathbf{Q}}_{\mathbf{\Sigma}} = \begin{bmatrix} \mathbf{Q}_{\mathbf{\Sigma}_1} & & \mathbf{0} \\ & \ddots & \\ \mathbf{0} & & \mathbf{Q}_{\mathbf{\Sigma}_1} \end{bmatrix}, \quad (35)$$

where

$$\mathbf{P}_{\mathbf{\Sigma}_j} = \mathbf{G}_j \left(\mathbf{G}_j{}^T \mathbf{\Sigma}^{-1} \mathbf{G}_j\right)^{-1} \mathbf{G}_j{}^T \mathbf{\Sigma}^{-1}_{vv}, \quad (36)$$
$$\mathbf{Q}_{\mathbf{\Sigma}_j} = \mathbf{I} - \mathbf{P}_{\mathbf{\Sigma}_j}, \quad (37)$$

$\mathbf{P}_{\mathbf{\Sigma}_j} \in \mathbb{R}^{K \times K}$ and $\mathbf{P}_{\mathbf{\Sigma}_j} \in \mathbb{R}^{K \times K}$.

Let

$$\mathbf{Q}_{\mathbf{\Sigma}_j} = \begin{bmatrix} \bar{\mathbf{Q}}_{CC_j} & \bar{\mathbf{Q}}_{CD_j} \\ \bar{\mathbf{Q}}_{DC_j} & \bar{\mathbf{Q}}_{DD_j} \end{bmatrix}$$

where $\mathbf{Q}_{CC_j} \in \mathbb{R}^{(K-n)\times(K-n)}$, $\mathbf{Q}_{DD_j} \in \mathbb{R}^{n \times n}$ and $\mathbf{Q}_{CD_j}, \mathbf{Q}_{DC_j}^\top \in \mathbb{R}^{(K-n)\times n}$. Let

$$\sum_1^M \mathbf{Q}_{CC_j} = \mathbf{Q}_{CC}$$
$$\sum_1^M \mathbf{Q}_{CD_j} = \mathbf{Q}_{CD}$$
$$\sum_1^M \mathbf{Q}_{DC_j} = \mathbf{Q}_{DC}$$
$$\sum_1^M \mathbf{Q}_{DD_j} = \mathbf{Q}_{DD}$$

Similarly, for the covariance matrix As the covariance matrix $\mathbf{\Sigma}_{vv}$ is block diagonal, let

$$\mathbf{\Sigma}_{vv} = \begin{bmatrix} \mathbf{\Sigma}_{CC} & \mathbf{0} \\ \mathbf{0} & \mathbf{\Sigma}_{DD} \end{bmatrix},$$

where $\mathbf{\Sigma}_{CC} \in \mathbb{R}^{(K-n)\times(K-n)}$ and $\mathbf{\Sigma}_{DD} \in \mathbb{R}^{n \times n}$. Let

$$\sum_1^M \mathbf{\Sigma}_{CC} = \mathbf{\Sigma}_{CC}$$
$$\sum_1^M \mathbf{\Sigma}_{DD} = \mathbf{\Sigma}_{DD}$$

Then, from (25)

$$\begin{aligned} \mathbf{L}^\top \mathbf{B}^\top \mathbf{B}\mathbf{L} &= \mathbf{L}^\top \mathbf{Q}_0 \mathbf{L} \\ &= \mathbf{L}^\top \mathbf{\Sigma}^{-1} \bar{\mathbf{Q}}_{\mathbf{\Sigma}} \mathbf{L} \\ &= \left[\begin{array}{ccc} \mathbf{I} & \cdots & \mathbf{I} \end{array}\right] \begin{bmatrix} \mathbf{\Sigma}_{vv} & & \mathbf{0} \\ & \ddots & \\ \mathbf{0} & & \mathbf{\Sigma}_{vv} \end{bmatrix} \\ &\quad \begin{bmatrix} \mathbf{Q}_{\mathbf{\Sigma}_1} & & \mathbf{0} \\ & \ddots & \\ \mathbf{0} & & \mathbf{Q}_{\mathbf{\Sigma}_1} \end{bmatrix} \begin{bmatrix} \mathbf{I} \\ \vdots \\ \mathbf{I} \end{bmatrix} \\ &= \begin{bmatrix} \bar{\mathbf{Q}}_{CC} & \bar{\mathbf{Q}}_{CD} \\ \bar{\mathbf{Q}}_{DC} & \bar{\mathbf{Q}}_{DD} \end{bmatrix} \begin{bmatrix} \mathbf{\Sigma}^{-1}_{CC} & \mathbf{0} \\ \mathbf{0} & \mathbf{\Sigma}^{-1}_{DD} \end{bmatrix} \\ &= \begin{bmatrix} \mathbf{\Sigma}^{-1}_{CC}\mathbf{Q}_{CC} & \mathbf{\Sigma}^{-1}_{CC}\mathbf{Q}_{CD} \\ \mathbf{\Sigma}^{-1}_{DD}\mathbf{Q}_{DC} & \mathbf{\Sigma}^{-1}_{DD}\mathbf{Q}_{DD} \end{bmatrix}. \end{aligned} \quad (38)$$

By the method that $\mathbf{N}$ is generated, $\hat{\mathbf{N}}_D = \mathbf{N}$ is an integer vector. Hence, the cost function $c(\mathbf{N})$ can be written as

$$\begin{aligned} c(\mathbf{N}) &= \|\mathbf{B}\mathbf{L}(\mathbf{N} - \hat{\mathbf{N}})\|^2 \\ &= (\mathbf{N} - \hat{\mathbf{N}})^\top \mathbf{L}^\top \mathbf{B}^\top \mathbf{B}\mathbf{L}(\mathbf{N} - \hat{\mathbf{N}}) \\ &= \begin{bmatrix} \mathbf{N}_C - \hat{\mathbf{N}}_C \\ \mathbf{0} \end{bmatrix}^T \begin{bmatrix} \mathbf{\Sigma}^{-1}_{CC}\mathbf{Q}_{CC} & \mathbf{\Sigma}^{-1}_{CC}\mathbf{Q}_{CD} \\ \mathbf{\Sigma}^{-1}_{DD}\mathbf{Q}_{DC} & \mathbf{\Sigma}^{-1}_{DD}\mathbf{Q}_{DD} \end{bmatrix} \begin{bmatrix} \mathbf{N}_C - \hat{\mathbf{N}}_C \\ \mathbf{0} \end{bmatrix} \\ &= (\mathbf{N}_C - \hat{\mathbf{N}}_C)^\top \left(\mathbf{\Sigma}^{-1}_{CC}\mathbf{Q}_{CC}\right) \mathbf{N}_C - \hat{\mathbf{N}}_C). \end{aligned} \quad (39)$$

Comparison of (31) and (39) shows that if we can prove $\mathbf{\Sigma}^{-1}_{CC}\mathbf{Q}_{CC} = \mathbf{\Sigma}^{-1}_{\hat{\mathbf{N}}_C}$, then these two cost functions are equivalent.

From (27), we know that $\mathbf{BL\hat{N}} = \mathbf{B\bar{y}}$. Multiplying on the left by $\mathbf{B}^\top$ yields $\mathbf{B}^\top\mathbf{BLN} = \mathbf{B}^\top\mathbf{B\bar{y}}$ which provides the following constraint on the covariance

$$\mathbf{B}^\top\mathbf{BL}\mathbf{\Sigma}_{NN}\mathbf{L}^\top\mathbf{B}^\top\mathbf{B} \quad = \quad \mathbf{B}^\top\mathbf{B}\mathbf{\Sigma}\mathbf{B}^\top\mathbf{B} \tag{40}$$

where

$$\mathbf{\Sigma}_{NN} = \begin{bmatrix} \mathbf{\Sigma}_{\hat{\mathbf{N}}_C} & \mathbf{0} \\ \mathbf{0} & \mathbf{0} \end{bmatrix}$$

as there is no uncertainty in $\mathbf{N}_D$.

From (23) and (25), (40) can be written as

$$\mathbf{\Sigma}^{-1}\mathbf{\bar{Q}_\Sigma}\mathbf{L}\mathbf{\Sigma}_{NN}\mathbf{L}^\top\mathbf{\bar{Q}}_\mathbf{\Sigma}^\top\mathbf{\Sigma}^{-1} = \mathbf{\Sigma}^{-1}\mathbf{\bar{Q}_\Sigma}\mathbf{\Sigma}\mathbf{\bar{Q}}_\mathbf{\Sigma}^\top\mathbf{\Sigma}^{-1}. \tag{41}$$

Because $\mathbf{\Sigma}$ is nonsingular, Eqn. (41) reduces to

$$\begin{aligned} \mathbf{\bar{Q}_\Sigma}\mathbf{L}\mathbf{\Sigma}_{NN}\mathbf{L}^\top\mathbf{\bar{Q}}_\mathbf{\Sigma}^\top &= \mathbf{\bar{Q}_\Sigma}\mathbf{\Sigma}\mathbf{\bar{Q}}_\mathbf{\Sigma}^\top & (42) \\ &= \mathbf{\bar{Q}_\Sigma}\mathbf{\Sigma}. & (43) \end{aligned}$$

Therefore,

$$\mathbf{\bar{Q}_\Sigma}\left(\mathbf{L}\mathbf{\Sigma}_{NN}\mathbf{L}^\top\mathbf{\bar{Q}}_\mathbf{\Sigma}^\top - \mathbf{\Sigma}\right) = 0, \tag{44}$$

$$\mathbf{\bar{Q}}_{\mathbf{\Sigma}_j}\left(\mathbf{\Sigma}_{NN}\mathbf{Q}_{\mathbf{\Sigma}_j}^\top - \mathbf{\Sigma}\right) = 0, \tag{45}$$

which can be written as

$$\begin{bmatrix} \mathbf{\bar{Q}}_{CC_j} & \mathbf{\bar{Q}}_{CD_j} \\ \mathbf{\bar{Q}}_{DC_j} & \mathbf{\bar{Q}}_{DD_j} \end{bmatrix}\begin{bmatrix} \mathbf{\Sigma}_{\hat{\mathbf{N}}_C}\mathbf{Q}_{CC_j}^\top - \mathbf{\Sigma}_{CC} & \mathbf{\Sigma}_{\hat{\mathbf{N}}_C}\mathbf{Q}_{DC_j}^\top \\ \mathbf{0} & -\mathbf{\Sigma}_{DD} \end{bmatrix} = \mathbf{0}.$$

From Sylvesters rank inequality: If A is a m-by-n matrix and B n-by-k, then

$$rank(A) + rank(B) - n \leq rank(AB).$$

As $\mathbf{Q}_{\mathbf{\Sigma}_j} \in \mathbb{R}^{K\times K}$, $\mathbf{\Sigma}_{NN}\mathbf{Q}_{\mathbf{\Sigma}_j}^\top - \mathbf{\Sigma} \in \mathbb{R}^{K\times K}$ and $rank(\mathbf{\bar{Q}_\Sigma}) = (K - n)$. Therefore, $rank\left(\mathbf{\Sigma}_{NN}\mathbf{Q}_{\mathbf{\Sigma}_j}^\top - \mathbf{\Sigma}\right) \leq n$.

As the block $-\mathbf{\Sigma}_{DD}$ has rank n, therefore,

$$\begin{aligned} \mathbf{\Sigma}_{\hat{\mathbf{N}}_C}\mathbf{Q}_{CC_j}^\top - \mathbf{\Sigma}_{CC} &= \mathbf{0} \\ \mathbf{\Sigma}_{\hat{\mathbf{N}}_C}\mathbf{Q}_{CC_j}^\top - \mathbf{\Sigma}_{CC} &= \mathbf{0} \\ \mathbf{\Sigma}_{\hat{\mathbf{N}}_C}\mathbf{Q}_{CC_j}^\top &= \mathbf{\Sigma}_{CC} \\ \mathbf{\Sigma}_{\hat{\mathbf{N}}_C} &= \mathbf{\Sigma}_{CC}\mathbf{Q}_{CC_j}^{-\top} \\ \mathbf{\Sigma}_{\hat{\mathbf{N}}_C}^{-1} &= \mathbf{Q}_{CC_j}^\top\mathbf{\Sigma}_{CC}^{-1} \\ \mathbf{\Sigma}_{\hat{\mathbf{N}}_C}^{-\top} &= \mathbf{\Sigma}_{CC}^{-\top}\mathbf{Q}_{CC_j} \\ \mathbf{\Sigma}_{\hat{\mathbf{N}}_C}^{-1} &= \mathbf{\Sigma}_{CC}^{-1}\mathbf{Q}_{CC_j} \\ c(\mathbf{N}) &= (\mathbf{N}_C - \hat{\mathbf{N}}_C)^\top\mathbf{\Sigma}_{\hat{\mathbf{N}}_C}^{-1}\mathbf{N}_C - \hat{\mathbf{N}}_C), \end{aligned}$$

which completes the proof. ∎

To find the integer vector that minimizes Eqn. (31), we follow the idea of LAMBDA to find a matrix $\boldsymbol{Z} \in \mathbb{Z}^{(m-n)\times(m-n)}$, such that $\boldsymbol{Z}^{-1} \in \mathbb{Z}^{(m-n)\times(m-n)}$, and $(\boldsymbol{Z}\mathbf{\Sigma}_{\hat{\mathbf{N}}_C}\boldsymbol{Z}^\top)^{-1}$ is nearly diagonal. The procedure to find the Z-transformation is described in detail in [10].

Let $\hat{\mathbf{M}}_C = \mathbf{Z}\hat{\mathbf{N}}_C$, then the cost function written in terms of $\mathbf{M}_C$ is

$$J(\mathbf{M}_C) \quad = \quad (\mathbf{M}_C - \hat{\mathbf{M}}_C)^\top\mathbf{\Sigma}_{\hat{M}_C}^{-1}(\mathbf{M}_C - \hat{\mathbf{M}}_C), \tag{46}$$

where $\mathbf{\Sigma}_{\hat{M}_C} = \boldsymbol{Z}\mathbf{\Sigma}_{\hat{N}_C}\boldsymbol{Z}^\top$. Because $\mathbf{\Sigma}_{\hat{M}_C}^{-1}$ is nearly diagonal, $J(\mathbf{M}_C)$ can be minimized by rounding $\hat{\mathbf{M}}_C$ to the nearest integer; therefore, the integer-valued estimate of $\mathbf{N}_C$ can be computed as:

$$\begin{aligned} \hat{\mathbf{M}}_C &= \boldsymbol{Z}\hat{\mathbf{N}}_C & (47) \\ \check{\mathbf{M}}_C &= [\hat{\mathbf{M}}_C]_{roundoff} & (48) \\ \check{\mathbf{N}}_C &= \boldsymbol{Z}^{-1}\check{\mathbf{M}}_C & (49) \end{aligned}$$

At this point we have an integer vector candidate $\left[\check{\mathbf{N}}_C^\top \ \mathbf{N}_D^\top\right]^\top$. One such candidate will be generated for each iteration of the 'for' loop in Fig. 1. We can compare each integer vector candidate using eqn. (21). Selecting the candidate vector with fragment the lowest cost (subject to validity tests) as the best. By rounding off the float estimate $\hat{\mathbf{N}}_C$ in the decorrelated domain of $\hat{\mathbf{M}}_C$, we have a better chance to achieve optimal integer estimate $\check{\mathbf{N}}_C$.

From Eqn. (30), the integer candidates $\hat{\mathbf{N}}_D$ can be searched exhaustively over some finite range of integers as described in Fig. 1.

IV. Test Result

In the MATLAB simulation, the test is epoch-by-epoch with a set of 6 single difference GPS L1 ($\lambda \approx 0.19m$) carrier phase measurements at different noise level. For each noise level, 1000 measurement epochs with randomly picked satellite elevation and azimuth angles were generated. We compared the success rate of getting all the integers correctly with different number of GPS epochs. The success rates of GPS integer ambiguity resolution by using different number of epochs with different level of covariance are plot versus different noise level in Fig. 2.

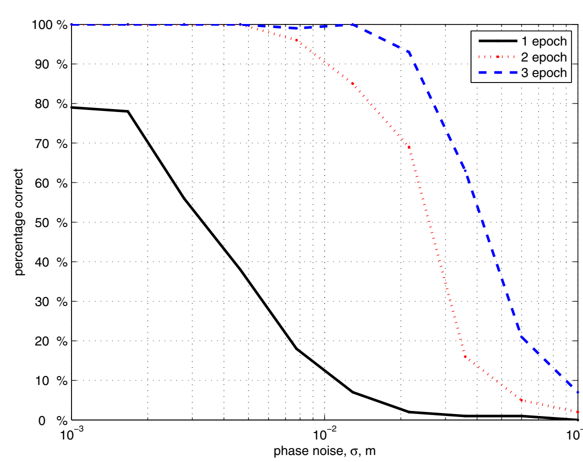

Fig. 2: Rate of Correct Integer Resolution vs. Phase Measurement Noise

Fig. 2 shows that by combing measurements from multiple epochs, we achieve a higher rate of estimating the correct integer vector at each covariance level.

A. Test over different number of satellites

The second set of test is performed to analyze the performance of the proposed approach as a function of the number of satellites. For each number of satellites, 1000 measurement epochs with randomly picked satellite elevation and azimuth angles were generated with the standard deviation of each phase measurement equal to $0.01m$. We compared the success rate of getting all the integers correctly in one, two and three epoches. The success rate of each scenario is plotted in Fig. 3.

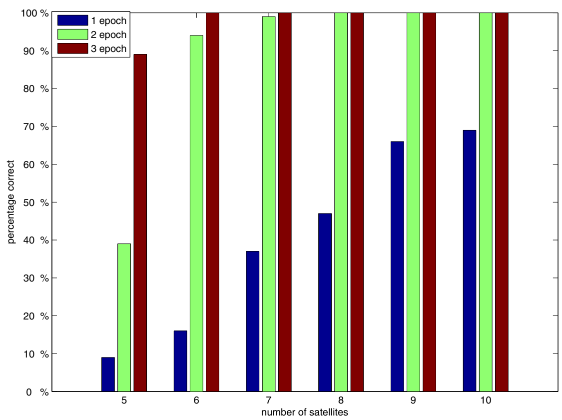

Fig. 3: Rate of Correct Integer Resolution vs. Number of Satellites

From Fig. 3, we can see that by combing measurements from multiple epochs, we achieve a substantially higher rate of getting the right integer vector, especially when there are few satellites. We can see that with as few as 5 satellites, we have a 90% chance of getting the right integer within 3 epochs if we combine the measurements over epochs. With 6 or more satellites, we will get the right integer within 2 epochs with probability of higher than 95%.

V. Conclusion and Future Work

A. Conclusion

In this paper, we extended the approach in [1] with GPS measurements from multiple epoches. We introduced a fast and efficient method for GPS integer ambiguity resolution with a theoretical derivation. Two sets of simulation result shows the effectiveness of the proposed approach.

B. Future work

Test with real-world data will be done in near future.

We note that the vehicle state at different epochs are not independent. Integrated GPS/INS (Inertial Navigation System) [3], [2] and integrated GPS/Encoders [2] are popular tools for localization. With such high rate sensors, we will have extra constraints on the vehicle states over epochs which can be used to facilitate GPS integer ambiguity resolution. In future research, we will extend the approach in this paper with auxiliary position estimate from INS or encoders.

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