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Closed Form Navigation Functions Based on Harmonic Potentials

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Abstract— **This paper proposes a new class of smooth closed form Navigation Functions that are derived from harmonic functions. The resulting functions are by construction free of local minima. Utilizing the underlying structure of harmonic functions a tuning controller is proposed to establish the non-degeneracy of critical points. The construction of this new class of Navigation Functions was made possible due to the recently introduced *Navigation Transformation*. In addition to the theoretical guarantees, the effectiveness of the proposed Navigation Functions is demonstrated through non-trivial computer simulations with systems with first and second order dynamics in a non-trivial workspace.**

I. Introduction

Harmonic potentials have always been considered as an attractive option for robotic navigation. The engineering intuition for using harmonic potentials for robotic navigation stems from the fact that potential flows of incompressible fluids that flow along the solutions of Laplace's equation, exhibit no local minima, apart from isolated stagnation points. Hence if a robot could be made to follow the trajectory of such a fluid particle, it would avoid collisions with nearby objects or getting trapped, and it would flow to an appropriately located sink at its destination configuration.

Following the above mentioned intuition, several solutions were given to the motion planning problem that were based on harmonic potentials. The first reported attempts in the robotic literature to use harmonic functions to solve the motion planning problem was reported in [2], [3] where the authors propose a numerical construction of a harmonic potential using Dirichlet and Neumann boundary conditions. In [5] an analytic construction of harmonic potential functions is proposed that is based on the panel method. In [4] harmonic potentials are implemented for path planning in dynamic environments. More recently Harmonic Functions have been used for multi-robot navigation [8].

In this paper we propose a closed form solution for the case of single robot navigation problem in a two-dimensional workspace, that was made possible due to the recently introduced Navigation Transformation, which initially appeared in [7]. Using this transformation, it is shown in the current paper that it is possible to construct a Navigation Function that is based on harmonic potentials for a 2-D workspace. The resulting function is proven to satisfy all the requirements set forth in [6] for a function to be a Navigation Function. This is the first to the author's knowledge closed form Navigation Function that is based on an underlying harmonic potential.

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There are several advantages of using closed form harmonic potentials to construct a closed form Navigation Function, that are in addition to the ones found in literature regarding closed form Navigation Functions. These include the capability to naturally decouple the influence of every obstacle in the environment and the destination configuration and most importantly for practical implementations, the absence of a tuning phase, allowing one to add and remove obstacles on the fly without worrying about introducing local minima.

The rest of the paper is organized as follows: Section II presents some preliminary notions while section III describes the Navigation Transformation. Section IV presents the construction of the proposed Navigation Function, while section V analyzes the properties of the constructed function. Section VI proposes and studies controllers that can be developed based on the proposed Navigation Function, while section VII presents simulation results. The paper concludes with section VIII.

II. Preliminaries

In this section we introduce the necessary terminology and definitions for the development of the methodology.

If K is a set, then $\bar{K}$ denotes the closure of the set, K^c denotes the complement of the set and $\overset{\circ}{K}$ the interior of the set. We denote with ∂K the boundary of K. Let S^n denote the n dimensional sphere. We will denote with $\mathcal{S}^n$ the n dimensional *sphere world* as this is defined in [6]. If we have a function $f(\cdot) : \mathbb{R}^n \to \mathbb{R}^n$, we denote the Jacobian matrix of this function as $J_f(\cdot)$ and the Jacobian determinant by $\|J_f(\cdot)\|$. Given a function $\phi : \mathbb{R}^n \to \mathbb{R}$ we denote by the matrix $\mathcal{H}_\phi(\cdot)$ the Hessian matrix of ϕ.

We will need the following:

Definition 1: Let $P_i \in \mathbb{R}^n$, $i \in \{1, \ldots M\}$ be M discrete elements of $\mathbb{R}^n$. Then a *point world* is defined as a manifold $\mathcal{P}^n \subseteq \mathbb{R}^n \setminus \bigcup\limits_{i=1}^{M} P_i$.

Definition 2: A *point world with spherical boundary* is a manifold $\tilde{\mathcal{P}}^n \subseteq \mathcal{P} \setminus \left(\overset{\circ}{S^n}\right)^c$ where $\bigcup\limits_{i=1}^{M} P_i \in \overset{\circ}{S^n}$.

We will restrict our attention to workspaces that are as follows:

Definition 3: The *workspace* $\mathcal{W} \subset \mathbb{R}^n$ is a manifold such that $\overset{\circ}{\mathcal{W}}$ is diffeomorphic to $\mathcal{P}^n$ or $\tilde{\mathcal{P}}^n$.

A workspace is *valid* if there is a non-zero minimum distance between different obstacles.

Let $\mathcal{O} = \partial\mathcal{W}$. Then $\mathcal{O}$ consists of the mutually disjoint sets of the *obstacle boundaries*, $\mathcal{O}_i, i \in \{1, \ldots M\}$ and the (also disjoint) "external" boundary $\mathcal{O}_0$, such that $\mathcal{O} = \bigcup\limits_{i=0\ldots M} \mathcal{O}_i$.

A system with holonomic kinematics is modelled as:

$$\dot{x} = u \tag{1}$$

where $x \in \mathbb{R}^n$ is the robot's position and $u \in \mathbb{R}^n$ is the velocity input. The initial configuration of the robot is denoted as x_0 and the destination configuration as x_d.

The dynamic model of a holonomic system is given by:

$$m\ddot{x} + c\dot{x} = f \tag{2}$$

where f is the force input to the system, m is the mass of the system and c is the viscous damping coefficient.

Instead of the original definition of a Navigation Function, proposed in [6], a smooth version will be considered:

Definition 4: Let $\mathcal{W} \subset \mathbb{R}^n$ be a compact connected smooth manifold with boundary. A map $\varphi : \mathcal{W} \to [0, 1]$, is a navigation function if it is:

1) Smooth on $\mathcal{W}$
2) Polar on $\mathcal{W}$, with minimum at $q_d \in \overset{\circ}{\mathcal{W}}$
3) Morse on $\mathcal{W}$
4) Admissible on $\mathcal{W}$

III. The Navigation Transformation

The definition and properties of the Navigation Transformation as well as a construction of a Navigation Transformation are described and studied in [7]. We will repeat in the current section some results from [7] for reasons of completeness:

The Navigation Transformation can be defined as follows:

Definition 5 ([7]): A Navigation Transformation is a diffeomorphism $\Phi : \overset{\circ}{\mathcal{W}} \to \mathcal{P}^n$, (or $\Phi : \overset{\circ}{\mathcal{W}} \to \tilde{\mathcal{P}}^n$) that maps $\mathcal{O}_i$ to M discrete elements $P_i, i \in \{1, \ldots M\}$ of $\overset{\circ}{S^n}$ and (if such exists) $\mathcal{O}_0$ to $\partial\tilde{\mathcal{P}}^n \setminus \bigcup\limits_{i \in \{1, \ldots M\}} P_i$.

As was shown in [7] a Navigation Transformation that operates on a *star-shaped* world [9] can be constructed as follows:

Let $h_\lambda : \mathcal{W} \to \mathcal{S}^n$ be the diffeomorphic transformation from the robot's workspace to a sphere world, as this is described in [9].

Let $q = h_\lambda(x)$. Then $b_i(q) = \|q - O_i\| - r_i$ where $i \in \{1, \ldots M\}$, is the distance from the boundary of the i'th sphere. O_i is the center of the i'th sphere in $\mathcal{S}^n$. Define the smooth function

$$\sigma_s(x) = \begin{cases} e^{-1/x} & x > 0 \\ 0 & x \leq 0 \end{cases}$$

Let

$$\mu_a = \min_{\substack{i,j \in \{1, \ldots M\} \\ i \neq j}} \{\|O_i - O_j\| - (r_i + r_j)\}$$

and

$$\mu_0 = \min_{i \in \{1, \ldots M\}} \{r_0 - \|O_i - O_0\| - r_i\}$$

Then define

$$\mu = \frac{1}{2} \min\{\mu_a, 2\mu_0\}$$

that is the minimum distance between any two spheres.

Define the smooth switch function:

$$\eta(x, \delta) = \frac{\sigma_s(x)}{\sigma_s(x) + \sigma_s(\delta - x)}$$

Define the function:

$$s(\delta, x) = \frac{x}{\delta}(1 - \eta(x, \delta)) + \eta(x, \delta)$$

Now define the contraction transformation:

$$v_i(q) = (1 - s(\mu, b_i(q)))(O_i - q)$$

Then the transformation from a sphere world to a point world is achieved through the transformation:

$$T(q) = q + \sum_{i=1}^{M} v_i(q)$$

Then the composition:

$$\Phi(x) = (T \circ h_\lambda)(x) \tag{3}$$

is a Navigation Transformation [7].

IV. Construction of the harmonic function based Navigation Function

A. Infeasibility of harmonic Navigation Functions

Let us first point out that a search for a closed form harmonic Navigation Function would be fruitless, due to the following:

Lemma 1: A harmonic Navigation Function is not possible.

Proof: This can be seen by implication of the Maximum Principle:

Definition 6 (Maximum Principle [1]): Suppose Ω is connected, u is real valued and harmonic on Ω, and u has a maximum or a minimum in Ω. Then u is constant.

Hence since a Navigation Function is both polar and admissible, a harmonic Navigation Function would have to be constant. ■

The result above in conjunction with the Navigation Transformation, motivates the construction a Navigation Function that is not harmonic itself but is based on harmonic functions.

B. Construction

The construction proposed in this paper is for the two-dimensional case. The basic idea here is to map the obstacle boundaries in a domain where a harmonic function with navigation-like properties can be constructed. We will refer to this domain as the *harmonic domain*. The next step is to generate the gradient vector field in the harmonic domain and pull-back a vector field from the harmonic domain to the initial workspace.

Let h be the robot's position and h_i be the position of the i'th obstacle's boundary in the harmonic domain. Then

$$\phi_i(h) = -a_i \ln(\|h - h_i\|)$$

is the i'th obstacle's potential, where a_i a positive parameter. Define the destination potential as

$$\phi_d(h) = a_d \ln(\|h - h_d\|)$$

where a_d a positive parameter and h_d denotes the destination configuration.

In order to ensure global attractivity of the destination, we impose the following condition:

$$\lim_{\|h\|\to\infty} \phi\left(h\right) = +\infty \tag{4}$$

This can be achieved by selecting $a_d = 1 + \sum_i a_i$. Hence by choosing $a_i = 1$ and $a_d = k \geq 1 + M$, condition (4) is satisfied.

The harmonic function is then:

$$\phi\left(h\right) = \phi_d\left(h\right) - \sum_{i=1}^{M} \phi_i\left(h\right) \tag{5}$$

The domain of this harmonic function is infinite. However the workspace where a robot is operating is usually bounded. The transformation $c : \mathcal{P}^n \to \tilde{\mathcal{P}}^n$ defined as:

$$c\left(q\right) \triangleq \rho \frac{q}{1 + \|q\|} \tag{6}$$

maps the harmonic domain to the sphere with radius ρ. The inverse transformation is

$$c^{-1}\left(x\right) = \frac{x}{\rho - \|x\|} \tag{7}$$

The Jacobian of the transformation is $\|J_c\| = \rho^2\left(1 + \|q\|\right)^{-3}$ which is non-zero and since both $c(\cdot)$, $c(\cdot)^{-1}$ and their derivatives are continuous within their domains, the transformation is a diffeomorphism.

Since by using the Navigation Transformation, we have mapped the workspace into the point world, we need to map the point world into the harmonic domain. This is achieved by implementing the transformation (7).

The image of the i'th obstacle's boundary in the harmonic domain becomes:

$$h_i = c^{-1}\left(p_i\right).$$

Define the switch function:

$$\sigma\left(x\right) \triangleq \frac{e^x}{1 + e^x} \tag{8}$$

This function maps the extended real number line to the interval $[0, 1]$.

Finally define the distortion function:

$$\sigma_d\left(x\right) \triangleq x^{2/k} \tag{9}$$

that will be used to render the destination configuration non-degenerate.

The final closed form Navigation Function is then constructed as the following composition:

$$\Theta\left(\cdot\right) = \left[\sigma_d \circ \sigma \circ \phi \circ c^{-1} \circ \Phi\right]\left(\cdot\right) \tag{10}$$

V. Analysis

In this section we will analyze the properties of the proposed function. In order to prove that the function (10) is a Navigation Function, we will need the following result:

Proposition 1 (Proposition 2.7 in [6]): Let $f_1, f_2 \subseteq \mathbb{R}$ be intervals, $\phi : \mathcal{P}^n \to f_1$ and $\sigma : f_1 \to f_2$ be analytic. Define the composition

$$\varphi \triangleq \sigma \circ \phi.$$

If σ is monotonically increasing on f_1, then the set of critical points of ϕ and φ coincide, $\mathcal{C}_\phi = \mathcal{C}_\varphi$ and the index of each point is identical, i.e. $\mathrm{index}(\phi)|_{\mathcal{C}_\phi} = \mathrm{index}(\varphi)|_{\mathcal{C}_\varphi}$.
Note that the functions in (5) and (8) are analytic. The following result is adapted from [6].

Proposition 2 (see also Proposition 2.6 in [6]): Let $\varphi : \mathcal{P}^n \to [0, 1]$ be a navigation function on $\mathcal{P}^n$, and $\eta : \mathcal{W} \to \mathcal{P}^n$ be a smooth diffeomorphism. Then

$$\tilde{\varphi} \triangleq \varphi \circ \eta$$

is a navigation function on $\mathcal{W}$.
The proof is the same as the one found in [6] since the requirements for the original proof are merely C^2 continuity.

Corollary 1 (see also Proposition 2.6 in [6]): Let $\varphi : \mathcal{P}^n \to [0, 1]$ be a smooth Morse function on $\mathcal{P}^n$, and $\eta : \mathcal{W} \to \mathcal{P}^n$ be a smooth diffeomorphism. Then

$$\tilde{\varphi} \triangleq \varphi \circ \eta$$

is a Morse function on $\mathcal{W}$, the critical points $\mathcal{C}_\varphi = \eta\left(\mathcal{C}_{\tilde{\varphi}}\right)$ of and index of the corresponding critical points is identical, i.e. $\mathrm{index}(\tilde{\varphi})|_{\mathcal{C}_{\tilde{\varphi}}} = \mathrm{index}(\varphi)|_{\mathcal{C}_\varphi}$.
The proof of this Corollary is part of the proof of Proposition 2.6 in [6] and will not be repeated here.

Proposition 3: For every valid workspace, there exists a finite $k_0(d, n)$ such that for any $k > k_0(d, n)$, function ϕ in (5) is Morse.

Proof: From the definition of harmonic functions we have that $\mathrm{trace}\left(\mathcal{H}_\phi\right) = \Delta\phi \triangleq 0$. Since the analysis is performed in 2 dimensions, in order for a harmonic function to have a degenerate critical point p_c, both eigenvalues of $\mathcal{H}_\phi\left(p_c\right)$ have to be zero. Since the Hessian is symmetric, this implies that the Hessian will be identically zero.

At a critical point we have that $\nabla\phi = 0$. Denote with $r_i = h - h_i$ and $r_d = h - h_d$ and $\hat{r}_i, \hat{r}_d$ their unit vectors respectively. From eq. (5), we have that at a critical point it will hold that:

$$k \frac{\hat{r}_d}{\|r_d\|} = \sum_{i=1}^{M} \frac{\hat{r}_i}{\|r_i\|} \tag{11}$$

The hessian of ϕ is given by:

$$\mathcal{H}_\phi = \frac{k}{\|r_d\|^2}\left(I - 2\hat{r}_d\hat{r}_d^T\right) - \sum_{i=1}^{M} \frac{1}{\|r_i\|^2}\left(I - 2\hat{r}_i\hat{r}_i^T\right) \tag{12}$$

From eq. (11) we get:

$$\frac{k^2}{\|r_d\|^2}\hat{r}_d\hat{r}_d^T = \sum_{i=1}^{M} \frac{1}{\|r_i\|^2}\hat{r}_i\hat{r}_i^T + 2A_s$$

where with the index s we denote the symmetric matrix, i.e. $A_s = \frac{1}{2}\left(A + A^T\right)$ and $A = \sum_{i \neq j} \frac{1}{\|r_i\|\|r_j\|}\hat{r}_i \hat{r}_j^T$.

Substituting the last in eq. (12) we get:

$$\mathcal{H}_\phi = \left(\frac{k}{\|r_d\|^2} - \sum_{i=1}^{M} \frac{1}{\|r_i\|^2}\right) I + 2\frac{k-1}{k}\sum_{i=1}^{M}\frac{1}{\|r_i\|^2}\hat{r}_i\hat{r}_i^T - \frac{4}{k}A_s$$

Let $m = \underset{i}{argmin}\{r_i\}$. Moreover assume that $d(k) = \min_{i \neq j}\{\|h_i - h_j\|\}$ to be the minimum allowable distance between h_i, h_j under the current selection of k. In order to show that $\mathcal{H}_\phi$ is non-degenerate, it suffices to show that there is at least one non-zero eigenvalue. We have that:

$$\hat{r}_m^T \mathcal{H}_\phi \hat{r}_m = \frac{k}{\|r_d\|^2} + \frac{k-2}{k}\frac{1}{\|r_m\|^2} - \dots$$
$$\dots - \sum_{i=1, i \neq m}^{M} \frac{1 - 2\frac{k-1}{k}\left(\hat{r}_i^T \hat{r}_m\right)^2}{\|r_i\|^2} - \frac{4}{k}\hat{r}_m^T A_s \hat{r}_m$$

As can be seen by eq. (11), as k increases, the same will be happening for $\frac{1}{\|r_m\|}$, since the rest $\frac{1}{\|r_i\|}$'s are bounded by $\frac{1}{d}$, where $d > d(k)$ is a fixed value denoting the current minimum distance between h_i's. Assuming $k > 3$ we have that:

$$\hat{r}_m^T \mathcal{H}_\phi \hat{r}_m > \frac{k}{\|r_d\|^2} + \frac{1}{3}\frac{1}{\|r_m\|^2} - \frac{n-1}{d^2} - \frac{4}{3}\frac{n-1}{dr_m}$$
$$= \frac{k}{\|r_d\|^2} + \frac{1}{3}\frac{1}{\|r_m\|}\left(\frac{1}{\|r_m\|} - \frac{4(n-1)}{d}\right) - \frac{n-1}{d^2}$$

Hence for every choice of d, there is a choice of $k \geq k_0$ such that $\|r_m\|$ can become arbitrarily small rendering the above expression positive. ■

Remark 1: Proposition 3 does not exclude the possibility of ϕ being Morse for values of $k < k_0$.

We have the following major result:

Proposition 4: For a valid workspace, there exists a positive k_0 such that for any $k > k_0$, function (10) is a Navigation Function.

Proof: Property (1) of Definition 4 is satisfied since $\Theta(\cdot)$ is a composition of smooth functions.

For Properties (2) and (3), first let us note that ϕ is undefined at singular points. However function $\varphi = \sigma_d \circ \sigma \circ \phi$ can be continuously extended to include the singular points where $+\infty$ is mapped to $+1$ and $-\infty$ is mapped to 0 as can seen by taking the limit of function (8) as x tends to $\pm\infty$. So even though function ϕ which is a harmonic function has no minima or maxima in its interior (property of harmonic functions [1]), function φ when continuously extended to include the destination configuration, it will have a minimum there. Since h_d is the unique singular point where $\phi \to -\infty$ then h_d will be the unique minimum of the continuously extended φ. Hence and φ is polar. This is true since the critical points that are inherited from the interior of the harmonic function ϕ by means of Proposition 1 are necessarily either saddle points or degenerate critical points. Since according to Proposition 3, function ϕ is Morse for $k > k_0$, there will be no degenerate critical points of ϕ. This in turn according to Proposition 1 implies that φ excluding h_d, will be Morse with the same critical points and with identical index of each critical point. To establish the non-degeneracy of the destination configuration, note that

$$\varphi = \frac{\|r_d\|^2}{\left(\prod_{i=1}^{n}\|r_i\| + \|r_d\|^k\right)^{2/k}}$$

Hence at the destination configuration we have that

$$\mathcal{H}_\varphi(h_d) = 2\prod_{i=1}^{M}\|r_i\|^{-2/k} \cdot I$$

which is a non-degenerate critical point. Then according to Corollary 1, since c^{-1} and Φ are diffeomorphisms, functions $\varphi \circ c^{-1}$ as well as Θ will be Morse (Property 3). The polarity (Property 2) is inherited by Θ from φ by means of Corollary 1.

Regarding Property 4, note that $\phi(h) \to \infty$, whenever h approaches the boundary of the harmonic domain. More specifically this happens when $\|h\| \to +\infty$ (according to condition 4) or when $h \to h_i$. However these are the points where the boundary of the workspace has been mapped through the diffeomorphic transformation $c^{-1} \circ \Phi$. Now at these points $\varphi \to 1$ due to the properties of function σ defined in 8. Hence $\Theta(\partial\mathcal{W}) = 1$ and the admissibility of $\Theta(\cdot)$ (Property (4)) has been established. ■

VI. Controller Design

Define

$$c_0(h) \triangleq \|\nabla\phi(h)\|^2$$

$$\theta(h) \triangleq -\frac{k}{\|r_d\|^2} - \sum_{i=1}^{M}\frac{1}{\|r_i\|^2} + 2\sum_{i=1}^{M}\frac{\left(\hat{r}_i^T \hat{r}_d\right)^2}{\|r_i\|^2}$$

$$\hat{u} = \frac{1}{\sqrt{2}}\begin{bmatrix} 1 & 1 \\ -1 & 1 \end{bmatrix}\hat{r}_d$$

$$\eta(h) = -\sum_{i=1}^{M}\frac{1}{\|r_i\|^2} + 2\sum_{i=1}^{M}\frac{\left(\hat{r}_i^T \hat{u}\right)^2}{\|r_i\|^2}$$

$$H_\varepsilon(x) = \begin{cases} 0 & x > \varepsilon \\ 1 & x \leq \varepsilon \end{cases}$$

We propose the following parameter update law:

Proposition 5: Let $\varepsilon = 0$ and K a positive gain. Then parameter k under the parameter update law:

$$\dot{k} = K \cdot H_\varepsilon(c_0(h)) \cdot H_\varepsilon(|\theta(h)| + |\eta(h)|) \qquad (13)$$

will remain bounded.

Proof: Choose h_c to be a critical point satisfying eq. (11) and as k is increasing, move h_c such that eq. (11) is satisfied. This means that $H_\varepsilon(c_0(h)) = 1$. Let $k(0) = M+1$. If h_c is a non-degenerate critical point, then the Hessian (eq. 12) will be non-degenerate. This means that the Hessian has a couple of non-zero eigenvalues of equal magnitude and the Hessian is sign indefinite. This implies that there exists a direction $\hat{v}$ such that $\hat{v}^T \mathcal{H}_\phi \hat{v} = 0$ and due to the symmetry of the Hessian and the orthogonality of the corresponding eigenvectors, it will also be true that $\hat{v}^{\perp T} \mathcal{H}_\phi \hat{v}^{\perp} = 0$. However selecting the direction $\hat{z} = \frac{1}{\sqrt{2}} \begin{bmatrix} 1 & 1 \\ -1 & 1 \end{bmatrix} \hat{v}$ we will get that $\hat{z}^T \mathcal{H}_\phi \hat{z} \neq 0$. Hence creating two quadratic forms $\hat{v}^T \mathcal{H}_\phi \hat{v}$ and $\hat{z}^T \mathcal{H}_\phi \hat{z}$ we are guaranteed that at least one of the two will be non-zero. Now observe that $\theta(h_c) = \hat{r}_d^T \mathcal{H}_\phi \hat{r}_d (h_c)$ and that $\eta(h_c) = \hat{u}^T \mathcal{H}_\phi \hat{u} (h_c)$. According to the previous analysis at least one of the two will be non-zero, hence $H_\varepsilon (|\theta(h_c)| + |\eta(h_c)|) = 0$ and $\dot{k} = 0$. Now in the case of a degenerate critical point the hessian will become zero, so both $\theta(h_c) = 0$ and $\eta(h_c) = 0$. Hence $H_\varepsilon (|\theta(h_c)| + |\eta(h_c)|) = 1$ and $\dot{k} = K$. However, according to Proposition 3 there exists a value of $k = k_0$, such that all the critical points are non-degenerate. This implies that under the proposed parameter adaptation law, parameter k will become at most k_0. ∎

Remark 2: Due to ϕ being a smooth function, there exists an ε-neighborhood around the critical point where the Hessian will have a behavior similar to that at the critical point. Hence for practical implementations, by choosing an ε small enough, the proposed parameter update law will perform "on-the-fly" tuning of ϕ when the system is in the neighborhood of a critical point that is (or is close to be) degenerate.

We have the following result about the kinematic system:

Proposition 6: Assume a workspace with n obstacles. Then system (1) under the control law:

$$u = -k_1 \nabla \Theta (x)$$

where k_1 a positive gain and Θ is a Navigation Function based on harmonic potentials as defined in (10), implementing the parameter update law defined in Proposition 5 with initial condition $k(0) = M + 1$, converges globally asymptotically.

Proof: By construction Θ with $k = M + 1$ is free of local minima. However if a system starts in the attractive submanifold of a critical point, it will converge to this critical point, where the parameter update law eq. (13) will be activated. Since k will increase the current position will no longer be a critical point, since the critical point condition eq. (11) will no longer be satisfied. If this procedure is repeated, an upper bound will be reached on k where the Θ is now guaranteed to be a Morse function. Hence the attractive submanifold for the saddle point will become a set of measure zero and by using $V = \Theta$ as a Lyapunov function candidate, we have that:

$$\dot{V} = -k_1 \|\nabla \Theta\|^2 \overset{a.e.}{<} 0$$

and the system will be globally asymptotically stable, almost everywhere[1]. ∎

For the dynamic system we have the following result:

[1]a.e.: almost everywhere except from a set of measure zero

Proposition 7: system (2) under the control law:

$$f = -k_1 \nabla \Theta (x) - c_1 \dot{x}$$

where k_1, c_1 are positive gains and Θ is a Navigation Function based on harmonic Potentials as defined in (10), implementing the parameter update law defined in Proposition 5 with initial condition $k(0) = M + 1$, converges globally asymptotically.

Proof: By construction Θ with $k = M + 1$ is free of local minima. However if a system starts in the attractive submanifold of a critical point, it will converge to this critical point, where the parameter update law eq. 13 will be activated. Since k will increase the current position will no longer be a critical point, since the critical point condition eq. (11) will no longer be satisfied. If this procedure is repeated, an upper bound will be reached on k where the Θ is now guaranteed to be a Morse function. Hence the attractive submanifold for the saddle point will become a set of measure zero and by using $V = k_1 \Theta + \frac{m}{2} \dot{x}^T \dot{x}$ as a Lyapunov function candidate, we have that:

$$\dot{V} = \dot{x}^T (k_1 \nabla \Theta + m \ddot{x})$$

Using eq. (2) we have that:

$$\dot{V} = \dot{x}^T (-(c + c_1) \dot{x}) = -(c_0 + c_1) \dot{x}^T \dot{x} \leq 0$$

By substituting the control law on eq. (2) we see that for $\dot{x} = 0$ this implies necessarily that $\nabla \Theta = 0$. But this is only true at the destination and at a set of measure zero. Hence we can write that:

$$\dot{V} \overset{a.e.}{<} 0$$

and the system is globally asymptotically stable. ∎

VII. Simulation Results

In order to demonstrate the effectiveness of the proposed approach, we have set up two sets of simulations, one with a kinematic system and one with a dynamic system.

In the first case, the control law of Proposition 6 was applied to system (1). The system was initially located at $x_0 = [3\,3.1]^T$ and the destination was $x_d = [0\,0]^T$. The gain was selected as $k_1 = 1$. Figure 1 depicts the workspace and the trajectory of the system under the proposed control law. As can be seen from Figure 1 the proposed methodology is successful in driving the system to its destination. In this simulation parameter k remained at its initial value, i.e. $k = 3$ throughout the evolution of the system. This means that either the system was sufficiently away from the attractive manifold of critical points or that Θ was indeed a Morse function when $k = 3$.

In the second case, the control law of Proposition 7 was applied to system (2). The system is initially located at $x_0 = [3\,3.1]^T$ and the destination is $x_d = [0\,0]^T$. The controller parameters were selected as $k_1 = 2$ and $c_1 = 0.2$. System parameters were selected as $m = 1$, $c = 1$. Figure 1 depicts the workspace and the trajectory of the system under the proposed control law. As can be seen from Figure 2 the proposed methodology is successful in driving the dynamic

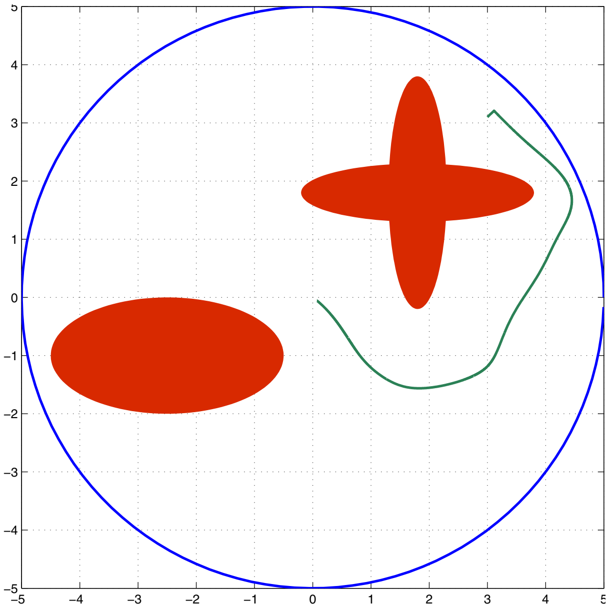

Fig. 1. Workspace and trajectory for the kinematic system

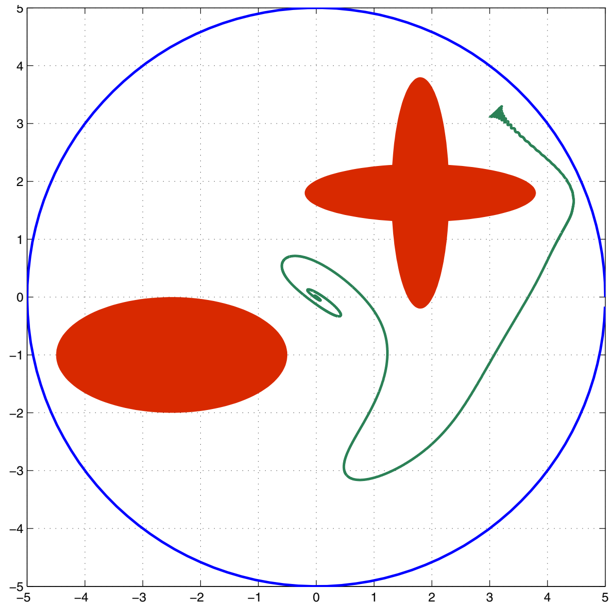

Fig. 2. Workspace and trajectory for the dynamic system

system to its destination. And in this simulation parameter k remained unchanged throughout the evolution of the system. This means that either the system was sufficiently away from the attractive manifold of critical points or that Θ was indeed a Morse function when $k = 3$.

A 3-D plot of the function Θ is shown in Figure 3.

VIII. Conclusions

In this paper a novel class of closed form Navigation Functions has been proposed that are based on harmonic potentials. This is the first time to the author's knowledge that closed form harmonic potentials functions are used to construct Navigation Functions and this was made possible due to the recently introduced Navigation Transformation. The constructed Navigation Functions are guaranteed by construction to be free of local minima due to the properties of the underlying harmonic functions without the need to tune any parameter. Moreover, it is shown that the proposed functions are guaranteed to be Morse functions under an appropriate choice of a parameter. A parameter tuning controller with performance guarantees is proposed to automatically tune the parameter in case the system evolves in the vicinity of a non-degenerate critical point. The performance of the proposed methodology is demonstrated through non-trivial computer simulations with systems with first and second order dynamics in a non-trivial workspace.

Further research includes extending the methodology to arbitrary dimensions, and to the case of moving obstacles and of multiple robots.

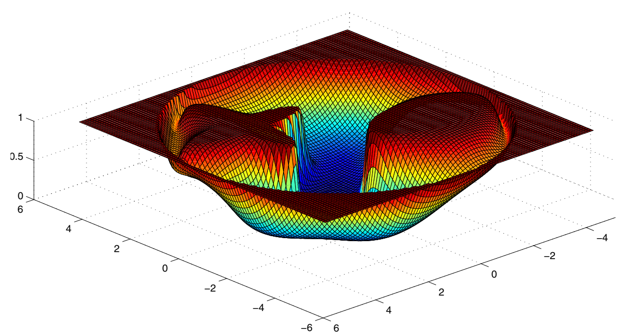

Fig. 3. Three-Dimensional representation of the harmonic function based Navigation Function Θ

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