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System Theory over Random Consensus Networks: Controllability and Optimality Properties

Marzieh Nabi-Abdolyousefi and Mehran Mesbahi

Abstract— **In this paper we present a framework for designing feedback controllers for directing a group of agents adopting consensus-type algorithms over random networks. In this direction, we first examine the pertinent necessary and sufficient conditions for controllability and observability of protocols over random networks. We then proceed to explore conditions on the underlying distribution for guaranteeing optimal infinite horizon linear quadratic regulators. The implementation of the proposed methodology is then discussed via an illustrative example.**

Index Terms— **Linear quadratic regulator (LQR), consensus, cooperative control, controllability, random networks**

I. Introduction

Distributed cooperative control for multiple autonomous dynamic system is an active research topic in systems and control theory [1]–[6]. The distributed nature of the control architecture necessitates an information-exchange network that affects the system theoretic properties of the over all networked system. It thus becomes pertinent to examine the behavior of the system as a function of the network structure, which in many situations of practical interest is dynamic. There are a number of venues to model dynamic networks, including viewing them as state-dependent, switching, and random. The latter approach can for example be used to model failures in the information exchange links between the agents. On the one hand, significant communication delays and data loss across the network are also common features of real systems. Such delays and packet drops have a negative impact on the performance of the designed networked systems. These types of inefficiencies can be considered with stochastic models. On the other hand, randomness might also be included by design in the operation of networked systems. For example, constraints on battery sources can motivate the network to switch on and off a group of sensors randomly during certain intervals. The randomness assumption on the existence of the information-exchange link between two dynamic units offers a natural modeling framework for situations of practical interest. However, this framework may be considered conservative in general. The ability of a pair of agents to pass relative states information among themselves depends on a number of factors, e.g., link failures, and packet drop-outs. In such a setup, the available bandwidth, power, and sensor geometry determine the parameters in the probabilistic model.

In this paper, we take a probabilistic approach to design a feedback control mechanism over a random network adopting a consensus protocol. Related to our work is the paper by Cao and Ren [8] that has studied optimal linear consensus algorithms for multi-vehicle systems in both continuous-time and discrete-time settings. We have been particularly inspired by Kalman in [7] which examined the linear-quadratic-regulator (LQR) for a random linear dynamic system in discrete-time setting. The organization of the paper is as follows. First, we explore the notion of controllability of random networks. Equipped with the stochastic notion of controllability and algebraic and probabilistic conditions for ensuring it, we then proceed to discuss the linear quadratic regulator problem for the controlled consensus over random networks. The rate of convergence of the protocol is also studied for special cases. An example demonstrating the application of the developed approach to control of random networks concludes the paper.

We use standard notation and terminology. We denote by $\mathcal{G} = (\mathcal{V}, \mathcal{E})$ the undirected simple graph with vertex set $\mathcal{V}$ and edge set $\mathcal{E}$. Two vertices u and v are called adjacent if $\{u, v\} \in \mathcal{E}$. The normalized Laplacian matrix for the graph $\mathcal{G}$ is denoted by $L(\mathcal{G})$; $\mathcal{G}(n, p)$ denotes the sample space of random graphs on n vertices while the existence of a pair of vertices in the set $\mathcal{E}$ is determined randomly with probability $p \in (0, 1]$ [9]–[11]. The operators $\mathbb{E}(w)$ and $\mathbb{P}\{w = \bar{w}\}$ refer to the expected value (ensemble average) of the random variable w and the probability that the random variable w is equal to $\bar{w}$, respectively. The notation $M(w_t)$ denotes that matrix M is a function of a random variable w_t which is written as M_t for the simplicity of the notation.

The transpose of vectors and matrices are denoted by the prime; thus $(x')' = x$. The norm $||x||$ is equal to $(x'x)^{1/2}$ and for a positive semidefinite matrix M, we use the special notation $||x||_M = (x'Mx)^{1/2}$. The eigenvalues of a matrix M are denoted by $\lambda_i[M]$. The following operations will prove to be useful throughout the paper.

Definition 1.1: Let $m_i \in \mathbb{R}^m$ denote the columns of the matrix $M \in \mathbb{R}^{m \times p}$ such that $M = [m_1, m_2, \ldots, m_p]$. Then forming the $mp-$vector by stacking the columns of M on top of each other defines the operator $\mathbf{vec}(M)$, i.e.,

$$\mathbf{vec}(M) = \begin{bmatrix} m_1 \\ \vdots \\ m_p \end{bmatrix} \in \mathbb{R}^{mp};$$

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The authors are with the Department of Aeronautics and Astronautics, University of Washington, Seattle, WA 98195-2400 USA (emails: mnabi+mesbahi@uw.edu)

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we refer to the resulting vector as M_v. The operator **mat** maps from vectors to matrices of prescribed dimensions, i.e., $m\times p$, formed by concatenating entries of the mp-vector, read from top to bottom, from left to right and putting them in p columns; thus $\mathbf{mat}(M_v) = M = [m_1, m_2, \ldots, m_p]$.

The computation of covariances in the linear quadratic design presented in this paper is facilitated by using the tensor notation. The tensor product $x \otimes y$ is defined as the outer product $\mathbf{vec}(xy')$. The tensor product $M \otimes N$ is equal to the Kronecker product of M and N. The tensor product of two linear transformation M and N is given by

$$(M \otimes N)(x \otimes y) = Mx \otimes Ny = \mathbf{vec}(Mxy'N').$$

For any three matrices M, N, and R by which the product matrix MNR is defined, we have

$$\mathbf{vec}(MNR) = (R' \otimes A)\mathbf{vec}(N).$$

We note that one can apply the $\mathbf{mat}$ operator on both sides of a vector identity to obtain a matrix identity.

II. PROBLEM STATEMENT

Consider the graph $\mathcal{G} = (\mathcal{V}, \mathcal{E})$ with $|\mathcal{V}| = n$ and the weighted consensus protocol

$$\dot{x}_i(t) = \sum_{\{i,j\}\in\mathcal{E}} \frac{1}{\sqrt{\mathbf{deg}\ i\ \mathbf{deg}\ j}}(x_j(t) - x_i(t)), \quad (1)$$

adopted by n-nodes, where $\mathbf{deg}\ i$ is the degree of node i and x_i is the state of the i-th node, e.g., its position, speed, heading, voltage, etc., evolving according to the weighted sum of the differences between the i-th node's state and its neighbors. Next, let a group of agents $\mathcal{I} \subset \mathcal{V}$ with cardinality $|\mathcal{I}| = r_{\mathcal{I}}$, "excite" the underlying coordination protocol by injecting signals to the network. Hence, the original consensus protocol from node i's perspective assumes the form

$$\dot{x}_i(t) = \sum_{\{i,j\}\in\mathcal{E}} \frac{1}{\sqrt{\mathbf{deg}\ i\ \mathbf{deg}\ j}}(x_j(t) - x_i(t)) + B_i u_i(t), \quad (2)$$

where $B_i = \beta_i$ if $i \in \mathcal{I}$, and zero otherwise. Without loss of generality, we can always assume that $\beta_i = 1$ and modify the control signal $u_i(t)$ as $\beta_i u_i(t)$ if necessary. The weights are defined to be $\frac{1}{\sqrt{\mathbf{deg}\ i\ \mathbf{deg}\ j}}$ for each neighbor node j. We arrive at the compact form of a linear time-invariant system,

$$\dot{x}(t) = -L(\mathcal{G})x(t) + Bu(t), \quad (3)$$

where $L(\mathcal{G}) \in \mathbb{R}^{n\times n}$ is the normalized Laplacian matrix. The input matrix $B \in \mathbb{R}^{n\times r_{\mathcal{I}}}$ where the j^{th} column has 1 at $\{ij\}$ entry if $i \in \mathcal{I}$ and zero otherwise and $u(t) = [u_1', u_2', \ldots u_{r_{\mathcal{I}}}']'$. Let us call $F = \{v \in \mathcal{V}$ such that $v \notin \mathcal{I}\}$ as the set of follower nodes and relabel the graph such that the nodes in $\mathcal{I}$ are the first $r_{\mathcal{I}}$ nodes and partition $L(\mathcal{G})$ as follows

$$L(\mathcal{G}) = \begin{bmatrix} \Delta_{\mathcal{I}} & \delta_{\mathcal{I}} \\ \delta_{\mathcal{I}}' & L_F \end{bmatrix},$$

where $\Delta_{\mathcal{I}}$, $\delta_{\mathcal{I}}$, and L_F capture the information-exchange links between the input nodes, the links between the input nodes and the set of followers, and the links between the followers, respectively.

A. Random Networks

In a random network, the existence of an edge between a pair of vertices in the set $\mathcal{V}$ is determined randomly with probability $p \in (0, 1]$ and independent of other edges. The sample space of such random graphs will be denoted by $\mathcal{G}(n, p)$. Note that the value of edge probability can be the same or distinct for all potential edges. This probability can also be fixed, or in more interesting scenarios, a function of the order of the graph, $p(n)$. Having embedded a random network in the dynamic system (3), it is convenient to consider an arbitrary sampling of the time axis at intervals $\delta > 0$ and monitor the trajectory $x(t) := x(\delta t)$ where t's are now assumed to be integers. We thus consider a random interactive network with associated normalized Laplacian matrix $L(w_t)$, where w_t is a sequence of mutually independent random events. It is assumed also that the random process is stationary, i.e., that the probability distribution of $L(w_t)$ does not depend explicitly on t.

The dynamics considered in (3) can be expressed in the sampled-data form

$$x(t+1) = A(w_t)x(t) + B(w_t)u(t), \quad (4)$$

where

$$\begin{aligned} A(w_t) &= e^{-L(w_t)\Delta}, \text{ and} \\ B(w_t) &= \left(\int_{(t-1)\Delta}^{t\Delta} e^{-L(w_\tau)\tau} d\tau\right) B. \end{aligned}$$

In order to consider the sampled-data setting for the Dirichlet dynamics, we need to partition the matrix exponential $e^{-L(w_t)\Delta}$. If the input nodes set, $\mathcal{I}$, is labeled as the first $r_{\mathcal{I}}$ nodes, the matrix exponential $e^{-L(w_t)\Delta}$ can be partitioned as

$$e^{-L(w_t)\Delta} = \begin{bmatrix} * & B(w_t)' \\ B(w_t) & A(w_t) \end{bmatrix}. \quad (5)$$

In a more general case, when the input nodes are not the first $r_{\mathcal{I}}$ nodes, the partitioning can be accomplished in the same manner as (5). Therefore, the dynamic system can be expressed as (4) with matrices described in (5). It is assumed also that (4) is stationary, i.e., the probability distributions of $A(w_t)$ and $B(w_t)$ do not depend on t. Whether we consider the consensus (3) or Dirichlet dynamics (5), we have a dynamic system described by random matrices $A(w_t)$ and $B(w_t)$, changing at every time interval based on the independent random events w_t. For the simplicity of notation, $A(w_t)$ and $B(w_t)$ will subsequently be referred to as A_t, and B_t, respectively.

III. CONTROLLABILITY OF STOCHASTIC SYSTEMS

Controllability and its dual notion of observability for networked systems are of fundamental importance among the many topics of interest in coordinated control. The basic issue in controllability for protocols evolving on random networks is whether it is possible to transfer any initial state to the desired state, in some stochastic sense, by applying judicious control input. This issue comprises the focus of this section.

In [13], it has been proven that a weak observability condition is necessary for the Kalman filter over a random network to converge almost surely (a.s.). Following the duality principle, in this section, we explore the notion of controllability of random networks. Equipped with the stochastic notion of controllability and algebraic and probabilistic conditions for ensuring it, we then proceed to discuss the linear quadratic regulator problem for the controlled consensus over random networks. We now introduce the weak controllability condition as follows.

Definition 3.1: Let $S_t = B_t B_t'$. Then the linear system (4), or equivalently, the sequence $\{(A_t, B_t),\ t \in \mathbb{Z}\}$, is called weakly controllable if for some $t \geq 1$,

$$\begin{aligned}\mathbb{P}\{\mathbf{det}\{S_t \quad &+ \quad A_t S_{t-1} A_t' + \dots \\ &+ \quad (A_t \dots A_2) S_1 (A_2' \dots A_t')\} \neq 0\} \neq 0. \quad (6)\end{aligned}$$

To explore the notion of controllability for random networks, let us first provide general definitions on stochastic controllability.

Definition 3.2: A stochastic system is said to be:

- Weakly state controllable if for all $x_0,\ x_1 \in \mathbb{R}^n$, and all $\epsilon \geq 0$, there exists a random time T a.s. finite and a control law u defined on $[0, T]$ such that
$$\mathbb{P}\{||x(T; x_o, u) - x_1|| \leq \epsilon\} > 0,$$
where $x(T; x_o, u)$ denotes the value at $t = T$ of the trajectory starting in x_o at $t = 0$ under the control u.
- State controllable if the above probability can be made equal to one.
- Strongly state controllable if the hitting time $T_H = \inf\{t > 0; ||x(t; x_o, u) - x_1|| \leq \epsilon\}$ has a finite expectation, i.e., $\mathbb{E}\{T_H\} < +\infty$.

The above notions of controllability are referred to as "weakly controllable," "controllable," and "strongly controllable," respectively, in the jump-parameter systems literature [14], [15]. This convention is partially adopted to avoid confusion with Definition 3.1. As we proved in [13], the condition in Definition 3.1 is the appropriate notion to guarantee that the state error defined as $x(T, x_0, u) - x_1$ converges almost surely, that is, for all $x_0,\ x_1 \in \mathbb{R}^n$, and $\epsilon \geq 0$, there exists T such that

$$\begin{aligned}\mathbb{P}\{||x(T, x_0, u) - x_1|| > \epsilon\} \quad &= \quad 0, \\ \text{or} \quad \mathbb{P}\{||x(T, x_0, u) - x_1|| \leq \epsilon\} \quad &= \quad 1. \quad (7)\end{aligned}$$

Thus weak controllability implies the system is state controllable with probability 1. In the next step, we give an insight that weakly state controllability might not be the appropriate notion for the development of a feedback regulator theory over random networks. In order to do this, we first notice that state controllability, strong state controllability and weak state controllability are analogous to controlled version of recurrence, positive recurrence, and nondegeneracy (or weak recurrence), respectively, of the Markov chain defined by (4). To clarify the analogy, let us start with recalling the classification of states in a Markov chain. A Markov chain on $\mathcal{D}$, the domain of its states, is called

- recurrent, if for all $i \in \mathcal{D}$ and for all t_0, there exists an almost surely finite random time $t > t_0$ such that $\mathbb{P}\{x(t) = i | x(t_0) = i\} = 1$, or equivalently, a state i is recurrent if $\mathbb{P}\{x(t) = i \ \text{ for infinitely many } t\} = 1$. The Markov chain is called recurrent if this condition holds for all $i \in \mathcal{D}$,
- weak recurrent or nondegenerate, if $\mathbb{P}\{x(t) = i | x(t_0) = i\} > 0$, or equivalently, a state i is weak recurrent if $\mathbb{P}\{x(t) = i \ \text{ for infinitely many } t\} > 0$. Then, the Markov chain is called weakly recurrent if this condition holds for all $i \in \mathcal{D}$,
- transient, if $\mathbb{P}\{x(t) = i | x(t_0) = i\} = 0$, or equivalently, if a state i is transient if $\mathbb{P}\{x(t) = i \ \text{ for infinitely many } t\} = 0$. The Markov chain is called transient if this holds for all $i \in \mathcal{D}$,
- positive recurrent, if in addition, the return time $T_R = \inf\{t > 0, x(t) = i, x(0) = i\}$ has a finite expectation.

Theorem 3.3 below shows that every state is either recurrent or transient. Let us define the random variable T_i as the first passage time to state i as

$$T_i(w_t) = \inf\{t \geq 1 : x(t) = i\}. \quad (8)$$

We now define inductively the rth passage time $T_i^{(r)}$ to state i by $T_i^{(0)}(w_t) = 0$, $T_i^{(1)}(w_t) = T_i(w_t)$, and, for $r = 0, 1, 2, \dots$,

$$T_i^{(r+1)}(w_t) = \inf\{t \geq T_i^{(r)}(w_t) + 1 : x(t) = i\}. \quad (9)$$

Let us also introduce the *number of visits* V_i to i, which can be written in terms of the indicator function

$$V_i = \sum_{t=0}^{\infty} 1_{\{x(t)=i\}}, \quad (10)$$

and note that

$$\mathbb{E}(V_i) = \sum_{t=0}^{\infty} \mathbb{P}\{x(t) = i\}. \quad (11)$$

Now we are ready to state the theorem.

Theorem 3.3: [7] The following dichotomy holds:

- If $\mathbb{P}\{T_i < \infty\} = 1$, then the state i is recurrent.
- If $\mathbb{P}\{T_i < \infty\} < 1$, then the state i is transient.

In particular, every state is either transient or recurrent.

Proof: If $f_i = \mathbb{P}\{T_i < \infty\}$, then we can prove by induction that we have $\mathbb{P}\{V_i > r\} = f_i^r$. Assuming this, if $\mathbb{P}\{T_i < \infty\} = 1$, then

$$\mathbb{P}\{V_i = \infty\} = \lim_{r \to \infty} \mathbb{P}\{V_i > r\} = 1$$

so i is recurrent and $\mathbb{E}(V_i) = \infty$.

On the other hand, if $\mathbb{P}\{T_i < \infty\} < 1$, then

$$\mathbb{E}(V_i) = \sum_{t=0}^{\infty} \mathbb{P}\{V_i > r\} = \sum_{t=0}^{\infty} f_i^r = 1/(1 - f_i) < \infty, \quad (12)$$

and hence $\mathbb{P}\{V_i = \infty\} = 0$ and i is transient. ∎

Therefore if the state i is weakly recurrent and

$$1 > \mathbb{P}\{x(t) = i \ \text{ for infinitely many } t\} > 0,$$

essentially the analogous controlled Markov chain built on (4) does not hit the desired state infinitely often. Now the question is whether weak state controllability or state controllability is a necessary condition for designing a feedback controller. In order to answer this question, let us restate the control problem of interest. The considered control problem is in fact a regulator problem, where we consider choosing a suitable control function $u(t)$ to ensure that every initial state $x(t_0) = x_0$ is returned to the reference signal $x = 0$ in a way that a performance index is minimized and the reference state is asymptotically stable. The reference signal $x = 0$ of (4) is asymptotically stable in the mean square sense (or in the norm) if

$$\begin{aligned} \mathbb{E}(||x(t,x_0)||^2) &< \infty \quad \text{for all } t \geq 0 \\ \text{and} \quad \lim_{t\to\infty} \mathbb{E}(||x(t,x_0)||^2) &= 0 \quad \text{for all} \quad x_0. \end{aligned} \tag{13}$$

Similarly, we have asymptotic stability with probability 1 if the relations

$$\begin{aligned} ||x(t,x_0)|| &< \infty, \quad \text{for all } t \geq 0 \\ \text{and} \quad \lim_{t\to\infty} ||x(t,x_0)||^2 &= 0 \end{aligned} \tag{14}$$

hold with probability 1.

Let $\mathfrak{C}_w$ consists of states that satisfy the weak state controllability in Definition 3.2. In the next step, following the same line of reasoning used in the classification of Markov chains, we prove that all states from the weak state controllability subset, $\mathfrak{C}_w$, do not hit the desired subspace infinitely often.

Theorem 3.4: If $x_1 \in \mathfrak{C}_w$, equivalently having $\mathbb{P}\{||x(T;x_o,u) - x_1|| \leq \epsilon\} > 0$, then there is no T a.s. finite, such that $x(T;x_o,u)$ hits $\mathfrak{C}_w$ infinitely often.

Proof: Let us introduce $T_1 = \inf\{t \geq 1 : x(t) = x_1\}$ and V_1 as the number of times that $x(T;x_o,u)$ hits the disk $D_\epsilon(x_1)$ when $||x(T+\delta;x_o,u) - x_1|| \leq ||x(T;x_o,u) - x_1||$ for $\delta > 0$. By $D_\epsilon(x_1)$, we mean a disk with radius ϵ centered at x_1. Then, define V_1 as

$$V_1 = \sum_{t=0}^{\infty} 1_{\{||x(T;x_o,u)-x_1|| \leq \epsilon\}} \tag{15}$$

and note that

$$\mathbb{E}(V_1) = \sum_{t=0}^{\infty} \mathbb{P}\{||x(T;x_o,u) - x_1|| \leq \epsilon\}. \tag{16}$$

We claim that $\mathbb{P}\{T_1 < \infty\} < 1$. The claim holds since $x_1 \in \mathfrak{C}_w$ and we know that $\mathbb{P}\{||x(T;x_o,u) - x_1|| \leq \epsilon\} > 0$. If $f_1 = \mathbb{P}\{T_1 < \infty\} < 1$, then

$$\mathbb{E}(V_1) = \sum_{t=0}^{\infty} \mathbb{P}\{V_1 > r\} = \sum_{t=0}^{\infty} f_1^r = 1/(1-f_1) < \infty,$$

therefore, $\mathbb{P}\{V_1 = \infty\} = 0$ and the system does not hit $D_\epsilon(x_1)$ infinitely often. ■

Therefore, for the control problem one should require that all states be in state controllability subset and not the weak controllability subset.

The next theorem provides an algebraic condition, in the probabilistic sense, for checking the weak controllability defined in Definition 3.1.

Theorem 3.5: The system (4) is weakly controllable if and only if for some $t \geq 1$,

$$\begin{aligned} \mathbb{P}\{\textbf{rank } (B_t, A_tB_{t-1}, A_tA_{t-1}B_{t-2}, \ldots, \\ A_t \ldots A_2B_1) = n\} \neq 0. \end{aligned} \tag{17}$$

Proof:

Sufficiency: The stochastic linear system (4) can be written as

$$\begin{aligned} x(t+1) &= A_tx(t) + B_tu(t) \\ &= A_tA_{t-1}\ldots A_0x(0) + A_t\ldots A_1B_0u(0) \\ &+ \ A_t\ldots A_2B_1u(1) + \ldots + B_tu(t) \\ &= A_tA_{t-1}\ldots A_0x(0) + \end{aligned}$$

$$\begin{bmatrix} A_t\ldots A_1B_0, & A_t\ldots A_2B_1, & \ldots, & B_t \end{bmatrix} \begin{bmatrix} u(0) \\ u(1) \\ \vdots \\ u(t) \end{bmatrix}. \tag{18}$$

If for some $t \geq 1$,

$$\mathbb{P}\{\textbf{rank } (B_t, A_tB_{t-1}, A_tA_{t-1}B_{t-2}, \ldots A_t\ldots A_2B_1) = n\}$$

is not identically zero, then for that specific t, the input vector $\begin{bmatrix} u'(0) & u'(1) & \ldots & u'(t) \end{bmatrix}'$ can be determined with probability 1, making the system (18) controllable.

Necessity: From (18), one may conclude that the controllable subspace is the range of $C = \begin{bmatrix} A_t\ldots A_1B_0, & A_t\ldots A_2B_1, & \ldots, & B_t \end{bmatrix}$ for a given t. The selected $u(i)$ in this way cannot generate a subspace larger than the range of the controllability matrix C. Therefore, **rank** $C \leq n$. If **rank** $C < n$, then there exists some x_1 and $t > 0$ that does not belong to the range of C and, for that x_1, one has

$$\mathbb{P}(||x(T,x_0,u) - x_1|| \leq \epsilon) = 0,$$

for any u, T, and x_0, which contradicts the assumption. Therefore, **rank** $C = n$. ■

In [13], we considered the scenario when A_i's are invertible. In this case, if $\sum_{i=1}^{t} B_iB_i'$ is full rank, the rank condition in (17) can be replaced by checking the irreducibility of $\bar{A} = \mathbb{E}(A_t)$ which holds for certain classes of networks such as Erdős-Renyi and Watts-Strogatz models [11].

IV. LQR OVER RANDOM NETWORKS

The control problem we now consider is choosing a suitable control function $u(t)$ to ensure that every initial state $x(t_0) = x_0$ is returned to the reference signal $x = 0$ in a way that the performance index

$$\begin{aligned} \mathbb{E}\{\rho^T||x(T,x_0)||_S^2 \quad + \quad \sum_{t=t_0}^{T-1} \rho^t||x(t,x_0)||_Q^2\}, \\ \rho > 0, Q > 0, S \geq 0, \end{aligned} \tag{19}$$

is minimized and the reference state is stable, in a probabilistic sense, as previously discussed. Since w_t is a sequence of mutually independent random events, $u(t)$ depends only on $x(t)$ and the knowledge of $x(t-1), x(t-2), \ldots$ carries no additional information about the future evolution of (4). Let the control law be the feedback

$$u(t) = -K(t)x(t), \tag{20}$$

leading to the closed loop matrix A_{c_t} defined as

$$A_{c_t} = A_t - B_tK(t). \tag{21}$$

Given the feedback law in (20), the system (4) generates the random sequence $x(t_0), x(t_0+1), \ldots$, referred to as the "motion" of (4) and denoted by $x(t \geq t_0, x_0)$.

It is well known that if the control law is stationary, the motion of (4) is asymptotically stable in the mean square sense if and only if

$$|\lambda_i[\mathbb{E}(A'_{c_t} \otimes A'_{c_t})]| < 1 \quad (i = 1, \ldots, n^2). \tag{22}$$

In [7], it has been shown that, for the specific system considered here, the same condition also implies asymptotic stability with probability 1.

Let the minimum value of (19) be $V(x_0, t_0, T)$, $t_1 = t_0 + 1$, $x(t_1) = x_1$. The principle of optimality [17] and the mutual independence of w_t now imply that

$$\begin{aligned} V(x_0, t_0, T) &= \min_u \{\rho \mathbb{E}[V(x_1, t_1, T] + ||x_0||_Q^2\}, \\ &= \min_u \{\rho \mathbb{E}[\rho V(Ax_0 + Bu, t_1, T)] + ||x_0||_Q^2\}. \end{aligned}$$

We see from (19) that $V(x, T, T) = ||x||_S^2$. Assume by induction that

$$\begin{aligned} V(x, t, T) = ||x||^2_{P(t,T)} = \min_u \{\rho \mathbb{E}[||Ax &+ Bu||^2_{P(t+1,T)}] \\ &+ ||x||_Q^2\}, \end{aligned}$$

where $P(T, T) = S$. The solution of the optimization problem in (19) for $T < \infty$ has been introduced in [7] as

$$\begin{aligned} u(t) &= K(t)x(t) \\ &= -\mathbb{E}[B'P(t+1,T)B]^{\dagger}\mathbb{E}[B'P(t+1,T)A]x(t), \\ P(t,T) &= \rho \mathbb{E}[A'_{c_t} P(t+1,T) A_{c_t}] + Q, \end{aligned} \tag{23}$$

where A_{c_t} is defined in (21) and $\dagger$ denotes the generalized inverse operator. Considering the stationarity of w_t we can now write

$$P(t, T) \equiv P(0, T-t) \equiv P(T-t)$$

and from (23), it can be seen that

$$P(T+1) = \rho \mathbb{E}[A'_c P(T) A_c] + Q, \text{ and } P(0) = S. \tag{24}$$

The more interesting case for us is where $T = \infty$. Let us also set $S = 0$. The next theorem describes this case.

Theorem 4.1: [7] Equation (24) has a fixed point P_* if and only if the regulator problem has an optimal solution in the limit $T = \infty$; P_* is necessarily unique and all iterates of $P(t, P_0)$ of (24) starting at $P_0 \geq 0$ converge to P_*. Moreover, $||x_0||^2_{P_*}$ is the optimal index. The optimal control law is constant and is given by

$$K_* = (\mathbb{E}[B'_t P_* B_t)^{-1} \mathbb{E}[B'_t P_* A_t], \tag{25}$$

where

$$\begin{aligned} P_* &= \rho \mathbb{E}(A_{c_t} P_* A_{c_t}) + Q \\ &= \rho\, \mathbf{mat}[\mathbb{E}(A'_{c_t} \otimes A'_{c_t}) \mathbf{vec}(P_*)] + Q \\ &= \sum_{t=0}^{\infty} \rho^t\, \mathbf{mat}[(\mathbb{E}[A'_{c_*} \otimes A'_{c_*}])' \mathbf{vec}(Q)]. \end{aligned} \tag{26}$$

The existence of the fixed point P_* and the optimality of the solution (25) have been discussed in [7]. Now the essential question is whether there exists a control law which provides closed loop stability. The next theorem provides the necessary and sufficient condition to answer this question. It also provides necessary and sufficient conditions for when the regulator problem has an optimal solution.

Theorem 4.2: [7] Let us denote the maximum degree of stability for the system (4) as $\rho_{\max}$. The optimal regulator problem in the limit $T = \infty$ has a unique solution if and only if $\rho < \rho_{\max}$. The system (4) can be made asymptotically stable in the mean square sense if and only if $\rho_{\max} > 1$. Any optimal system with $1 \leq \rho < \rho_{\max}$ is asymptotically stable in the mean square sense and is therefore asymptotically stable with probability 1.

We note that $\rho_{\max}$ can be evaluated by successive iterations. Furstenburg and Kesten [16] have also suggested an expression for calculating $\rho_{\max}$. In this direction, let us define $\mu(K)$ as

$$\mu(K) = \lim_{q \to \infty} q^{-1} \{\mathbb{E} \log ||A_{c_q} \ldots A_{c_1}||\},$$

which can also be defined as

$$\lim_{q \to \infty} q^{-1} \log ||A_{c_q} \ldots A_{c_1}||\}$$

with probability 1. Given K, there is a corresponding $\mu(K)$, such that one can determine $\rho_{\max}$ as

$$\rho_{\max}^{-1/2} = \inf_K \{e^{\mu(K)}\}. \tag{27}$$

Let us now proceed to get a better understanding of the parameter $\rho_{\max}$. In order to estimate $\rho_{\max}$, the first step is estimating $||A_{c_t}|| = ||A_t - B_t K_t||$. Let us consider the case where $K_t = K_*$ and $B_t = I$ where I is $n \times n$ identity matrix. Therefore,

$$\begin{aligned} B_t K_* &= B_t[\mathbb{E}(B_t^T P_* B_t)]^{-1} \mathbb{E}(B_t^T P_* A_t) \\ &= [\mathbb{E}(P_*)]^{-1} \mathbb{E}(P_*) \mathbb{E}(A_t) = \mathbb{E}(A_t). \end{aligned}$$

Consequently,

$$||A_{c_t} - B_t K_*||_2 = ||A_t - \mathbb{E}(A_t)||_2 = \lambda_{\max}\{A_t - \mathbb{E}(A_t)\}.$$

The next step is to estimate $\lambda_{\max}\{A_t - \mathbb{E}(A_t)\}$. Assume $0 < \delta \ll 1$ and estimate $A_t \approx I - \delta L_t$. Therefore, $\mathbb{E}(A_t)$ can be estimated as $I - \delta \mathbb{E}(L_i)$. Now let us estimate $||A_t - B_t K_*||_2$ where $B_t = I$ as

$$\begin{aligned} ||A_t - B_t K_*||_2 &= \lambda_{\max}(A_t - \mathbb{E}(A_t)) \\ &= \lambda_{\max}(I - \delta L_t - I + \delta \mathbb{E}(L_t)) \\ &= -\delta \lambda_{\max}(L_t - \mathbb{E}(L_t)). \end{aligned}$$

Chung in [18] has shown that if $pn \gg \ln n$, then with probability at least $1 - 1/n$, one has

$$|\lambda_k(L_t) - \lambda_k(\mathbb{E}(L_t))| \leq 3\sqrt{\frac{6 \ln 2n}{pn}},$$

where L_t is the normalized Laplacian. Then,

$$\begin{aligned} -3\sqrt{\frac{6 \ln 2n}{pn}} &\leq \lambda_{\max}(L_t) - \lambda_{\max}(\mathbb{E}(L_t)) \\ &\leq \lambda_{\max}\left(L_t - \mathbb{E}(L_t)\right). \end{aligned} \tag{28}$$

Now by multiplying (28) by $-\delta$ we get

$$3\delta\sqrt{\frac{6 \ln 2n}{pn}} \geq -\delta \lambda_{\max}(L_t - \mathbb{E}L_t) = ||A_t - B_t K_*||_2.$$

Let us estimate $q^{-1}\mathbb{E}\{\log ||A_{c_q} \ldots A_{c_1}||\}$ as

$$\begin{aligned} q^{-1} \quad & \mathbb{E}\{\log ||A_{c_q} \ldots A_{c_1}||\} \leq q^{-1} \mathbb{E}\{\sum_{t=1}^{q} \log ||A_{c_t}||\} \leq \\ q^{-1} \quad & \{q \log \left(3\delta\sqrt{\frac{6 \ln 2n}{pn}}\right)\} = \log \left(3\delta\sqrt{\frac{6 \ln 2n}{pn}}\right). \end{aligned}$$

For the problem set-up in (4) with normalized Laplacian, we can show $\mu(K) < 0$ where the control gain $K = 0$. Therefore $\log\left(3\delta\sqrt{\frac{6\ln 2n}{pn}}\right) < 0$ and $3\delta\sqrt{\frac{6\ln 2n}{pn}} < 1$ which implies that

$$p > \frac{54\delta \ln 2n}{n} \quad \text{and} \quad p > \frac{\ln n}{n}.$$

Briefly, if $p > \max\{\frac{54\delta\ln 2n}{n}, \frac{\ln n}{n}\}$, then $\mu(K_*) \leq \log\left(3\delta\sqrt{\frac{6\ln 2n}{pn}}\right)$. Therefore,

$$\frac{1}{\sqrt{\rho_{\max}}} = \inf_K e^{\mu(K)} \leq e^{\mu(K_*)} \leq 3\delta\sqrt{\frac{6\ln 2n}{pn}},$$

which provides an upper bound for $\rho_{\max} \geq \frac{pn}{54\delta^2 \ln 2n}$.

In the next section, we implement the proposed controller for a randomly evolving network according to the Erdős-Renyi distribution.

V. An Example

Consider a group of seven agents, coordinating their orientations to achieve a particular alignment over a random information-exchange network. Values of K_* and P_* in (25) and (26) can be calculated off-line based on the distributions for the matrices A_t and B_t.

Fig. 1 shows the evolution of the information graph with the edge probability $p = 0.3$ for the first 6 seconds of the simulation. The crossed node at each interval acts as the input to the network, which is selected uniformly from the set of possible input nodes. Fig. 2 (a) and (b) demonstrate the convergence of the states to the reference signal $x = 0$ when the random network is running the consensus protocol with different rates of convergence dictated by the parameter ρ. The proposed analysis suggests that with probability at least $6/7$ the maximum degree of stability, $\rho_{\max}$, is greater than 1600 if the input matrix is the identity matrix.

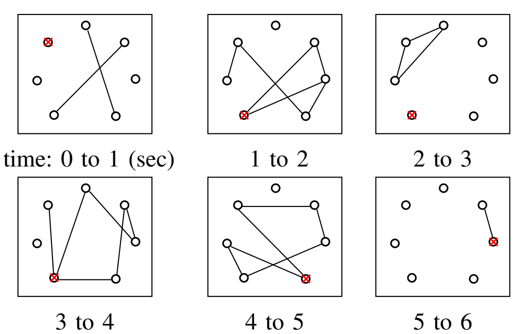

Fig. 1. Behavior of the random network evolving based on Erdős- Renyi distribution with $p = 0.3$ in the first 6 intervals. The crosses are the input nodes.

VI. Conclusion

The necessary and sufficient conditions for the controllability of random consensus networks generated based on Erdős-Renyi distribution are examined in this paper. Based on the controllability condition, the stability properties of the optimal linear quadratic regulator are also discussed.

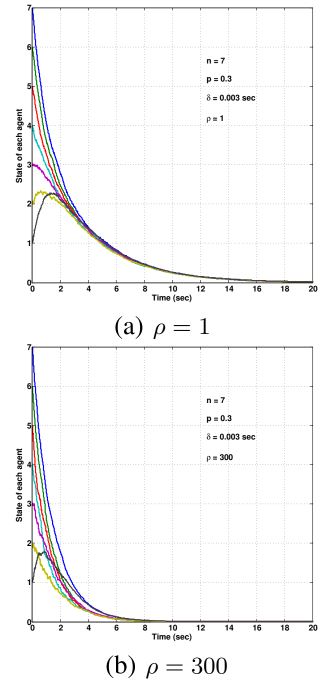

Fig. 2. Convergence of the states to the reference signal $x = 0$ with $p = 0.3$ with different rates of convergence.

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