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Synthesis of PID Controllers with Guaranteed Non-overshooting Transient Response

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Abstract— **This paper presents a new method for approximating the set of PID controllers satisfying a class of transient specifications. The problem of designing a controller to satisfy transient specifications such as the maximum allowable overshoot to a given input or the response being required to be within an envelope can be cast as a problem of guaranteeing the impulse response of an appropriate closed loop error transfer function to be non-negative. Stabilizing PID controllers for Linear Time Invariant (LTI) systems can be synthesized as a union of convex polygons in $k_i - k_d$ space for k_p's lying in a specific range. In this paper, we provide a method to restrict the stabilizing set for LTI systems further by using Widder's theorem and Markov-Lucaks representation for polynomials that are non-negative on the positive real axis. Widder's theorem provides necessary and sufficient conditions for the error response to be non-negative and upon an application of Widder's theorem, we obtain a sequence of polynomials, whose coefficients are polynomial functions of k_p, k_i and k_d to be non-negative. For every polynomial in the sequence and for a specified k_p, using Markov-Lucaks theorem and Minkowski's projection, we obtain a polynomial inequality in k_i and k_d that must be satisfied by every controller satisfying the desired transient specification. We also provide a method to arbitrarily tighten this set of desired controllers.**

I. INTRODUCTION

The problem of designing stabilizing controllers with guaranteed specific transient response specifications is important for practical applications. One such specification of the transient response is the amount of allowable overshoot/undershoot to a specific input signal such as a unit step input. Several results on the problem of achieving non-overshooting step response have been provided in [9]-[13]. For the discrete-time systems Deodhare and Vidyasagar [8] showed that a non-overshooting step response is achievable through synthesizing a deadbeat closed loop system. However, their results for the continuous-time Linear Time Invariant (LTI) systems can lead to controllers with irrational transfer functions. Darbha and Bhattacharyya [7] showed that a non-overshooting response can be achieved by proper, rational two parameter controllers. The results in [13], [5], and [6] shows the existence of a high order two-parameter stabilizing controller that can guarantee a monotonically increasing step response. The problem of controlling the transient response of discrete-time LTI systems using the non-negativity of polynomials with coefficients depend polynomially on the controller parameters is studied in [2]. Fixed order and PID controller synthesis for achieving transient specifications in LTI systems was considered in [14]. Fixed structure controllers such as PID controllers are widely being employed in industrial control applications. Recent results on the synthesis of PID controllers are provided in [1].

This paper in organized as follows: In section II, the main results are provided. In section III, an example of synthesis of a PID controller is presented. In section IV, we conclude by pointing the extension of the results to discrete-time LTI systems.

II. MAIN RESULTS

In this section, we will restrict ourselves to the response of finite-dimensional, continuous-time LTI systems to a unit step input. The procedure for synthesizing a controller to satisfy a desired transient specification such as overshoot to any other specified input signal is similar to the case when the input signal is a step input. Since it is also one of the most widely used transient specifications, we will concentrate on this transient specification and focus on synthesizing an outer approximation of the set of stabilizing PID controllers that satisfy the given transient specification. By an outer approximation, $\mathscr{S}_{outer}$, we mean the following: If C is a stabilizing PID controller that satisfies the desired transient specification, then $C \in \mathscr{S}_{outer}$. It is possible that $\mathscr{S}_{outer}$ has stabilizing PID controllers that do not satisfy the specification. We provide a way to "cut" off such controllers, thereby providing a way to refine our outer approximation. Since this procedure of cutting off undesirable controllers from the outer approximation can be repeated indefinitely, we also show that we can tighten the outer approximation arbitrarily.

A. An Outer Approximation for the Non-overshooting Step Response Region

Consider the unity feedback control system shown in Fig.1 where $P(s) = N_p(s)/D_p(s)$ represents the plant transfer function and $C(s)$ represents a PID controller; i.e. $C(s) = (k_d s^2 + k_p s + k_i)/s$.

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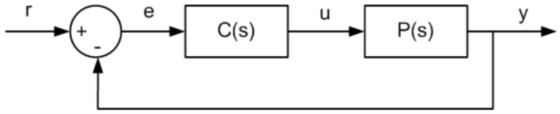

Fig. 1. Schematic of a unity feedback control system.

The set of all stabilizing PID controllers in the $k_i - k_d$ space, for a given value of k_p in an admissible range, can be constructed as the union of convex polygons [1]. Each polygon is the solution to a set of linear inequalities in the controller parameters k_i and k_d. Now the problem is to restrict the stability region further to find the set of PID controllers that also satisfy non-overshooting step response of the closed loop system.

The error function, defined as $e(t) = r(t) - y(t)$, is non-negative for $t \geq 0$ when response is non-overshooting. The work of Berstein and Widder in [3] provides necessary and sufficient conditions for the non-negative impulse response of a rational, proper transfer function in terms of the derivatives of the transfer function. These conditions require that a sequence of polynomials must have no real, positive zeros:

Theorem 1. Given $D(s,\mathbf{K})$ is Hurwitz and $H(s,\mathbf{K}) = \frac{N(s,\mathbf{K})}{D(s,\mathbf{K})}$, denote by $h(t)$ the impulse response of $H(s,\mathbf{K})$. Then $h(t) \geq 0$ for all $t \geq 0$ if and only if

$$H_k(s,\mathbf{K}) = (-1)^k \ \frac{d^k H(s,\mathbf{K})}{ds^k}$$

must have no real, positive zeros for $\forall k \geq 0$ and $\forall s \geq 0$.

The necessity of this result can be easily seen by recollecting that the Laplace transformation of $th(t)$ is $-\frac{dH(s)}{ds}$ and that for any integer $k \geq 0$, $t^k h(t) \geq 0$ if and only if $h(t) \geq 0$. The sufficiency part of the result will be useful for showing that the procedures we develop can be used to arbitrarily tighten the outer approximation we are seeking.

The Markov-Lucaks theorem [4] provides a sum-of-square representation for non-negative polynomials on any interval of the real axis:

Theorem 2. A polynomial $h(x) = \sum_{q=0}^{n} a_q x^q$ is non-negative on the interval $[0,\infty)$ if and only if there exists polynomials $f(x)$ of degree at most $\frac{n}{2}$ and $g(x)$ of degree at most $\frac{n-1}{2}$ such that $h(x) = f^2(x) + x g^2(x)$.

For the purpose of characterizing the non-overshooting step response of the feedback control system shown in Fig.1, one may consider the error transfer function

$$E(s,\mathbf{K}) = \frac{N_E(s,\mathbf{K})}{D_E(s,\mathbf{K})} = \frac{sD_p(s)}{(k_d s^2 + k_p s + k_i)N_p(s) + sD_p(s)} \frac{1}{s}. \tag{1}$$

Using theorem 1, the error signal $e(t)$ is non-negative for all $t \geq 0$ if and only if

$$E_k(s,\mathbf{K}) = (-1)^k \ \frac{d^k E(s,\mathbf{K})}{ds^k} \geq 0 \ , \ \forall k \geq 0 \text{ and } \forall s \geq 0. \tag{2}$$

Let us write $E_k(s,\mathbf{K})$ as

$$\begin{aligned} E_k(s,\mathbf{K}) &= \frac{N_{E_k}(s,\mathbf{K})}{D_{E_k}(s,\mathbf{K})} \\ &= \frac{\alpha_n(\mathbf{K})s^n + \alpha_{n-1}(\mathbf{K})s^{n-1} + \ldots + \alpha_1(\mathbf{K})s + \alpha_0}{D_{E_k}(s,\mathbf{K})}. \end{aligned} \tag{3}$$

where $D_{E_k}(s,\mathbf{K})$ is of the form $(D_E(s,\mathbf{K}))^{2k}$ and thus is non-negative for all $s \geq 0$. Using theorem 2, in order to have $N_{E_k}(s,\mathbf{K}) \geq 0$ there must exists $f_k(s,\mathbf{K})$ and $g_k(s,\mathbf{K})$ such that

$$N_{E_k}(s,\mathbf{K}) = f_k^2(s,\mathbf{K}) + s g_k^2(s,\mathbf{K}). \tag{4}$$

This is equivalent to the existence of positive semi-definite matrices $F_k(\mathbf{y}) \succeq 0$ and $G_k(\mathbf{z}) \succeq 0$ such that, for an appropriate l_k,

$$\begin{aligned} N_{E_k}(s,\mathbf{K}) = {} & \alpha_{r_k}(\mathbf{K})s^{r_k} + \alpha_{r_k-1}(\mathbf{K})s^{r_k-1} + \ldots + \alpha_1(\mathbf{K})s + \alpha_0 \\ = {} & [1,s,s^2,\ldots,s^{l_k}]F_k(\mathbf{y})[1,s,s^2,\ldots,s^{l_k}]^T \\ & + s[1,s,s^2,\ldots,s^{l_k}]G_k(\mathbf{z})[1,s,s^2,\ldots,s^{l_k}]^T \end{aligned} \tag{5}$$

where $\mathbf{y}$ and $\mathbf{z}$ are the vectors of the Markov-Lucaks variables and

$$\begin{aligned} F_k(\mathbf{y}) &= y_1 F_{k,1} + y_2 F_{k,2} + \ldots, \\ G_k(\mathbf{z}) &= z_1 G_{k,1} + z_2 G_{k,2} + \ldots \end{aligned} \tag{6}$$

are symmetric matrices of dimension (l_k+1) by (l_k+1). The right hand side of (5) is linear in Markov-Lucaks variables; however, the left hand side is linear in the controller parameters $\mathbf{K}$ for the first derivative of $E(s,\mathbf{K})$, is quadratic in $\mathbf{K}$ for the second derivative of $E(s,\mathbf{K})$ and so on.

Thus, one can construct a sequence of polynomial matrix inequalities where for the first derivative of the transfer function it reduces to Linear Matrix Inequalities (LMIs), for the second derivative of the transfer function reduces to Quadratic Matrix Inequalities (QMIs), and so on. Let us rewrite (5) as follows, after equating the same powers of s,

$$\mathbb{E}_{k,j}(k_p^*, k_i, k_d) = \mathbb{L}_{k,j}(\mathbf{y},\mathbf{z}), \ k = 0,1,2,\ldots \ , \ j = 0,1,2,...,r_k \tag{7}$$

where $\mathbb{E}_{k,j}(k_p^*, k_i, k_d)$ and $\mathbb{L}_{k,j}(\mathbf{y},\mathbf{z})$ denote the corresponding coefficients of s in (5) for the k-th derivative of the error transfer function with r_k as the highest power of s. The terms $\mathbb{E}_{k,j}(k_p^*, k_i, k_d)$ are polynomials on the controller parameters $\mathbf{K}$ and the terms $\mathbb{L}_{k,j}(\mathbf{y},\mathbf{z})$ are linear on the Markov-Lucaks variables $\mathbf{y},\mathbf{z}$. We also observe that $\mathbb{L}_k(\mathbf{0},\mathbf{0}) = \mathbf{0}$ for $k = 0,1,2,\ldots$.

Now, the problem of finding the set of stabilizing PID controllers with guaranteed non-overshooting step response can be expressed as a sequence of feasibility problems. For a given value of k_p^*, one may solve the following feasibility problem to obtain the set of all feasible (k_i, k_d) for which the stability and the non-overshooting step response of the closed loop system are guaranteed:

Feasibility Problem: Find all feasible values of $k_i, k_d, \mathbf{y}, \mathbf{z}$ subject to

$$\begin{aligned}&\mathbb{E}_{k,j}(k_p^*, k_i, k_d) = \mathbb{L}_{k,j}(\mathbf{y}, \mathbf{z}),\\ &F_k(\mathbf{y}) \succeq 0,\ G_k(\mathbf{z}) \succeq 0,\ k = 0,1,2,\dots,\ j = 0,1,2,\dots,r_k,\\ &\mathbb{S}_q(k_p^*, k_i, k_d) \leq 0,\quad q = 1,2,\dots,m \end{aligned} \tag{8}$$

where the last constraint is the stability constraint determined by m number of linear inequalities in k_i and k_d which is either a single convex polygon or a union of convex polygons. If we were to represent by $\mathscr{S}_{outer}^k$, the set of (k_p, k_i, k_d) satisfying the above set of inequalities, then, for every integer $k \geq 0$, the set $\mathscr{S}_{outer}^k$ is an outer approximation; furthermore, the set $\mathscr{S}_{des} := \cap_k \mathscr{S}_{outer}^k$ is also an outer approximation. From the sufficiency portion of Widder's theorem, in fact, it is the desired set of stabilizing PID controllers satisfying the given transient specification.

The non-overshooting step response region is obtained by considering an infinite number of derivatives of the error transfer function; thus, an outer approximation of the actual region may be obtained by solving the feasibility problem (8) for a finite number of the derivatives of the error transfer function.

Since solving (8) requires finding all feasible values of the controller parameters and the Markov-Lucaks variables, a more practical and efficient approach is to find an outer approximation of such a (possibly non-convex) set with sufficient number of cutting hyperplanes.

Lemma 1. Let k be given. For a given k_p^*, let k_i^* and k_d^* be stabilizing integral and derivative gains. If there is no solution corresponding to (8), then there exists a valid inequality in k_i and k_d to the set $\mathscr{S}_{des} \cap \{k_p = k_p^*\}$.

Proof. Suppose that for the selected values of k_p^*, k_i^* and k_d^*; (8) is infeasible. This is equivalent to the infeasibility of the LMI obtained by augmenting the constraints $k_i = k_i^*$, $k_d = k_d^*$ to (8). By the theorem of alternatives, there exists vector λ of an appropriate dimension and symmetric positive semi-definite matrices $Q_{k,1} \succeq 0$, $Q_{k,2} \succeq 0$, such that the following alternative problem is feasible:

$$\begin{aligned}&g(\lambda, Q_{k,1}, Q_{k,2}) < 0,\\ &Q_{k,1} \succeq 0,\ Q_{k,2} \succeq 0,\ k = 0,1,2,\dots\end{aligned} \tag{9}$$

with

$$\begin{aligned}g(\lambda, Q_{k,1}, Q_{k,2}) = \inf_{\mathbf{y},\mathbf{z}\in\mathfrak{D}} \{&\lambda \cdot [\mathbb{E}_k(k_p^*, k_i^*, k_d^*) - \mathbb{L}_k(\mathbf{y}, \mathbf{z})]\\ &+ (Q_{k,1} \cdot F_k(\mathbf{y})) + (Q_{k,2} \cdot G_k(\mathbf{z}))\}\end{aligned}$$

where $\mathfrak{D}$ is the domain of the constraints in the original feasibility problem (8). Let $\mathscr{F}_{dual}(k_p^*, k_i^*, k_d^*)$ be the set of dual variables $(\lambda, Q_{k,1}, Q_{k,2})$ for which the function $g(\lambda, Q_{k,1}, Q_{k,2})$ is well defined. This set is non-empty by the theorem of alternatives. If $(\lambda, Q_{k,1}, Q_{k,2}) \in \mathscr{F}_{dual}$, then one may express

$$\begin{aligned}g(\lambda, Q_{k,1}, Q_{k,2}) = &\lambda \cdot [\mathbb{E}_k(k_p^*, k_i^*, k_d^*) - \underbrace{\mathbb{L}_k(\mathbf{0}, \mathbf{0})}_{\mathbf{0}}]\\ &+ Q_{k,1} \cdot F_k(\mathbf{0}) + Q_{k,2} \cdot G_k(\mathbf{0}).\end{aligned} \tag{10}$$

The following is a valid inequality for the set $\mathscr{S}_{des}$:

$$\sum_j \lambda_j \mathbb{E}_{k,j}(k_p^*, k_i, k_d) \geq 0. \tag{11}$$

Remark 1. One can even find a deep cut by solving the following problem:

$$\begin{aligned}&\min\ \sum_j \lambda_j \mathbb{E}_{k,j}(k_p^*, k_i^*, k_d^*)\\ &\text{subject to}\ \ (\lambda, Q_{k,1}, Q_{k,2}) \in \mathscr{F}_{dual}\end{aligned} \tag{12}$$

which is a semi-definite program. The optimal dual variable is such that $\sum_j \lambda_j^* \mathbb{E}_{k,j}(k_p^*, k_i^*, k_d^*)$ is the most negative; the cut for eliminating the controller (k_p^*, k_i^*, k_d^*) is $\sum_j \lambda_j^* \mathbb{E}_{k,j}(k_p^*, k_i, k_d) \geq 0$, for the given k.

Remark 2. We observe that the deep cut for a given $k_p = k_p^*$ may be plotted in the $k_i - k_d$ plane using the cut inequality:

$$\sum_j \lambda_j^* \mathbb{E}_{k,j}(k_p^*, k_i, k_d) \geq 0.$$

Let $\mathscr{S}_{outer}$ be the current best outer approximation of the desired set $\mathscr{S}_{des}$. If we write $\mathscr{S}_{outer}(k_p^*) := \mathscr{S}_{outer} \cap \{(k_p, k_i, k_d) : k_p = k_p^*\}$, then, we may update the current best outer approximation of the desired set $\mathscr{S}_{des}$ through

$$\begin{aligned}\mathscr{S}_{outer}(k_p^*) \leftarrow &\mathscr{S}_{outer}(k_p^*) \cap\\ &\{(k_p, k_i, k_d) : k_p = k_p^*, \sum_j \lambda_j^* \mathbb{E}_{k,j}(k_p^*, k_i, k_d) \geq 0\}.\end{aligned} \tag{13}$$

We can arbitrarily tighten the outer approximation as follows: Let $(k_p^*, k_i^*, k_d^*) \in \mathscr{S}_{outer}(k_p^*)$. If $(k_p^*, k_i^*, k_d^*) \notin \mathscr{S}_{des}$, then Widder's theorem allows us to identify an integer $k \geq 0$ for which (8) is infeasible when $k_p = k_p^*, k_i = k_i^*, k_d = k_d^*$. Correspondingly, using lemma 1, we can identify a cut (polynomial inequality) which is satisfied by every controller in $\mathscr{F}_{des}$ but is not satisfied by the controller: $k_p = k_p^*, k_i = k_i^*, k_d = k_d^*$. Hence, we can arbitrarily tighten the outer approximation.

B. First Outer Approximation of the Non-overshooting Step Response Region by Cutting Hyperplanes

In this subsection, we will enforce the requirement that the error transfer function and its first derivative have no real, positive zeros. Clearly, this is a necessary (but not sufficient) condition for the error response to be non-negative. In essence, we are trying to find $\mathscr{S}_{des} := \cap_{k=0}^1 \mathscr{S}_{outer}^k$. The non-negativity of $N_{E_1}(s, \mathbf{K})$ can be expressed as the following feasibility problem:

Feasibility Problem: Find all feasible values of $k_i, k_d, \mathbf{y}, \mathbf{z}$ subject to

$$\begin{aligned}&\mathbb{L}_{1,j}(k_p^*, k_i, k_d) = \mathbb{L}_{1,j}(\mathbf{y}, \mathbf{z}),\ j = 0,1,2,...,r_1,\\ &F_1(\mathbf{y}) \succeq 0,\ G_1(\mathbf{z}) \succeq 0,\\ &\mathbb{S}_q(k_p^*, k_i, k_d) \leq 0,\ q = 1,2,...,m \end{aligned} \tag{14}$$

where $\mathbb{L}_{1,j}(k_p^*, k_i, k_d)$ is the j-th linear polynomial in k_i and k_d obtained from equating the coefficients of the same powers of s in (5) with r_1 as the highest power of s. Now, (14) and its dual problem are SDP problems. To find a cutting hyperplane, one may choose (k_i^*, k_d^*) inside the stability region for which (14) is infeasible. From the dual problem of (14), one may find $\lambda, Q_{1,1} \succeq 0, Q_{1,2} \succeq 0$. Using lemma 1, the cut is given by:

$$\sum_j \lambda_j \mathbb{L}_{1,j}(k_p^*, k_i, k_d) \geq 0. \tag{15}$$

C. Second Outer Approximation of the Non-overshooting Step Response Region by Cutting Hyperboloids

In order to tighten the outer approximation determined by (14), one may also consider the non-negativity of the second derivative of the error transfer function. In this case the problem is to determine the non-negativity of the following polynomials

$$\begin{aligned}&E(s,\mathbf{K}) \geq 0,\\ &E_1(s,\mathbf{K}) \geq 0,\\ &E_2(s,\mathbf{K}) \geq 0\end{aligned} \tag{16}$$

to find the second outer approximation $\mathscr{S}_{des} := \cap_{k=0}^{2} \mathscr{S}_{outer}^{k}$. The polynomial $N_{E_2}(s,\mathbf{K})$ has coefficients depend quadratically on the controller parameters $\mathbf{K}$. The non-negativity of $N_{E_2}(s,\mathbf{K})$ can be expressed as the following feasibility problem:

Feasibility Problem: Find all feasible values of $k_i, k_d, \mathbf{y}, \mathbf{z}$ subject to

$$\begin{aligned}&\mathbb{Q}_{2,j}(k_p^*, k_i, k_d) = \mathbb{L}_{2,j}(\mathbf{y}, \mathbf{z}),\ j = 0,1,2,...,r_2,\\ &F_2(\mathbf{y}) \succeq 0,\ G_2(\mathbf{z}) \succeq 0,\\ &\mathbb{S}_q(k_p^*, k_i, k_d) \leq 0,\ q = 1,2,...,m \end{aligned} \tag{17}$$

where $\mathbb{Q}_{2,j}(k_p^*, k_i, k_d)$ is the j-th quadratic polynomial in k_i, k_d obtained from equating the coefficients of the same powers of s in (5) with r_2 as the highest power of s. To find a cutting hyperboloid, one may choose (k_i^*, k_d^*) inside the stability region for which (17) is infeasible. From the dual problem of (17), one may find $\lambda, Q_{2,1} \succeq 0, Q_{2,2} \succeq 0$. Using lemma 1, the cut is given by:

$$\sum_j \lambda_j \left[\mathbb{Q}_{2,j}(k_p^*, k_i, k_d)\right] \geq 0. \tag{18}$$

D. Estimate of the Minimum Possible Overshoot with a PID Controller

Let $\gamma > 0$ denote the maximum allowable percentage overshoot to a unit step with a PID controller. In other words, we want the impulse response of the following error transfer function to be non-negative:

$$E(s,\mathbf{K}) = \frac{1+\gamma}{s} - \frac{1}{s}\frac{(k_d s^2 + k_p s + k_i)N_p(s)}{sD_p(s) + (k_d s^2 + k_p s + k_i)N_p(s)}$$

which may be expressible through $\bar{\gamma} := \frac{1}{\gamma}$ as:

$$\bar{E}(s,\mathbf{K}) := \bar{\gamma}E(s,\mathbf{K}) = \frac{(1+\bar{\gamma})D_p(s) + (k_d s^2 + k_p s + k_i)N_p(s)}{s\underbrace{(sD_p(s) + (k_d s^2 + k_p s + k_i)N_p(s))}_{\Delta_{cl}(s)}}.$$

We will now require the impulse response of $\bar{E}(s,\mathbf{K})$ to be non-negative and use the methodology developed in the earlier subsections to construct an outer approximation $\mathscr{S}_{outer}$ for the desired set of controllers. In fact, if we have an outer approximation, $\mathscr{S}_{linear}$ defined by linear constraints in terms of variables $\bar{\gamma}, k_p, k_i$ and k_d, (for example, we can obtain one through Markov-Lucaks theorem by requiring that none of the zeros of $\bar{E}(s,\mathbf{K})$ to be real and positive), then one can also obtain a lower bound on minimum possible overshoot, by solving the following linear optimization problem:

$$\max_{\bar{\gamma}, k_p, k_i, k_d} \bar{\gamma}$$

subject to the constraint $(\bar{\gamma}, k_p, k_i, k_d) \in \mathscr{S}_{linear}$.

III. EXAMPLES

To illustrate the method developed in the previous section consider the following unstable plant

$$P(s) = \frac{s+1}{s^3 + 2s^2 + s + 3}. \tag{19}$$

Now, it is of interest to find the entire set of PID controllers that satisfy the stability of the closed loop system and guarantee a non-overshooting step response. Following the signature method in [1] one can obtain the stability region at $k_p^* = 5$ as a polygon determined by

$$\begin{aligned}&k_i > 0,\\ &k_d - 0.2k_i + 0.5 > 0.\end{aligned} \tag{20}$$

The stability region for this example is shown in Fig.2.

The error transfer function for the step input will be

$$E(s,\mathbf{K}) = \frac{s^3 + 2s^2 + s + 3}{s^4 + (k_d + 2)s^3 + (k_p + 1 + k_d)s^2 + (k_p + k_i + 3)s + k_i}. \tag{21}$$

The first outer approximation of the non-overshooting step response region can be obtained by requiring $E(s,\mathbf{K}) \geq 0$ and $E_1(s,\mathbf{K}) \geq 0$. The numerator of (21) must have no real, positive zeros for all $s \geq 0$. Since $s^3 + 2s^2 + s + 3 \geq 0$ for all $s \geq 0$, this does not add any further constraint than the

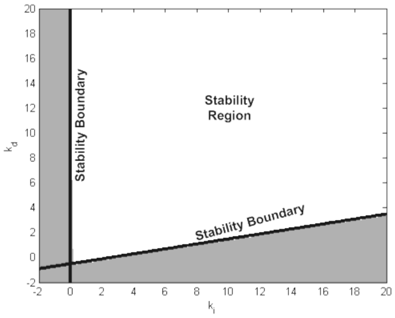

Fig. 2. The stability region at $k_p = 5$.

stability conditions to the problem.

Now, consider the non-negativity of the first derivative of the error transfer function. This means that:

$$\begin{aligned} N_{E_1}(s,\mathbf{K}) \; = \; & s^6+4s^5+(6-k_p+k_d)\,s^4 \\ & +(-2k_p+10+2k_d-2k_i)\,s^3 \\ & +(-k_p+13+10k_d-5k_i)\,s^2 \\ & +(6+6k_d-4k_i+6k_p)\,s \\ & +(2k_i+3k_p+9)\geq 0, \quad \forall s\geq 0. \end{aligned} \tag{22}$$

The non-negativity of (22) is guaranteed through the existence of the semi-definite matrices:

$$F_1(\mathbf{y}) = \begin{bmatrix} y_1 & y_2 & y_3 & y_4 \\ y_2 & y_5 & y_6 & y_7 \\ y_3 & y_6 & y_8 & y_9 \\ y_4 & y_7 & y_9 & y_{10} \end{bmatrix} \succeq 0, \tag{23}$$

$$G_1(\mathbf{z}) = \begin{bmatrix} z_1 & z_2 & z_3 & z_4 \\ z_2 & z_5 & z_6 & z_7 \\ z_3 & z_6 & z_8 & z_9 \\ z_4 & z_7 & z_9 & z_{10} \end{bmatrix} \succeq 0 \tag{24}$$

where, using (5), the entries of the matrices $F_1(\mathbf{y})$ and $G_1(\mathbf{z})$ are related to the controller parameters by the following set of linear equations:

$$\begin{aligned} y_1 &= 2k_i+3k_p+9, \\ 2y_2+z_1 &= 6+6k_d-4k_i+6k_p, \\ 2z_2+y_5+2y_3 &= -k_p+13+10k_d-5k_i, \\ 2y_6+2y_4+2z_3+z_5 &= -2k_p+10+2k_d-2k_i, \\ 2z_4+y_8+2y_7+2z_6 &= 6-k_p+k_d, \\ 2y_9+z_8+2z_7 &= 4, \\ y_{10}+2z_9 &= 1, \\ z_{10} &= 0. \end{aligned} \tag{25}$$

The set of all feasible (k_i,k_d), assuming $k_p = 5$, satisfying (20), (23), (24) and (25) is the first outer approximation of the stable non-overshooting step response region in the space of a PID controller parameters for the plant (19).

Pick $k_i^* = 100$ and $k_d^* = 40$ which is inside the stability region. These values of the controller parameters result the SDP determined by (20), (23), (24) and (25) to be infeasible which means that the selected point (k_p^*, k_i^*, k_d^*) is outside of the first outer approximation. Thus, there exists a cutting hyperplane, which is the deepest cut, corresponding to this point. Fig.3 shows this point and its corresponding cutting hyperplane.

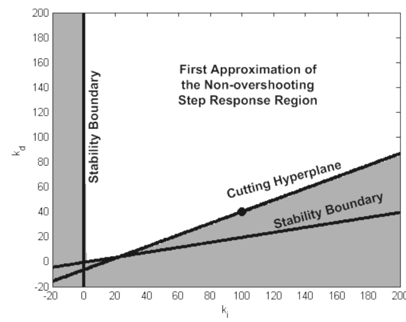

Fig. 3. The first approximation of the non-overshooting step response region by a cutting hyperplane corresponding to $k_p = 5, k_i = 100, k_d = 40$.

For the second outer approximation, the feasibility problem corresponding to $N_{E_2}(s,\mathbf{K}) \geq 0$ will be:

Feasibility Problem: Find all feasible values of $k_i, k_d, \mathbf{y}, \mathbf{z}$ subject to

$$F_2(\mathbf{y}) = \begin{bmatrix} y_1 & y_2 & y_3 & y_4 & y_5 \\ y_2 & y_6 & y_7 & y_8 & y_9 \\ y_3 & y_7 & y_{10} & y_{11} & y_{12} \\ y_4 & y_8 & y_{11} & y_{13} & y_{14} \\ y_5 & y_9 & y_{12} & y_{14} & y_{15} \end{bmatrix} \succeq 0, \tag{26}$$

$$G_2(\mathbf{z}) = \begin{bmatrix} z_1 & z_2 & z_3 & z_4 & z_5 \\ z_2 & z_6 & z_7 & z_8 & z_9 \\ z_3 & z_7 & z_{10} & z_{11} & z_{12} \\ z_4 & z_8 & z_{11} & z_{13} & z_{14} \\ z_5 & z_9 & z_{12} & z_{14} & z_{15} \end{bmatrix} \succeq 0, \tag{27}$$

$$\begin{aligned} y_1 = \; & 8{k_i}^2+4k_ik_p+6{k_p}^2+24k_i \\ & +36k_p+54-6k_dk_i, \\ 2y_2+z_1 = \; & -6k_dk_i+18{k_p}^2+54k_d \\ & -24k_i+72k_p+54 \\ & +18k_dk_p+6{k_i}^2+12k_ik_p, \\ 2z_2+y_6+2y_3 = \; & 126+90k_d-18k_i+72k_p \\ & +18{k_d}^2-6k_dk_i+54k_dk_p \\ & +6{k_i}^2-6k_ik_p+18{k_p}^2, \\ 2y_7+2y_4+2z_3+z_6 = \; & 46k_dk_p+2{k_i}^2-2k_ik_p \\ & +50{k_d}^2+130k_d-62k_i \\ & +104k_p-36k_dk_i+128, \\ y_{10}+2y_8+2y_5+2z_4+2z_7 = \; & 120+36k_p-6k_dk_p+180k_d \\ & -102k_i+42{k_d}^2-24k_dk_i, \\ 2y_9+2y_{11}+2z_5+z_{10}+2z_8 = \; & -60k_i-30k_p-6k_dk_p \\ & +102k_d+102-6k_dk_i+6{k_d}^2, \end{aligned}$$

$$
\begin{aligned}
2z_9+2z_{11}+2y_{12}+y_{13} = & \; 58-22k_p+22k_d-14k_i \\
& +2{k_d}^2-2k_dk_p, \\
z_{13}+2z_{12}+2y_{14} = & \; 6k_d-6k_p+30, \\
y_{15}+2z_{14} = & \; 12, \\
z_{15} = & \; 2. \qquad (28)
\end{aligned}
$$

In order to obtain an initial approximation prior to enforcing the constraints of the second derivative of the error transfer function, we found 50 cuts (or hyperplanes) that corresponding to 50 different stabilizing controllers that did not have non-overshooting step response. From this approximate set, we picked $k_i^*=100$ and $k_d^*=50$. These values of the controller parameters result in the SDP defined by (26), (27) and (28) to be infeasible. This means that the selected point (k_p^*, k_i^*, k_d^*) is outside $\mathscr{S}_{outer}^2$. Thus, there exists a cutting hyperboloid, or simply a cut, which is the deepest cut, corresponding to this point. Fig.4 shows this point and its corresponding cut.

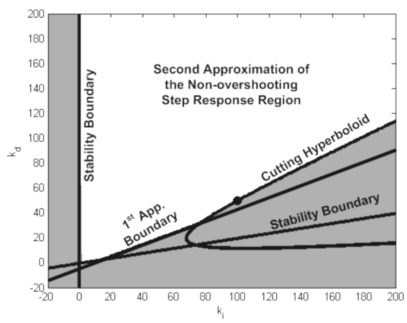

Fig. 4. The second approximation of the non-overshooting step response region by a cutting hyperboloid corresponding to $k_p=5, k_i=100, k_d=50$.

As Fig.4 shows, the outer approximation becomes tighter as more derivatives of the error transfer function are considered.

IV. CONCLUSIONS

In this paper, we presented a method for constructing an outer approximation of the set of stabilizing PID controllers that guaranteed a non-overshooting step response of the closed loop system. This is accomplished through solving a sequence of SDPs based on Markov-Lucaks theorem and Widder' theorem. The results of this paper readily generalized to discrete-time LTI systems also through the counterpart of Widder's theorem for discrete-time LTI systems given in [2].

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