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Coordinated networked estimation strategies using structured systems theory

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Abstract— **In this paper, we consider linear networked estimation strategies using the results from structured systems theory. We are interested in estimating a linear dynamical system where the observations are distributed over a network of agents. In this context, we devise both state fusion and observation fusion strategies that guarantee a stable estimator. We assume global observability, i.e., given all of the observations, the dynamical system is observable. To derive our results, we employ the genericity properties of dynamical systems that are studied in the structured systems theory. The genericity properties rely on the graphical properties of the dynamical systems and their outputs, and thus, depend on the zero and non-zero pattern of the system and output (observation) matrices. In particular, we study the generic observability of networked estimators and derive results on the topology of the agent communication graph to ensure a stable estimator. We then focus on the design of local estimator gains that results into iterative procedures to solve a Linear Matrix Inequality (LMI) with structural constraints.**

I. INTRODUCTION

Networked estimation theory is essential in the study of autonomous and multi-agent systems that are deployed to estimate a physical phenomenon and/or monitor the behavior of a hazardous environment. Such networked systems typically consists of power-constrained, relatively cheap sensors and/or robots that can implement local communication and have limited measurements of the underlying system. The goal of the agents is to implement a reasonable estimator of the underlying dynamics under such constraints.

A variety of solutions exist for networked estimation. The literature on this subject exists from earlier work in [1], [2] and references therein, where parallel Kalman filter architectures are considered, generally, for all-to-all connected networks, to more recent work in [3], [4], [5], [6], where consensus-based strategies are discussed for sparsely connected networks. Consensus-based schemes are challenged with a large number of consensus iterations in between every two time-steps of the dynamics, and, in general, are impractical as a large number of local communication is resource-heavy at the agents.

Recently, distributed estimators with a focus on single consensus-step, only one information exchange among the agents, is considered ranging from scalar systems [7] to vector systems [8], [9], [10]. In general, none of the related work has addressed the notion of network observability as single-step observability [8], [10] and/or local observability [11] (each agent observable with its measurements) is assumed. The problem of observability, in the most general setting, is non-trivial in networked estimation as it requires knowledge the exact agent communication graph and the exact fusion rule (weights assigned to each neighbor) in order to construct the observability Gramian. Furthermore, when the parameters of the underlying system depends on their operating point, for example, in linearized models, an apriori computation of observability is not possible.

In this paper, we use the structured systems theory [12], [13], results to address the generic observability of a system where the emphasis is on the sparsity pattern of the system matrices and the non-zero elements are considered as *free parameters*. If such *generic* properties are true for one choice of free parameters, they are true for almost all choices of free parameters; the space where they are not true is some proper algebraic variety in the parameter space with Lebesgue measure zero [14], [12]. Hence, the generic properties provide a novel technique to address observability and/or controllability of networked estimators/controllers, as they are a departure from the conventional algebraic rank-based methods to structural graph-based techniques. Related work on using structured systems theory for distributed systems can be found in [15] where a global system is divided into subsystems and generic subsystem properties are considered. On the other hand, this paper concerns with networked systems that result from inter-agent communication.

The next question after formulating the observability of networked estimators is to design the local estimator gains. Since the estimation is distributed, the local observer gains constitute a block-diagonal global gain matrix. This structure on the gain matrices prevents us from directly using standard Linear Matrix Inequality (LMI) and Lyapunov-based arguments. A particular approach that we employ is to use a linear approximation of the underlying non-convex trace objective [16]. An iterative procedure can be implemented on this linear approximation resulting into structured gain for the networked system [16]. A similar technique is used in [17] to study a wireless control network.

We now describe the rest of the paper. Section II provides preliminary material on basic dynamical system estimation and structured systems theory, whereas, Section III presents the problem formulation. We present the generic observability in Section IV and consider the estimator gain design in Section V with simulations in Section VI. Finally, Section VII concludes the paper.

II. BACKGROUND AND PRELIMINARIES

Consider a discrete-time linear dynamical system represented by the following state-space model,

$$\mathbf{x}_{k+1} = A\mathbf{x}_k + \mathbf{v}_k, \tag{1}$$

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$\mathbf{x}_k \in \mathbb{R}^n$ is the state vector, $A = \{a_{ij}\} \in \mathbb{R}^{n\times n}$ is the system matrix, and $\mathbf{v}_k \sim \mathcal{N}(0, V)$ is the system noise. We assume that the dynamical system in (1) is monitored by a network of N agents such that agent i has the following observation model,

$$\mathbf{y}_k^i \quad = \quad C_i\mathbf{x}_k + \mathbf{r}_k^i, \tag{2}$$

where $\mathbf{y}_k^i \in \mathbb{R}^{p_i}$ is the output vector at agent i, $\mathbf{r}_k^i \sim \mathcal{N}(0, R_i)$ is the output noise, and C_i is the output matrix at agent i.

A. Centralized Kalman filter

We use the following notation,

$$\mathbf{y}_k = \begin{bmatrix} \mathbf{y}_k^1 \\ \vdots \\ \mathbf{y}_k^N \end{bmatrix}, \quad C = \begin{bmatrix} C_1 \\ \vdots \\ C_N \end{bmatrix}, \quad \mathbf{r}_k = \begin{bmatrix} \mathbf{r}_k^1 \\ \vdots \\ \mathbf{r}_k^N \end{bmatrix}, \tag{3}$$

where $\mathbf{r}_k \sim \mathcal{N}(0, R)$ is the global observation noise, $R = \text{blockdiag}[R_1, \ldots, R_N]$, and $\mathbf{C} = \{c_{ij}\}$ is the global output matrix. Let $\widehat{\mathbf{x}}_{k|k}^c$ be the centralized Kalman estimator [18] at time k given all the observations, $\mathbf{y}_k$, up to time k. It can be shown that the error in the centralized Kalman estimator,

$$\widehat{\mathbf{e}}_{k|k}^c = \mathbf{x}_k - \widehat{\mathbf{x}}_{k|k}^c, \tag{4}$$

is given by

$$\widehat{\mathbf{e}}_{k|k}^c = (A - KCA)\widehat{\mathbf{e}}_{k-1|k-1}^c + \eta_k, \tag{5}$$

where K_c is the centralized Kalman gain and the vector η_k collects the remaining terms that are independent of $\widehat{\mathbf{e}}_{k-1|k-1}^c$. It is well known that the centralized Kalman error, $\widehat{\mathbf{e}}_{k|k}^c$ can be made stable if and only if (A, C) is observable. In other words, a Kalman gain matrix, K_c, exists such that $\rho(A - KCA) < 1$, if and only if (A, C) is observable, where $\rho(\cdot)$ is the matrix spectral norm.

B. Structured systems theory

Conventional notion of n-step (A, C)-observability is where the following Gramian,

$$\mathcal{O} = \left[\begin{array}{cccc} C^T & A^TC^T & \ldots & (A^{n-1})^TC^T \end{array}\right]^T, \tag{6}$$

has full rank or the matrix $\mathcal{O}^T\mathcal{O}$ is invertible. Checking for full rank or invertibility relies on the exact values of each element in the matrices A and C. However, in many practical applications, only the sparsity (zero and non-zero pattern) of these matrices is known and non-zero elements are subject to change.

Structured systems theory studies the zero and non-zero pattern of system matrices, A and C, and consider the non-zero elements as free parameters. For a number of system theoretic properties, e.g., controllability and observability among others, it turns out that if a property is true for one particular choice of these free parameters, it is true for *almost all* choices of the free parameters, and, therefore, is called a *generic* property of the system. The choices of free parameters where such generic properties do not hold lie in some proper algebraic variety in the parameter space, see [14], [12]. A proper algebraic variety has Lebesgue measure zero, justifying the use of almost all, for more details, see [12], and the references therein. In the following, we will review the generic observability of linear systems. Other generic properties are elaborated in [12] and in the references therein. Below, we present the structured system theoretic approach to generic observability.

Generic observability: To describe the generic observability, we first revisit some basic graph theoretic definitions. Let $X = \{x_1, \ldots, x_n\}$ denote the state set, and let $Y = \{y_1, \ldots, y_p\}$ (where $p = p_1 + \ldots + p_N$) denote the output set of the system in (1)-(3). Let $G = (V, E)$ define the *system digraph*, where $V = X \cup Y$ is the vertex set, and E is the edge set containing directed edges, $(v_1, v_2) \in E$, of the form $v_1 \to v_2$ with $v_1, v_2 \in V$. The edge set E is defined as $E_A \cup E_C$, where $E_A = \{(x_j, x_i) \mid a_{ij} \neq 0\}$ and $E_C = \{(x_j, y_i) \mid c_{ij} \neq 0\}$. A *path* of length ℓ from $v_1 \in V$ to $v_\ell \in V$ is such that there exists a sequence of vertices, $v_2, v_3, \ldots, v_{\ell-1}$ with each subsequent edge, $(v_1, v_2), (v_2, v_3), \ldots, (v_{\ell-1}, v_\ell) \in E$. Here v_1 is the *begin-vertex* of the path and v_ℓ is its *end-vertex*. A path is *simple* if each vertex contained in a path occurs only once. A path is *disjoint* from another path if they consist of disjoint set of vertices. A set of paths is *mutually disjoint* when each two of them are disjoint. A simple path is said to be a Y-*topped* path if the path has its end-vertex in Y. A set of mutually disjoint Y-topped paths is called a Y*-topped family*. A *cycle* is a simple path that begins and ends with the same vertex. A set of mutually disjoint cycles is called a *cycle family*. With these definitions, we now describe the generic observability.

Theorem 1: A dynamical system is generically observable if and only if: (i) every state vertex is the begin-vertex of a Y-topped path; and (ii) there exists a disjoint union of a Y-topped path family and a cycle family that covers all of the states vertices.

The above theorem is provided for generic controllability in [12], where other equivalent graph-theoretic conditions to generic controllability (observability) are also defined that we omit here. The generic controllability theorem[1] has been proved in [13]. Following is an immediate corollary to the above theorem.

Corollary 1: A dynamical system with each diagonal entry being non-zero in the system matrix, A, is generically observable if and only if every state vertex is the begin-vertex of a Y-topped path.

It can be easily verified that a non-zero diagonal leads to a disjoint cycle family with $(x_1, x_1), \ldots, (x_n, x_n)$ as its elements. This cycle family covers all of the state vertices and the condition (ii) of Theorem 1 is implied by a system matrix with a non-zero diagonal.

Examples: We now provide some examples to illustrate the concepts described before. Consider the following tridiagonal

[1]We are not familiar with a reference where a generic observability theorem is proved. Nevertheless, since observability and controllability are dual to each other, the theorem can be readily established as (A, C)-observability is implied by (A^T, C^T)-controllability.

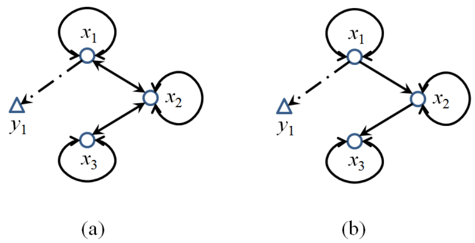

Fig. 1. (a) A generically observable system. (b) An unobservable system.

system ($n = 3$) with a single scalar observation, i.e.,

$$A = \begin{bmatrix} \times & \times & 0 \\ \times & \times & \times \\ 0 & \times & \times \end{bmatrix}, \qquad C = \begin{bmatrix} \times & 0 & 0 \end{bmatrix}. \tag{7}$$

The system digraph, G, is shown in Fig. 1(a). From Corollary 1, we only need to verify the condition (i) in Theorem 1. To this end, note that (x_1, y_1), $\{(x_2, x_1), (x_1, y_1)\}$, and $\{(x_3, x_2), (x_2, x_1), (x_1, y_1)\}$ are three Y-topped paths satisfying the condition (i). Hence, the above system is generically observable. In fact, it can be easily verified that any multi-diagonal system is generically observable with a single scalar observation.

As another example, consider the following lower triangular system,

$$A = \begin{bmatrix} \times & 0 & 0 \\ \times & \times & 0 \\ 0 & \times & \times \end{bmatrix}, \qquad C = \begin{bmatrix} \times & 0 & 0 \end{bmatrix}, \tag{8}$$

whose digraph, G, is shown in Fig. 1(b). Here, again we only need to verify the condition (i) in Theorem 1, and we note that a Y-topped path does not exist with both x_2 and x_3 being the begin-vertices. It can be verified that the only single scalar observation that makes this system observable is through the output matrix $C = [0\ 0\ \times]^T$. These examples can also be verified using the conventional observability definition.

III. Problem formulation

Consider the discrete-time linear dynamical system of (1) monitored by a network of N agents (2). We assume that the agents are able to communicate according to a topology, i.e., $i \leftrightarrow j$ implies that agent i and agent j are connected. Let the agent communication graph be denoted by, $G_a = (V_a, E_a)$, where $V_a = \{1, \ldots, N\}$ is the vertex set and $E_a = \{(i, j) \mid i \leftrightarrow j\}$ is the edge set. Let $\mathcal{D}_i = \{i\} \cup \{j \mid (i, j) \in E\}$ denote the extended neighborhood of agent i.

Let $\widehat{\mathbf{x}}^i_{k|m}$ be the state estimate at time k and agent i given the outputs until time m from agent i and its neighbors, $j \in \mathcal{D}_i$.. We implement the following (Kalman-type) estimator at agent i:

Predictor and state fusion: The local predictor at agent i is given by

$$\widehat{\mathbf{x}}^i_{k|k-1} = \sum_{j \in \mathcal{D}_i} W_{ij} A \widehat{\mathbf{x}}^j_{k-1|k-1}, \tag{9}$$

where the state fusion is carried out by the diagonal weight matrix $W_{ij} \in \mathbb{R}^{n \times n}$ that agent i assigns to each element $j \in \mathcal{D}_i$, such that $W_{ij} \geq 0$ ($\geq$ represents an element-wise operation) and $\sum_{j \in \mathcal{D}_i} W_{ij} = I_n$. Clearly, for no state fusion, we can choose $W_{ii} = I_n$.

Estimator and output fusion: The local estimator at agent i is given by

$$\widehat{\mathbf{x}}^i_{k|k} = \widehat{\mathbf{x}}^i_{k|k-1} + K^i_k \sum_{j \in \mathcal{D}_i} u_{ij} C_j^T (\mathbf{y}^j_k - C_j \widehat{\mathbf{x}}^i_{k|k-1}), \tag{10}$$

where K^i_k is the local estimator gain and the output fusion is carried out by the weight $u_{ij} \in \mathbb{R}$. Clearly, for no output fusion, we can choose $u_{ii} = 1$ and $u_{ij} = 0$ for $j \neq i$.

We assume that the dynamical system is globally observable, i.e., (A, C) is observable (with C defined in (3)). If $W_{ii} = I_n$ for each agent i (this implies $W_{ij} = 0, \forall j \neq i$ as they add to identity), i.e., there is no state fusion, then each local Kalman filter (9)-(10) is observable if and only if $(A, \sum_{j \in \mathcal{D}_i} u_{ij} C_j^T C_j)$ is observable at each agent i. Clearly, global observability of the system *does not* imply this local observability through output exchange alone.

In this paper, we are interested in the observability of local Kalman filters by choosing appropriate local weight matrices W_{ij}. In other words, what communication strategy should the agents employ to implement the local predictor in (9) such that the networked estimator is observable.

We next consider the design of local estimator gain matrices, K^i_k, at each agent, i. In particular, this is equivalent to solving a (Lyapunov thoeretic) Linear Matrix Inequality (LMI) with structural (block-diagonal) constraints on the objective. Observability (controllability) guarantees an unconstrained solution to the related LMI, but a constrained (structured) solution prevents one to directly apply these results. In Section V, we provide a method to solved structured LMIs that is based on a iterative procedure after a linear approximation of the underlying non-convex trace objective.

IV. Generic local observability

In this section, we show that the local Kalman filters, not observable through output fusion alone, can be made observable by using state fusion. To this end, we derive the error dynamics at each local estimator (9)-(10) and then build the networked estimator error. Define

$$\mathbf{e}^i_k = \mathbf{x}_{k|k} - \mathbf{x}^i_{k|k}, \tag{11}$$

to be the estimation error at agent i and $\mathbf{e}_k = [(\mathbf{e}^1_k)^T, \ldots, (\mathbf{e}^N_k)^T]^T$ to be the network estimator error, then it can be verified that

$$\mathbf{e}_k = (W(I \otimes A) - K_k D_C W(I \otimes A))\mathbf{e}_{k-1} + \mathbf{q}_k, \tag{12}$$

where $W = \{W_{ij}\}$ is the global weight matrix, i.e., a collection of the local weight matrices,

$$\begin{aligned} K_k &= \text{blockdiag}[K^1_k, \ldots, K^N_k], \\ D_C &= \begin{bmatrix} \sum_{j \in \mathcal{D}_1} u_{1j} C_j^T C_j & & \\ & \ddots & \\ & & \sum_{j \in \mathcal{D}_N} u_{Nj} C_j^T C_j \end{bmatrix}, \end{aligned}$$

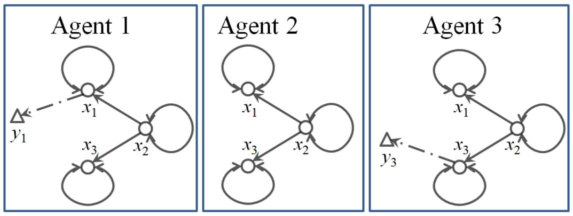

Fig. 2. System digraphs at agent 1, 2, and 3.

and $\mathbf{q}_k$ collects the remaining terms, a weighted linear function of the system and output noise alone. Note that when $W_{ii} \neq I_n$, each local error process, $\mathbf{e}_k^i$, is coupled with the error processes at the rest of the agents through the agent communication graph, G_a.

A. Illustration

We now show that the addition of state fusion by choosing a non-identity W_{ii} at each agent can lead to the stability of the networked estimator error assuming the system is globally (A, C) (with C defined in (3)) observable. Comparing (12) to (5), we note that the stability of the networked estimation error is given by the observability of the following:

$$(W(I_N \otimes A), D_C). \tag{13}$$

Note here that the only constraint on the local weight matrices, W_{ij}, is that they are diagonal and stochastic (non-negative and sum to identity). For the sake of clarity, we assume that $u_{ij} = 0$, $\forall\, i \neq j$, to specifically study the effect of fusion on local predictors in the following discussion.

Consider $N = 3$ agents and the system matrix, A, to be of the form,

$$A = \begin{bmatrix} \times & \times & 0 \\ 0 & \times & 0 \\ 0 & \times & \times \end{bmatrix}, \tag{14}$$

with the output matrices given by

$$\begin{bmatrix} C_1 \\ C_2 \\ C_3 \end{bmatrix} = \begin{bmatrix} \times & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & \times \end{bmatrix}. \tag{15}$$

The system digraph corresponding to each agent is shown in Fig. 2. Clearly, none of the agents is observable with the given observation matrices. This is because agent 2 has no observation and x_3 at agent 1 and x_1 at agent 3 are not the begin vertices of any Y-topped path. We now allow the following agent communication graph, G_a: $1 \leftrightarrow 2 \leftrightarrow 3$; i.e., choose the following weight matrix, W,

$$W = \begin{bmatrix} \times & \times & 0 \\ \times & \times & \times \\ 0 & \times & \times \end{bmatrix} \otimes I_n. \tag{16}$$

With this W, the networked system now becomes nN dimensional and the observability is according to (13). The new system matrix is now given by $W(I_N \otimes A)$ and has the

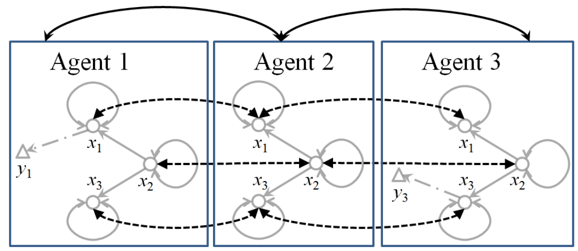

Fig. 3. System digraph for the networked system in Fig. 2 with agent communication graph, $1 \leftrightarrow 2 \leftrightarrow 3$.

following sparsity pattern,

$$W(I_N \otimes A) = \begin{bmatrix} \times & \times & 0 & \times & \times & 0 & 0 & 0 & 0 \\ 0 & \times & 0 & 0 & \times & 0 & 0 & 0 & 0 \\ 0 & \times & \times & 0 & \times & \times & 0 & 0 & 0 \\ \times & \times & 0 & \times & \times & 0 & \times & \times & 0 \\ 0 & \times & 0 & 0 & \times & 0 & 0 & \times & 0 \\ 0 & \times & \times & 0 & \times & \times & 0 & \times & \times \\ 0 & 0 & 0 & \times & \times & 0 & \times & \times & 0 \\ 0 & 0 & 0 & 0 & \times & 0 & 0 & \times & 0 \\ 0 & 0 & 0 & 0 & \times & \times & 0 & \times & \times \end{bmatrix} \tag{17}$$

and the digraph is shown in Fig. 3, where the new links added because of the W are shown in black-dashed and the original links are greyed out. Note that the previous un-observable networked estimator of Fig. 2 is now observable as each state is the begin-vertex of some Y-topped path. For the sake of clarity, we have omitted some of the redundant links added by state fusion.

B. Main results

With the help of the above discussion, we now provide the following results.

Lemma 1: Let the system be globally (A, C) observable with each diagonal entry being non-zero in the system matrix, A. If the agent communication graph, G_a, is strongly connected then the networked estimator in (9)-(10) is generically observable.

Proof: From Corollary 1, we only need to verify the condition (i) of Theorem 1. To this end, note that a strongly connected communication graph on the agents implies that each lth state variable is strongly connected among the agents. In other words, there exists a path from each x_l^i to any other x_l^j. Since (A, C) is observable, x_l^j from some j is the begin-vertex of a Y-topped path that makes x_l^j at every $j = 1, \ldots, N$ to be the begin-vertex of some Y-topped path. Since the previous is true for all $l = 1, \ldots, n$, the network estimator in (9)-(10) is generically observable. ■

The above lemma leads to a general theorem given below.

Theorem 2: Let the system be globally (A, C) observable such that there exists a disjoint union of cycle family that covers all of the n states vertices in A. Then the networked estimator in (9)-(10) is generically observable, if the agent communication graph, G_a, is strongly connected.

Proof: The proof follows from Corollary 1 and Lemma 1. ■

Similar results can be extended when the matrix A has other structures.

C. Discussion

The results provided in the previous section are independent of what fusion rule (e.g., Metropolis-Hastings [19]) is chosen in (9). This is because using structured system theoretic arguments, it is possible to consider the observability problem *generically* such that it is true for almost all possible choices of the fusion rule (weight matrices). Nevertheless, the structure of the underlying agent communication remains relevant and leads to infrastructure or network design questions. Furthermore, generic properties are, in general, easily verified, see [13], [20], [21], [22] for related algorithms.

V. Estimator gain design

We now consider the design of the local estimator gain in (10). Notice that observability guarantees a *full* gain matrix such that the estimation error is a stable process, in particular, if $(W(I_N \otimes A), D_C)$ is generically observable then for almost all non-zero elements in the corresponding matrices, there exists a *full* matrix, $\overline{K}$, such that

$$\rho(W(I_N \otimes A) - \overline{K} D_C W(I_N \otimes A)) < 1. \tag{18}$$

It is well-known that a full gain matrix, $\overline{K}$, can be obtained by solving the following Linear Matrix Inequality (LMI) (after some manipulation)

$$X - \overline{A}^T X \overline{A} \succ 0, \tag{19}$$

for some $X \succ 0$ ('$\succ$' denotes positive-definiteness) with an appropriate $\overline{A}$. However, this LMI cannot be directly solved when the gain matrix has a structure.

In the case of networked estimation, the gain matrix, $\overline{K}$, is not full, but is given by K_k, see (12), that is block-diagonal with N, $n \times n$ blocks. For the rest of this section, we assume that a constant estimator gain matrix is applied, i.e., the matrix K_k is independent of time, k, and denote it by K. Hence, we would like to find $X \succ 0$ such that

$$\begin{aligned} X &- \widehat{A}^T X \widehat{A} \succ 0, & (20)\\ \widehat{A} &= (W(I_N \otimes A) - K D_C W(I_N \otimes A)), & (21) \end{aligned}$$

or equivalently,

$$\begin{bmatrix} X & \widehat{A}^T X, \\ X\widehat{A} & X \end{bmatrix} \succ 0, \tag{22}$$

with $X \succ 0$. Since the above is non-linear in the design parameter (product of X and K), we note that $\rho(\widehat{A}) < 1$ [17], if and only if there exists $X, Y \succ 0$ such that

$$\begin{bmatrix} X & \widehat{A}^T, \\ \widehat{A} & Y \end{bmatrix} \succ 0, \tag{23}$$

with $X = Y^{-1}$. The LMI in the above theorem is linear in the design parameter K but the constraint involved $X = Y^{-1}$ is non-convex. Here, we use the approach in [16] to approximate $X = Y^{-1}$ with a linear function. In particular, the matrices, $X, Y \succ 0$, satisfy $X = Y^{-1}$, if and only if they are optimal points of the following optimization problem [16].

$$\min\ tr(XY) \text{ subject to } \begin{bmatrix} X & I \\ I & Y \end{bmatrix} \succeq 0, \tag{24}$$

with $X, Y \succ 0$. The above discussion can be summarized in the following lemma.

Lemma 2: If the networked estimator (9)-(10) is generically observable, then a structured gain matrix, K, is the solution of the following optimization.

$$\begin{gathered} \min\ tr(XY), \\ \begin{bmatrix} X & \widehat{A}^T, \\ \widehat{A} & Y \end{bmatrix} \succ 0, \qquad \begin{bmatrix} X & I, \\ I & Y \end{bmatrix} \succeq 0, \\ K \text{ is block-diagonal}, \qquad X, Y \succ 0. \end{gathered} \tag{25}$$

Notice that since the second LMI is equivalent to $X = Y^{-1}$, the minimum trace is achieved at $X = Y^{-1}$ and the optimal value is nN.

Furthermore, the trace operator over the product of X and Y can be replaced with a linear approximation [23], [16]

$$\phi_{\text{lin}}(X, Y) = tr(Y_0 X + X_0 S), \tag{26}$$

and an iterative algorithm can be used to minimize $tr(XY)$, under the block-diagonal constraints on the estimator gain matrix, K. The iterative algorithm [16] is as follows:

(i) Find feasible points X_0, Y_0, K. If no such points exist, Terminate.
(ii) Find X_{t+1}, Y_{t+1} by minimizing $tr(Y_t X + X_t Y)$ under the constraints in (25).
(iii) Terminate when $\rho(\widehat{A}) < 1$ or according to a desirable stopping criterion.

A. Discussion

Let

$$s_{t+1} = tr(Y_t X_(t+1)) + X_{t+1} Y), \tag{27}$$

then it is shown in [16] that s_t is a decreasing sequence that converges to $2nN$; the convergence to $2nN$ is because as $t \uparrow$, $X_{t+1} \rightarrow Y^{-1}$. However, a characterization on the convergence rate is a topic of further research. A stopping criterion in (iii) of the above iterative procedure can also be established in terms of reaching within $2nN + \varepsilon$ of the trace objective. The iterative procedure given above, similar to the cone-complementarity linearization algorithm in [16], is a centralized algorithm and has to be implemented at a center. However, the center has to implement this process only once, off-line, and then it may broadcast appropriate estimator gains to each agent. Afterwards, the center plays no role in the implementation of local estimators at each agent; each agent, subsequently, observes and performs *in-network* operations to implement the estimator.

A same time-scale algorithm can also be implemented, where the above iterative procedure is implemented at the same time-scale k as of the dynamical system in (1). With this approach, the estimator gain, K_{k+1}, becomes a function of k and may be transmitted to each agent at each time-step k. This is helpful when the implementation is assumed in real-time.

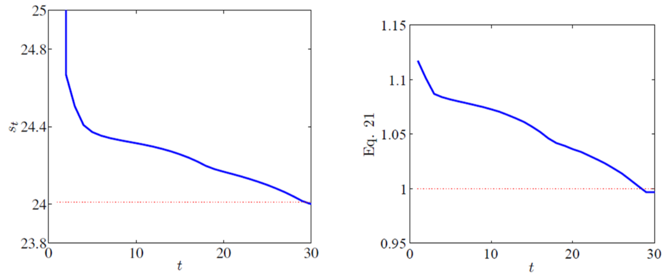

Fig. 4. (left) The linearized trace objective function. (right) Spectral radius corresponding to the error dynamics with block-diagonal gain.

VI. SIMULATIONS

We consider the dynamical system evolved with the system matrix in (14) with $N = 4$ agents with G_a given by $1 \leftrightarrow 2 \leftrightarrow 3 \leftrightarrow 4$. We assume the output matrices at C_1 and C_4 to be the same as C_1 and C_3 in (15) with no outputs at agents 2 and 3. We further choose the same fusion rule for each state, $x_l,\ l = 1, \ldots, n$, at each agent, and generate W_{ij} with random positive numbers such that the resulting local weight matrices, W_{ij}, are positive, diagonal, and sum to identity, I_4. Note that with $N = 4$ agents, none of the agent is either independently observable, (A, C_i), or locally observable through output fusion, $(A, \sum_{j\in\mathcal{D}_i} u_{ij} C_j^T C_j)$, alone. However, since the agent communication graph, $G_a, 1 \leftrightarrow 2 \leftrightarrow 3 \leftrightarrow 4$, is strongly connected and A has non-zero elements on all of its diagonals, the resulting networked system is generically observable. This can also be verified by computing the rank of the underlying observability Gramian after putting any elements in A and W such that the sparsity pattern is not violated.

We then implement the iterative procedure to compute structured gains and the results are shown in Fig. 4, where we have used a stopping criterion of $s_t < 24.01$. Note that the optimal value of the linearized trace objective is $2nN = 24$. Fig. 4 (left) shows the spectral radius of $\widehat{A}$ as a function of the iterations of the iterative procedure, whereas, Fig. 4 (right) shows the cost of the objective linearized trace function. The block-diagonal estimator gain matrices that constitute K can now be made available to the agents to implement the networked estimator.

VII. CONCLUSIONS

In this paper, we use the results from structured systems theory to formulate generic observability for networked estimators. We show that a strongly connected agent communication graph suffices the networked generic observability when the underlying system matrix have particular structures. For other system matrix, our theory is easily extendible to similar generic arguments. We then consider the design of local estimator gains that is the equivalent to structured LMI solutions that can be solved using well-known iterative procedures. Our approach provides a foundation to build and further explore the networked systems theory.

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