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Stabilization of Switched Linear Stochastic Dynamical Systems Under Limited Mode Information

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Abstract— **Almost sure asymptotic stabilization problem of continuous-time switched linear stochastic dynamical systems is considered. The mode signal, which manages the transition between subsystems, is modeled as a Markov chain. Mode information is assumed to be only available at certain time instances. We propose a control law that depends on the sampled information of the mode signal, which is constructed from the available mode samples. Based on our stability analysis for switched linear stochastic systems, we obtain sufficient conditions under which the proposed control law guarantees stability of the zero solution. Finally, we present an illustrative numerical example to demonstrate the efficacy of our results.**

I. Introduction

Stochastic hybrid system models can accurately describe various real life processes from finance, physics and engineering fields that are subject to noise and random environmental variations. There has been increasing amount of studies concerning the stability of stochastic hybrid systems. Particularly, researchers have extensively explored stability of Markov jump systems, which are composed of deterministic subsystems and a probabilistic mode signal (e.g., [1]–[3] and the references therein). Some researchers have combined probabilistic mode signals with stochastic subsystem dynamics to obtain more general stochastic hybrid system models, which are often called "switching diffusion processes". Switching diffusion processes have found applications in population studies [4]–[7] and finance [5], [8]. Stochastic stability properties of switching diffusion processes is explored in several works [9]–[16].

Stabilization of stochastic hybrid systems has also been a topic of interest. Specifically, the stabilization of continuous-time Markov jump linear systems is addressed in [17]; stabilization of Markov jump systems with delays is explored in [18] and [19]; furthermore, several almost sure stabilization results are provided for switching diffusion processes in [10] and [12].

In most of the studies that deal with stabilization of switched stochastic systems, proposed control laws depend on full information of the mode signal of the switched system. As a result, these control laws may not be appropriate when the mode information is sampled and only available at sampling instances. In this paper, we explore the stabilization problem under sampled mode information for continuous-time switched linear stochastic dynamical systems. First, we provide stability analysis for switched linear stochastic dynamical systems without control input. These systems are composed of stochastic subsystems which include Brownian motion in the dynamics. The mode signal, which manages the transition between these subsystems, is modeled as a finite-state Markov chain. Based on our stability analysis, we propose a stabilizing control law that depends on the mode signal. Next, we consider the case where the mode signal information is sampled and hence available only at certain time instances. The intervals between these time instances are assumed to be independent and exponentially distributed. By using "sample and hold" technique, we construct a good representation of the mode signal from the available mode samples. Furthermore, we propose a control law that depends only on the sampled mode information. In this setting, the problem at hand is similar to the one in [20], where the authors investigate mean-square stabilizability of Markov jump systems with additive noise under a control law that depends on an estimate of the mode signal. Moreover, the closed-loop system under the control law that we propose resembles a fault tolerant control system with normal/faulty modes and a "fault detection and isolation scheme" which is explored in [21] and [22]. In this sense, the investigation of the stability of this closed-loop system is also important due to possible applications in the field of fault-tolerant control systems as well. Based on our stability analysis for switched linear stochastic dynamical systems, we obtain sufficient conditions under which the proposed control law achieves stabilization with probability one.

The paper is organized as follows. In Section II, the notation used in the paper is explained; moreover, a review of Markov chains, Poisson processes, and the definition of almost sure asymptotic stability are given. In Section III, we present the mathematical model for continuous-time switched linear stochastic dynamical systems, and provide sufficient conditions of stability. Furthermore we propose a stabilizing control law that depends on the mode signal. We investigate feedback stabilization of switched linear stochastic systems under limited mode information in Section IV. A numerical example is provided in Section V to demonstrate the utility of our results. Finally, we conclude the paper in Section VI.

II. Mathematical Preliminaries

In this section we introduce notation, several definitions, and some key results concerning stochastic dynamical systems that are necessary for developing the main results of this paper. Specifically, $\mathbb{R}$ denotes the set of real numbers,

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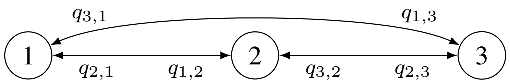

Fig. 1. Transition diagram of a 3-state Markov chain

$\mathbb{R}^n$ denotes the set of $n \times 1$ real column vectors, $\mathbb{R}^{n\times m}$ denotes the set of $n \times m$ real matrices, $\mathbb{N}$ and $\mathbb{N}_0$ respectively denote positive and nonnegative integers, and $\|\cdot\|$ denotes the Euclidean vector norm. Furthermore, we write $(\cdot)^{\mathrm{T}}$ for transpose and $\mathrm{tr}(\cdot)$ for trace of a matrix, I_n for the identity matrix of dimension n, $\lambda_{\min}(M)$ (resp., $\lambda_{\max}(M)$) for the minimum (resp., maximum) eigenvalue of the Hermitian matrix M, and $J_n^i \in \mathbb{R}^{n\times n}$ for the matrix with the (i,i) entry being 1 and the rest of the entries being zero. Finally, ∇V denotes the vector of the first order spatial derivatives of a twice continuously differentiable scalar V, that is, $\nabla V = \left[\frac{\partial V}{\partial x_1}, \frac{\partial V}{\partial x_2}, \ldots, \frac{\partial V}{\partial x_n}\right]$ and $\nabla(\nabla V)$ denotes the matrix of the second-order spatial derivatives of V, that is,

$$\nabla(\nabla V) = \begin{bmatrix} \frac{\partial^2 V}{\partial x_1 \partial x_1} & \cdots & \frac{\partial^2 V}{\partial x_1 \partial x_n} \\ \vdots & \ddots & \vdots \\ \frac{\partial^2 V}{\partial x_n \partial x_1} & \cdots & \frac{\partial^2 V}{\partial x_n \partial x_n} \end{bmatrix}.$$

Let $(\Omega, \mathcal{F}, \mathbb{P})$ be a probability space. A filtration $\{\mathcal{F}_t\}_{t\geq 0}$ on this probability space is a family of σ-algebras such that

$$\mathcal{F}_s \subset \mathcal{F}_t \subset \mathcal{F}, \quad 0 \leq s < t.$$

A stochastic process $\{x(t)\}_{t\geq 0}$ is adapted to the filtration $\{\mathcal{F}_t\}_{t\geq 0}$ if the random variable $x_t : \Omega \to \mathbb{R}^n$ is $\mathcal{F}_t$-measurable, that is,

$$\{\omega \in \Omega : x_t(\omega) \in B\} \in \mathcal{F}_t, \quad t \geq 0,$$

for all Borel sets $B \subset \mathbb{R}^n$.

A. Markov Chains

A finite-state Markov chain is a piecewise-constant stochastic process that takes values from a finite set $I \triangleq \{1, 2, \ldots, M\}$. Mathematically, it is defined to be the $\mathcal{F}_t$-adapted right-continuous stochastic process $\{r(t) \in I\}_{t\geq 0}$, with $r(0) = r_0 \in I$. A Markov chain is characterized by a generator matrix $Q \in \mathbb{R}^{M\times M}$, which determines the transition rates between each pair of states $i, j \in I$ such that

$$\mathbb{P}[r(t+\Delta t) = j | r(t) = i] = \begin{cases} q_{i,j}\Delta t + o(\Delta t), & i \neq j, \\ 1 + q_{i,j}\Delta t + o(\Delta t), & i = j, \end{cases}$$

where $q_{i,j}$ denotes the (i,j)th element of the matrix Q. Note that $q_{i,j} \geq 0$, $i \neq j$ and $q_{i,i} = -\sum_{j\neq i} q_{i,j}$, $i \in I$. A Markov chain can be represented by a state transition diagram. For instance, a 3-state Markov chain is represented by a graph of 3 nodes as shown in Fig. 1. The nodes in the figure represent the states of the Markov chain, the arrowed edges represent a possible transition between the states in the direction of the arrows, and the labels on the edges indicate the transition rates between the paired states. A finite-state Markov chain is called "irreducible" if it is possible to reach from any state to another state with one or more transitions. Thus, a finite-state Markov chain is irreducible if there exists a directed path from each node to another node in the state transition diagram. For example, the Markov chain presented in Fig. 1 is irreducible provided that $q_{i,j}$, $i, j \in \{1, 2, 3\}$, are nonzero. For all finite-state, irreducible Markov chains there exists a unique stationary probability distribution $\pi \triangleq [\pi_1, \ldots, \pi_M]^{\mathrm{T}} \in \mathbb{R}^M$ such that $\pi^{\mathrm{T}} Q = 0$, $\pi_i > 0$, $i \in I$, and $\sum_{i\in I} \pi_i = 1$ [23].

In this study, the mode signal, which manages the transition between subsystems (modes) of the switched system, is modeled as a finite-state Markov chain.

B. Poisson Processes

A Poisson process is a continuous-time stochastic process that counts the number of occurrences of some events. Mathematically, it is defined to be the $\mathcal{F}_t$-adapted stochastic process $\{N(t) \in \mathbb{N}_0\}_{t\geq 0}$ with $N(0) = 0$, where $N(t)$ denotes the number of events that occur in the time interval $(0, t]$. Probability of the occurrence of an event in a short time interval $(t, t+\Delta t]$ is given by

$$\mathbb{P}[N(t+\Delta t) = k+1 \,|\, N(t) = k] = \lambda\Delta t + o(\Delta t), k \in \mathbb{N}_0,$$

where $\lambda > 0$ denotes the intensity of occurrences. Length of intervals between consecutive events are distributed by the exponential distribution with parameter λ. A Poisson process has "stationary and independent increments". "Independent increments" property suggests that occurrences of events in non-overlapping intervals are independent. Moreover, as a result of "stationary increments" property, the number of events in any time interval is distributed with Poisson distribution depending only on the length of the interval. For Poisson processes, the probability of occurrences of more than one event at a time is zero. Additionally, only finite number of events occur in finite time intervals, almost surely.

C. Almost Sure Asymptotic Stability

In our analysis we adopt almost sure asymptotic stability notion. The zero solution $x(t) \equiv 0$ of a stochastic system is asymptotically stable almost surely if

$$\mathbb{P}[\omega \in \Omega : \lim_{t\to\infty} \|x_t(\omega)\| = 0] = 1, \tag{1}$$

for $t \geq 0$ with a fixed initial condition $x_0(\cdot)$. This notion is also called "asymptotic stability with probability one" [5].

III. Stability and Stabilization of Switched Linear Stochastic Dynamical Systems

In this section, we first provide the mathematical model for switched linear stochastic dynamical systems. We obtain sufficient conditions of almost sure asymptotic stability. Then, we consider switched linear stochastic dynamical systems with control input. Based on our stability analysis, we propose a piecewise-continuous control strategy that achieves stabilization of the zero solution of continuous-time switched linear stochastic dynamical systems.

A. Sufficient Conditions of Almost Sure Asymptotic Stability

Consider the continuous-time switched linear stochastic dynamical system given by

$$\mathrm{d}x(t) = A_{r(t)}x(t)\mathrm{d}t + D_{r(t)}x(t)\mathrm{d}W(t), \quad t \geq 0, \tag{2}$$

with initial conditions $x(0) = x_0$ and $r(0) = r_0$, where $\{x(t)\}_{t\geq 0}$ is the $\mathbb{R}^n$-valued $\mathcal{F}_t$-adapted state vector, $\{W(t)\}_{t\geq 0}$ is an $\mathbb{R}$-valued $\mathcal{F}_t$-adapted Wiener process, $A_i,\, D_i \in \mathbb{R}^{n\times n}$, $i \in I \triangleq \{1, 2, \ldots, M\}$, are subsystem matrices. The dynamical system (2) is assumed to have $M \geq 1$ number of subsystems (modes). The transition between the modes is characterized by the piecewise constant $\mathcal{F}_t$-adapted mode signal $\{r(t) \in I\}_{t\geq 0}$, which is assumed to be an irreducible Markov chain with generator matrix $Q \in \mathbb{R}^{M\times M}$ with a stationary probability distribution $\pi \in \mathbb{R}^M$. We assume that the Wiener process $\{W(t) \in \mathbb{R}\}_{t\geq 0}$ and the mode signal $\{r(t) \in I\}_{t\geq 0}$ are mutually independent stochastic processes.

The stability of the dynamical system given by (2) can be analyzed using a quadratic Lyapunov-like function.

Theorem 3.1: Consider the switched linear stochastic system given by (2). If there exist $P > 0$ and scalars $\zeta_i \in \mathbb{R}$, $i \in I$, such that

$$0 \geq A_i^{\mathrm{T}}P + PA_i + D_i^{\mathrm{T}}PD_i - \zeta_i P, \quad i \in I, \tag{3}$$

$$\sum_{i\in I}\pi_i\Big(\zeta_i - \frac{\lambda^2_{\min}(D_i^{\mathrm{T}}P + PD_i)}{2\lambda^2_{\max}(P)}\Big) < 0, \tag{4}$$

then the zero solution $x(t) \equiv 0$ of the system described by (2) is asymptotically stable almost surely.

Proof: The proof is omitted due to space limitations. □

We employ the stability result presented in Theorem 3.1 for investigating almost sure feedback stabilization problem in the following sections.

B. Feedback Stabilization

In this section, we develop a stabilizing control law for switched linear stochastic dynamical systems. Consider the continuous-time switched linear stochastic system with control input given by

$$\mathrm{d}x(t) = A_{r(t)}x(t)\mathrm{d}t + B_{r(t)}u(t)\mathrm{d}t + D_{r(t)}x(t)\mathrm{d}W(t), \tag{5}$$

for $t \geq 0$, with initial conditions $x(0) = x_0$ and $r(0) = r_0$, where $u(t) \in \mathbb{R}^m$ is the control input and $B_i \in \mathbb{R}^{n\times m}$, $i \in I$, are input matrices.

The stabilization problem here is to design a feedback control law which guarantees the almost sure asymptotic stability of the zero solution $x(t) \equiv 0$. By assuming that information on the mode signal $\{r(t) \in I\}_{t\geq 0}$ is available to the controller for $t \geq 0$, we propose a control law of the form $u(t) = K_{r(t)}x(t)$, where $K_i \in \mathbb{R}^{m\times n}$ denotes the state feedback gain for the ith mode. Note that the feedback matrix is switched when there is a mode transition. As a result, the control input may have discontinuities at mode switching instances, which we denote by the sequence $\{t_1,\, t_2, \ldots\}$.

Corollary 3.2: Consider the continuous-time switched linear stochastic dynamical system given by (5). If there exist $P > 0$ and scalars $\zeta_i \in \mathbb{R}$, $i \in I$, such that

$$0 \geq A_i^{\mathrm{T}}P + PA_i + D_i^{\mathrm{T}}PD_i - 2PB_iB_i^{\mathrm{T}}P - \zeta_i P, \quad i \in I, \tag{6}$$

and (4) are satisfied, then the feedback control law

$$u(t) = -B_{r(t)}^{\mathrm{T}}Px(t) \tag{7}$$

guarantees that the zero solution $x(t) \equiv 0$ of the switched stochastic system (5) is asymptotically stable almost surely.

Proof: The result is a direct consequence of Theorem 3.1 with A_i replaced by $A_i - B_iB_i^{\mathrm{T}}P$, $i \in I$. □

The proposed control law (7) is a function of the mode signal $\{r(t) \in I\}_{t\geq 0}$, and hence cannot be used for stabilization when the mode information is available only at certain time instances or when it is not available at all. For the case where the mode signal information is not available, one can seek a control law of the form

$$u(t) = Kx(t), \tag{8}$$

which does not depend on the mode signal $\{r(t) \in I\}_{t\geq 0}$. On the other hand, when mode signal is sampled and only available at certain time instances, sampled mode information can also be employed in the control law.

IV. Feedback Stabilization Under Limited Mode Information

In this section we explore feedback stabilization problem for the case where the mode signal information $\{r(t) \in I\}_{t\geq 0}$ of the switched linear stochastic system (5) is available only at certain time instances, which we denote by the sequence $\{\tau_0 = 0, \tau_1, \tau_2, \ldots\}$. We assume that the length of time intervals between these instances are independent random variables that are distributed by exponential distribution with parameter $\lambda > 0$. As a result, these time instances correspond to occurrences of events of a Poisson process $\{N(t) \in \mathbb{N}_0\}_{t\geq 0}$ with the parameter $\lambda > 0$. We call λ the mode sampling intensity parameter.

The elements of the sequence $\{\tau_0,\, \tau_1,\, \tau_2,\, \ldots\}$ are characterized by

$$\tau_k \triangleq \inf\{\tau : N(t) \geq k\}, \quad k \in \mathbb{N}_0. \tag{9}$$

Note that when the mode sampling intensity λ is small, the length of the time intervals $(\tau_k, \tau_{k+1}]$, $k \in \mathbb{N}_0$, are likely to be large; therefore, the mode signal information is expected to be rarely available.

By employing the "sample and hold" technique we construct the sampled mode signal $\{\sigma(t) \in I\}_{t\geq 0}$ of the mode signal $\{r(t) \in I\}_{t\geq 0}$ by using only the available mode samples $\{r(\tau_0), r(\tau_1), r(\tau_2), \ldots\}$ as

$$\sigma(t) \triangleq r(\tau_{N(t)}), \quad t \geq 0. \tag{10}$$

At time instances $\{\tau_0, \tau_1, \tau_2, \ldots\}$, the sampled mode signal is equal to the actual mode signal of the plant, that is, $\sigma(\tau_k) = r(\tau_k)$, $k \in \mathbb{N}_0$. Furthermore, the sampled mode signal may be discontinuous at the time instance τ_k, $k \in \mathbb{N}$,

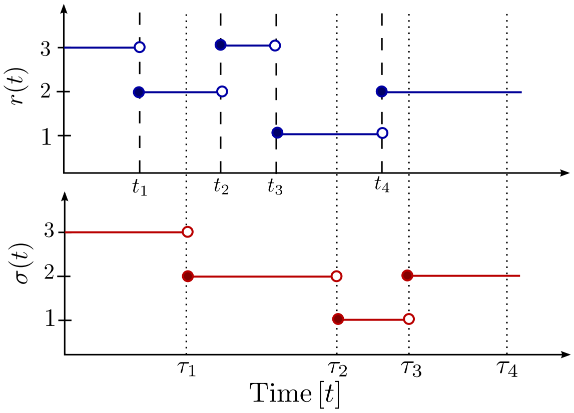

Fig. 2. Actual mode signal $r(t)$ and the sampled mode signal $\sigma(t)$ versus time

if a mode switch occurs in the time interval $(\tau_{k-1}, \tau_k]$. Fig. 2 shows a sample path of the actual mode signal $r(t)$ and the sampled mode signal $\sigma(t)$ of a switched system (5) with $M = 3$ modes. Note that when the mode sampling intensity parameter λ is sufficiently large, mode signal information samples will be frequently available; therefore, $\{\sigma(t) \in I\}_{t\geq 0}$ is likely to be a good representation of the mode signal.

Now, we show that under certain conditions, the zero solution of the switched linear system (5) can be stabilized by a controller that depends only on the sampled information of the mode signal rather than the actual mode signal. Specifically, we consider the control law of the form

$$u(t) = K_{\sigma(t)} x(t). \tag{11}$$

The closed-loop system (5) under the control law (11) is given by

$$\mathrm{d}x(t) = (A_{r(t)} + B_{r(t)} K_{\sigma(t)}) x(t) \mathrm{d}t + D_{r(t)} x(t) \mathrm{d}W(t). \tag{12}$$

We now verify that the closed-loop system (12) can be expressed as a switched linear stochastic dynamical system described by (2). For finite values of the mode sampling intensity parameter λ, the sampled mode signal is imperfect, that is, the actual mode signal $r(t)$ and the sampled mode signal $\sigma(t)$ may take different values when $t \neq \tau_k$, $k \in \mathbb{N}_0$. We define the bivariate stochastic process

$$\{\hat{r}(t)\}_{t\geq 0} \triangleq \{\big(r(t), \sigma(t)\big)\}_{t\geq 0}. \tag{13}$$

Under the assumption that the Poisson process $\{N(t) \in \mathbb{N}_0\}_{t\geq 0}$ and the mode signal $\{r(t) \in I\}_{t\geq 0}$ are independent stochastic processes, for any $i, j, k, l \in I$,

$$\begin{aligned} &\mathbb{P}[\hat{r}(t+\Delta t) = (j,l) \,|\, \hat{r}(t) = (i,k)] \\ &= \begin{cases} q_{i,j}\Delta t + o(\Delta t), & i \neq j,\ k = l, \\ 1 + q_{i,i}\Delta t + o(\Delta t), & i = j = k = l, \\ \lambda \Delta t + o(\Delta t), & i = j,\ k \neq l,\ i \neq k, \\ 1 + q_{i,i}\Delta t - \lambda\Delta t + o(\Delta t), & i = j,\ k = l,\ i \neq k, \\ o(\Delta t), & \text{otherwise}. \end{cases} \end{aligned} \tag{14}$$

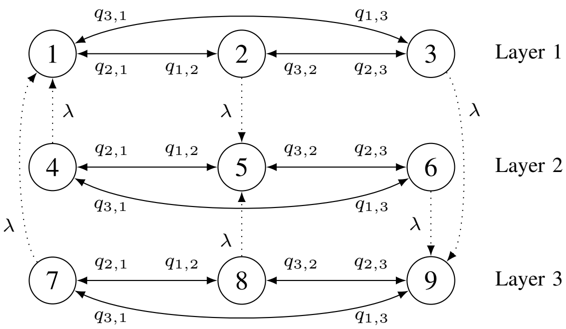

Fig. 3. Transition diagram of a Markov chain of 9 states with a special structure for $M = 3$

It follows that the bivariate stochastic process $\{\hat{r}(t)\}_{t\geq 0}$ is a Markov chain with M^2 states given by $\{(1,1), (1,2), \ldots, (1,M), (2,1), (2,2), \ldots, (2,M), \ldots, (M,1), (M,2), \ldots, (M,M)\}$. We enumerate the states in this order as $\hat{I} \triangleq \{1, 2, \ldots, M^2\}$. Furthermore, the generator of the Markov chain $\{\hat{r}(t) \in \hat{I}\}_{t\geq 0}$ is given by

$$\hat{Q} = \begin{bmatrix} T^1 & \lambda J_{\mathrm{M}}^2 & \cdots & \lambda J_M^M \\ \lambda J_M^1 & T^2 & \cdots & \lambda J_M^M \\ \vdots & \vdots & \ddots & \vdots \\ \lambda J_M^1 & \lambda J_M^2 & \cdots & T^M \end{bmatrix}, \tag{15}$$

where $T^i = Q - \lambda I_M + \lambda J_M^i$, $i \in I$.

The Markov chain $\{\hat{r}(t) \in \hat{I} = \{1, 2, \ldots, M^2\}\}_{t\geq 0}$ can be represented by a transition diagram with a special graph structure of M^2 nodes (Fig. 3). In this graph structure, the nodes are placed in M layers. The nodes in the ith layer are numbered as $\{(i-1)M+1, (i-1)M+2, \ldots, (i-1)M+M\}$. The graph structure of each separate layer resembles the transition diagram of the Markov chain $\{r(t) \in I\}_{t\geq 0}$. For example, an arrowed edge directed from the $((i-1)M+j)$th node to the $((i-1)M+k)$th node represents a possible transition from the state j to state k of the Markov chain $\{r(t) \in I\}_{t\geq 0}$. On the other hand, between two distinct layers i and j in the graph structure of the Markov chain $\{\hat{r}(t) \in \hat{I}\}_{t\geq 0}$, there exist two directed edges: one from the $((i-1)M+j)$th node in the ith layer to the $((j-1)M+j)$th node in the jth layer, and another one from the $((j-1)M+i)$th node in the jth layer to the $((i-1)M+j)$th node in the ith layer. The directed edge from the ith layer to the jth layer represents a possible change in the state of the sampled mode signal $\{\sigma(t) \in I\}_{t\geq 0}$ from i to j.

Since the mode signal $\{r(t) \in I\}_{t\geq 0}$ is irreducible, there exists a directed path between each pair of nodes within each layer of the transition diagram of the Markov chain $\{\hat{r}(t) \in \hat{I}\}_{t\geq 0}$. Furthermore, there exists a directed edge from each layer to another layer. It follows that there exists a directed path from each node to another node in the transition diagram of the Markov chain $\{\hat{r}(t) \in \hat{I}\}_{t\geq 0}$. We conclude that the Markov chain $\{\hat{r}(t) \in \hat{I}\}_{t\geq 0}$ is also irreducible. Consequently, there exists a unique stationary probability distribution $\hat{\pi} \in \mathbb{R}^{M^2}$ such that $\hat{\pi}^{\mathrm{T}}\hat{Q} = 0$, $\hat{\pi}_i > 0$, $i \in \hat{I}$, and $\sum_{i\in\hat{I}}\hat{\pi}_i = \sum_{i\in I}\sum_{j\in I}\hat{\pi}_{(i-1)M+j} = 1$

The Markov chain $\{\hat{r}(t) \in \hat{I}\}_{t\geq 0}$ is irreducible; therefore, we can express the closed-loop system (12) as a comparison system which is a switched linear stochastic dynamical system of M^2 modes described by (2) with subsystem matrices $A_{(i-1)M+j}$ replaced by $A_j - B_j K_i$, and $D_{(i-1)M+j}$ replaced by D_j, for $i, j \in I$. The transition between the modes of this comparison system is represented by the transition diagram of the Markov chain $\{\hat{r}(t) \in \hat{I}\}_{t\geq 0}$ with M layers.

We now state our main result on the almost sure asymptotic stabilization of the switched stochastic dynamical system (5) under sampled mode information. The result is based on the stability analysis for the comparison system (2) stated in Theorem 3.1.

Theorem 4.1: Consider the continuous-time switched linear stochastic dynamical system given by (5). If there exist $P > 0$ and scalars $\zeta_i \in \mathbb{R}$, $i \in I$, such that (6) and

$$\sum_{i\in I}\sum_{j\in I}\hat{\pi}_{(i-1)M+j}\Big(\beta_{i,j} - \frac{\lambda^2_{\min}(D_j^{\mathrm{T}}P + PD_j)}{2\lambda^2_{\max}(P)}\Big) < 0, \quad (16)$$

where

$$\beta_{i,j} = \begin{cases} \zeta_j, & i = j, \\ \zeta_j + \frac{2\lambda_{\max}(PB_jB_j^{\mathrm{T}}P)}{\lambda_{\min}(P)} \\ \quad - \frac{\lambda_{\min}(P(B_jB_i^{\mathrm{T}} + B_iB_j^{\mathrm{T}})P)}{\lambda_{\max}(P)}, & i \neq j, \end{cases} \quad (17)$$

and $\hat{\pi} \in \mathbb{R}^{M^2}$ is the unique stationary distribution of the Markov chain $\{\hat{r}(t) \in \hat{I} = \{1, 2, \ldots, M^2\}\}_{t\geq 0}$ characterized by the generator matrix $\hat{Q}$ given in (15), then the feedback control law (11) with the feedback gain matrix given by

$$K_{\sigma(t)} = -B^{\mathrm{T}}_{\sigma(t)}P, \quad (18)$$

guarantees that the zero solution $x(t) \equiv 0$ of the closed-loop system (5) and (11) is asymptotically stable almost surely.

Proof: The proof is omitted due to space limitations. □

The transition rates $q_{i,j}$, $i, j \in I$, as well as the mode sampling intensity λ affect the stability conditions of the closed-loop system under the control law (18). Note that the stationary distribution $\hat{\pi} \in \mathbb{R}^{M^2}$ also depends on the values of both $q_{i,j}$, $i, j \in I$, and λ. Therefore, the condition (16), which involves the stationary distribution $\hat{\pi} \in \mathbb{R}^{M^2}$, is satisfied only for certain values of $q_{i,j}$, $i, j \in I$, and λ.

When the transition rates $q_{i,j}$, $i, j \in I$, are large, the switchings between the modes of the system (5) are likely to be frequent. In this case, if the mode sampling intensity λ is very small, then the stationary probability distributions associated with the states $\{(i-1)M+j : i, j \in I, i \neq j\}$ are high. Furthermore, the sampled mode signal $\{\sigma(t) \in I\}_{t\geq 0}$ is expected to differ from the mode signal $\{r(t) \in I\}_{t\geq 0}$. On the contrary, when the mode switchings are statistically rare and the mode sampling intensity λ is sufficiently large, the stationary probability distributions associated with the states $\{(i-1)M+i : i \in I\}$ are high. Moreover, $\{\sigma(t) \in I\}_{t\geq 0}$ is expected to be a good representation of the mode signal $\{r(t) \in I\}_{t\geq 0}$.

V. Illustrative Numerical Example

In this section, we present a numerical example to demonstrate the efficacy of our approach. Specifically, we consider the switched linear stochastic dynamical system (5) with $M = 3$ modes described by the subsystem matrices given by

$$A_1 = \begin{bmatrix} 2 & -2 \\ 3 & 1.5 \end{bmatrix}, \quad B_1 = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix},$$
$$A_2 = \begin{bmatrix} 1.5 & 0 \\ 0 & 2 \end{bmatrix}, \quad B_2 = \begin{bmatrix} 0 & 0 \\ 0 & 0 \end{bmatrix},$$
$$A_3 = \begin{bmatrix} 2 & 1 \\ -0.5 & 3 \end{bmatrix}, \quad B_3 = \begin{bmatrix} -1 & 0.3 \\ 0.2 & 1 \end{bmatrix},$$

and $D_1 = D_2 = D_3 = I_2$. The mode signal $\{r(t) \in I \triangleq \{1, 2, 3\}\}_{t\geq 0}$ of the system is assumed to be a 3-state Markov chain characterized by the generator matrix

$$Q = \begin{bmatrix} -2 & 1 & 1 \\ 1 & -2 & 1 \\ 1 & 1 & -2 \end{bmatrix}. \quad (19)$$

The mode signal $\{r(t) \in I\}_{t\geq 0}$ is assumed to be available only at certain time instances. Furthermore, intervals between these time instances are assumed to be distributed independently by exponential distribution with the parameter $\lambda = 5$.

The bivariate stochastic process $\{\hat{r}(t) \in \hat{I} \triangleq \{1, 2, \ldots, 9\}\}_{t\geq 0}$ defined in (13) is a Markov chain with the unique invariant distribution given by

$$\hat{\pi}_{(i-1)M+j} = \begin{cases} 0.25, & i, j \in I,\ i = j, \\ \frac{0.25}{6}, & i, j \in I,\ i \neq j. \end{cases}$$

Note that the positive-definite matrix $P = 5I_2$ and the scalars $\zeta_1 = -4.3$, $\zeta_2 = 5$, $\zeta_3 = -3.3$ satisfy the conditions (6) and (16). Therefore, it follows from Theorem 4.1 that the control law (11) guarantees almost sure asymptotic stability of the zero solution $x(t) \equiv 0$ of the system given by (5).

With initial conditions $x(0) = [1, 1]^{\mathrm{T}}$ and $r(0) = 1$, Figs. 4 and 5 show sample paths of $x(t)$ and $u(t)$, respectively.

The piecewise-continuous control law (11) depends on the sampled mode signal information $\sigma(t)$. As a consequence, control profile is subject to jumps when $\sigma(t)$ changes its value at mode sampling instances. Note that both the mode sampling intensity and the frequency of mode switches directly affect the quality of the representation of the actual mode signal by the sampled mode signal. In this example, the sampling intensity $\lambda = 5$ is relatively high compared to the frequency of mode switches; consequently, the sampled mode signal $\sigma(t)$ closely matches the actual mode signal $r(t)$ (see Fig. 6).

VI. Conclusion

The stability of continuous-time switched linear stochastic systems was investigated. A quadratic Lyapunov-like function has been employed for obtaining sufficient almost sure asymptotic stability conditions. Moreover, feedback stabilization of the zero solution under sampled mode information

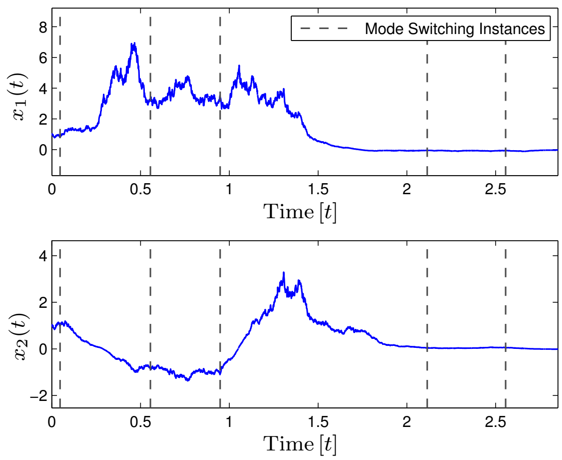

Fig. 4. State trajectory versus time

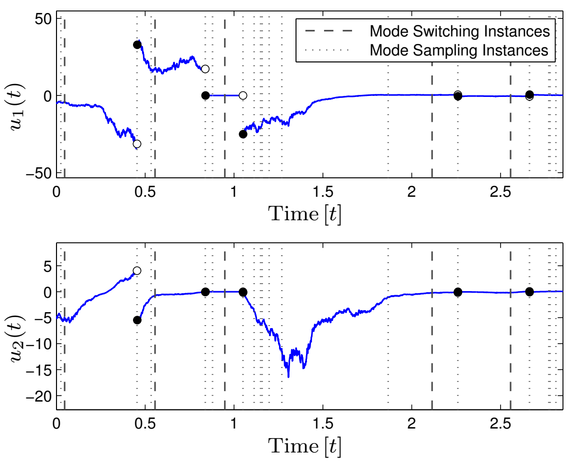

Fig. 5. Control input versus time

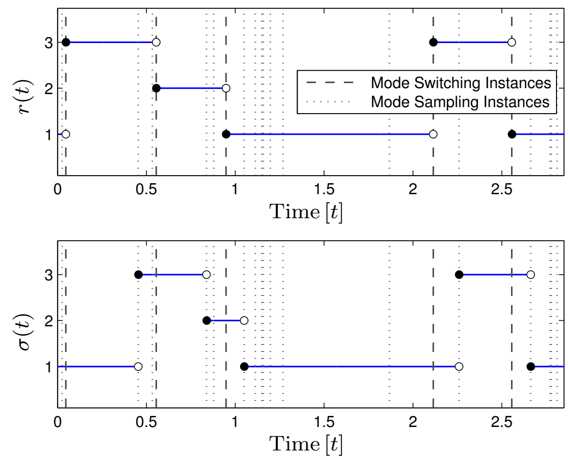

Fig. 6. Actual mode signal $r(t)$ and the sampled mode signal $\sigma(t)$ versus time

was explored. The intervals between mode sampling time instances are assumed to be exponentially distributed random variables. We proposed a piecewise-continuous control law that guarantees almost sure asymptotic stability of the zero solution. The proposed control law depends only on the sampled mode signal which is constructed from the available mode samples by using "sample and hold" technique. Future work includes extension of the results for the case where the random time intervals between mode sampling instances are characterized by a general probability distribution.

REFERENCES

[1] X. Feng, K. A. Loparo, Y. Ji, and H. J. Chizeck, "Stochastic stability properties of jump linear systems," *IEEE Trans. Autom. Contr.*, vol. 37, pp. 38–53, 1992.

[2] E. K. Boukas and H. Yang, "Stability of stochastic systems with jumps," *Math. Prob. Eng.*, vol. 3, pp. 173–185, 1996.

[3] P. Bolzern, P. Colaneri, and G. D. Nicolao, "On almost sure stability of continous-time Markov jump linear systems," *Automatica*, vol. 42, pp. 983–988, 2006.

[4] C. G. Cassandras and J. Lygeros, eds., *Stochastic Hybrid Systems*. Boca Raton: CRC Press, 2006.

[5] X. Mao and C. Yuan, *Stochastic Differential Equations with Markovian Switching*. Imperial College Press: London, 2006.

[6] Q. Luo and X. Mao, "Stochastic population dynamics under regime switching," *J. Math. Anal. Appl.*, vol. 334, pp. 69–84, 2007.

[7] C. Zhu and G. Yin, "On hybrid competetive Lotka-Volterra ecosystems," *Nonl. Anal.*, vol. 71, pp. 1370–1379, 2009.

[8] Q. Zhang, "Stock trading: An optimal selling rule," *SIAM J. Contr. Optim.*, vol. 40, pp. 64–87, 2001.

[9] D. V. Dimarogonas and K. J. Kyriakopulos, "Lyapunov-like stability of switched stochastic systems," in *Proc. Amer. Contr. Conf.*, (Boston, MA), pp. 1868–1872, 2004.

[10] C. Yuan and J. Lygeros, "On the exponential stability of switching diffusion process," *IEEE Trans. Autom. Contr.*, vol. 50, pp. 1422–1426, 2005.

[11] J. Luo, "Comparison principle and stability of Ito stochastic differential delay equations with Poisson jump and Markovian switching," *Nonl. Anal.*, vol. 64, pp. 253–262, 2006.

[12] X. Mao, G. G. Yin, and C. Yuan, "Stabilization and destabilization of hybrid systems of stochastic differential equations," *Automatica*, vol. 43, pp. 264–273, 2007.

[13] R. Z. Khasminskii, C. Zhu, and G. Yin, "Stability of regime-switching diffusions," *Stoc. Prc. Appl.*, vol. 117, pp. 1037–1051, 2007.

[14] G. G. Yin, B. Zhang, and C. Zhu, "Practical stability and instability of regime-switching diffusions," *J. Contr. Theo. Appl.*, vol. 6, pp. 105–114, 2008.

[15] G. Yin and F. Xi, "Stability of regime-switching jump diffusions," *SIAM J. Contr. Optim.*, vol. 48, pp. 4525–4549, 2010.

[16] G. Yin and C. Zhu, *Hybrid Switching Diffusions: Properties and Applications*. Springer-Verlag: New York, 2010.

[17] Y. Fang and K. A. Loparo, "Stabilization of continous-time jump linear systems," *IEEE Trans. Autm. Contr.*, vol. 47, pp. 1590–1602, 2002.

[18] E. K. Boukas, "Delay-dependent stability and output feedback stabilisation of Markov jump system with time-delay," *IEE Proc., Contr. Theo. Appl.*, vol. 149, pp. 379–386, 2002.

[19] Z. Fei, H. Gao, and P. Shi, "New results on stabilization of Markovian jump systems with time delay," *Automatica*, vol. 45, pp. 2300–2306, 2009.

[20] M. D. Fragoso and O. L. V. Costa, "Mean square stabilizability of continuous-time linear systems with partial information on the Markovian jumping parameters," *Stoc. Anal. App.*, vol. 22, pp. 99–111, 2005.

[21] M. Mahmoud, J. Jiang, and Y. Zhang, "Analysis of the stochastic stability for fault tolerant control systems," in *Proc. IEEE Conf. Dec. Contr.*, (Phoenix, AZ), pp. 3188–3193, 1999.

[22] F. Tao and Q. Zhao, "Synthesis of active fault-tolerant control based on Markovian jump system models," *IET Contr. Theo. App.*, vol. 1, pp. 1160–1168, 2007.

[23] J. Norris, *Markov Chains*. Cambridge University Press: New York, 2009.