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Robust Homogeneous Higher Order Sliding Mode Control

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Abstract— **This paper presents a new robust homogeneous higher order sliding mode control for nonlinear single-input single output (SISO) systems. The proposed law combines first order sliding mode control with homogeneous finite time stabilization. The performance of the control system has been evaluated in comparison with other homogeneity based control laws proposed by Hong [1] and Levant [2]. Simulation results highlight the better performance of our controller compared to others, and confirm its robustness property.**

I. INTRODUCTION

Sliding mode control (SMC) [3] is a powerful method to control high-order nonlinear dynamic systems operating under uncertainty conditions. The technique is based on applying discontinuous control on a system to reach a "sliding surface" (a surface comprising of the system trajectories) in finite time, and then to exactly keep the system dynamics on the surface. Naturally, the choice of the surface depends upon the control objective and other possible performance criteria [4]. When the trajectories reach the surface, the so-called "Sliding mode" is established [3], [4]. The existence of sliding mode makes the system robust against parametric uncertainties and external matched disturbances.

The robustness of standard sliding mode control is overshadowed by the chattering phenomenon, i.e. high frequency vibrations in the controlled system. These vibrations degrade the system's performance and may lead to instability. A number of methods have been proposed to reliably prevent chattering: among them, the boundary layer solution [5], observer-based solution [6]. Higher order sliding mode control (HOSMC) [7], [8], [9], [10], [11] has also been studied in the context of chattering. In HOSMC, the signum function acts on a higher order time derivative of sliding variable. This method, not only extends the properties of standard sliding mode to systems with higher relative degrees, but also reduces chattering effectively. In contemporary literature, Second order sliding mode controllers are the most popular. Several second order sliding mode algorithms have been proposed, some of the examples being [7], [12], [8], [13], [10].

Results on second order SMC have also been extended to arbitrary order SMCs (for example, in [9], [11] and [14]). The control problem can hence be considered as equivalent to the finite time stabilization of higher order integrator chains with bounded nonlinear uncertainties.

Finite time stabilization has been studied extensively in the context of r^{th} order integrator chains (for example, in [15]). Contemporary research works have however remained limited to linear uncertain systems. Hong [1] for example has studied continuous time-invariant feedback for global finite time stabilization. Their work has also been extended to a class of nonlinear uncertain systems [16], [17]. However the problem of practically applying their work remains unresolved as the control input is not explicit.

In this paper, we have proposed a control law that combines homogenous finite time stabilization control with sliding mode control. The problem of HOSMC has been formulated in input-output terms as ([18], [19], [15])

$$s^{(r)} = \varphi(.) + \gamma(.)u$$

where $\varphi(.)$ and $\gamma(.)$ are considered as bounded nonparametric uncertainties: in this case the system can be viewed as an uncertain linear system. The proposed constrictive algorithm is based on the modified form of the algorithm presented in [1], combined with first order sliding mode controller. The main idea is to apply the concept of finite time homogeneous stabilization to calculate a nonlinear manifold $\phi = f(s, \dot{s}, ...s^{(r-1)}) = 0$. Then a first order sliding mode control applied to this manifold ensures the establishment of higher order sliding mode in finite time with respect to the sliding variable s. Our contribution is twofold. First we have simplified the complex Hong's algorithm to calculate the finite time stabilization trajectory, by using "Tube Lemma". Then we have exploited the properties of sliding mode control to reach and exactly keep the trajectory in finite time.

The presented control law has some structural similarities to the algorithm proposed by Levant [10], [20], [2]. Therefore,we have also conducted a comparative study based on the nature of algorithms, and have presented the development and simulation results.

This paper has been divided as follows. The control problem has been formally presented in section 2. In section 3, we have presented the derivation of our control law. In section 4, we have presented the comparative study between our algorithm and Levant's algorithm. In section 5 we have presented simulation results and some conclusions have been presented in section 6.

II. PROBLEM FORMULATION

Let us consider an uncertain nonlinear system:

$$\begin{cases} \dot{x} = f(x,t) + g(x,t)u \\ y = s(x,t) \end{cases} \quad (1)$$

where $x \in \mathbb{R}^n$ and $u \in \mathbb{R}$ is the input control, a measured smooth output-feedback function (sliding variable). $f(x,t)$ and $g(x,t)$ are uncertain smooth functions. Assuming that

Assumption 1. The relative degree r of the system (1) with respect to s is constant and known and the associated zero dynamics are stable.

The control objective is to fulfill the constraint $s(x,t) = 0$ in finite time and to keep it exact by discontinuous feedback control. The r^{th} order sliding mode is defined by the following definition.

Definition 2.1 [10], [20]. Consider the nonlinear system (1), and let the system be closed by some possibly-dynamical discontinuous feedback. Then, provided that $s, \dot{s}, \ldots, s^{(r-1)}$ are continuous functions, and the set $S^r = \{x | s(x,t) = \dot{s}(x,t) = \ldots = s^{(r-1)}(x,t) = 0\}$, called "$r^{th}$ order sliding set", is non-empty and is locally an integral set in the Filippov sense [21], the motion on S^r is called "r^{th} order sliding mode" with respect to the sliding variable s.

The r^{th} order SMC approach allows the finite time stabilization to zero of the sliding variable s and its $r-1$ first time derivatives by defining a suitable discontinuous control function. If the system (1) is extended by the introduction of a fictitious variable $x_{n+1} = t, \dot{x}_{n+1} = 1$, and $f_e = (f^T 1)^T, g_e = g^T 0^T$ (where the last component corresponds to x_{n+1}), then the output s satisfies the equation [20]

$s^{(r)} = \varphi(.) + \gamma(.)u$, with $\gamma = L_{g_e} L_{f_e}^{r-1} s$ and $\varphi = L_{f_e}^r s$.

Assuming that:

Assumption 2. The solutions are understood in the Filippov sense [21], and system trajectories are supposed to be infinitely extendible in time for any bounded Lebesgue measurable input.

Assumption 3. Functions $\varphi(.)$ and $\gamma(.)$ are bounded uncertain functions, and, without loss of generality, let also the sign of control gain γ is strictly positive. Thus there exist $K_m \in \mathbb{R}^{+*}$, $K_M \in \mathbb{R}^{+*}$, $C_o \in \mathbb{R}^+$ such that

$$0 < K_m < \gamma < K_M, \quad |\varphi| \leq C_0$$

for $x \in X \subset R^n$, X being a bounded open subset of R^n within which the boundedness of the system dynamics is ensured. Assumption **Assumption 3** implies that results in the following sections of the paper can be considered as local. Then, the r^{th} order SMC of (1) with respect to the sliding variable s is equivalent to the finite time stabilization of

$$\begin{cases} \dot{z}_i = z_{i+1} \\ \dot{z}_r = \varphi(.) + \gamma(.)u \\ 1 \leq i \leq r-1, z = [z_1 z_2 \ldots z_r]^T := [s\dot{s} \ldots s^{(r-1)}]^T \end{cases} \tag{2}$$

Remark 2.1. Denoting the relative degree [22] of system (1) with respect to sliding variables s as ρ, the problem of higher order SMC for $r > \rho$ is a natural extension of the current work, through the extension of system (1) by $r - \rho$ length integrators chain. All the results displayed in following sections can then be applied to the extended system. For the sake of clarity, the current paper is only devoted to $r = \rho$ case.

III. Design of higher order sliding mode controller

As explained in the introduction, the problem has been formulated as the finite time stabilization of a sliding surface which is defined by integrator chain. Consider the nonlinear system (1) with a relative degree $\rho = r$ with respect to s. The r^{th} order sliding mode control with respect to s is equivalent to the finite time stabilization to zero of the uncertain linear system.

$$\begin{array}{rcl} \dot{Z}_1 & = & A_{11} Z_1 + A_{12} Z_2 \\ \dot{Z}_2 & = & \varphi + \gamma u \end{array} \tag{3}$$

where $Z_1 = [s \cdots s^{(r-2)}]^T$, $Z_2 = s^{(r-1)}$, $0 < K_m < \gamma < K_M$, $|\varphi| \leq C_0$ and $A_{11((r-1)\times(r-1))}$, $A_{12((r-1)\times 1)}$ defined by

$$A_{11} = \begin{bmatrix} 0 & 1 & \ldots & 0 & \ldots \\ \vdots & \ddots & \ddots & \ddots & \ddots \\ \vdots & \ddots & \ddots & \ddots & \ddots \\ 0 & \ddots & \ddots & \ldots & 1 \\ 0 & \ddots & \ddots & \ddots & 0 \end{bmatrix} \quad \text{and} \quad A_{12} = \begin{bmatrix} 0 \\ \vdots \\ 0 \\ 0 \\ 1 \end{bmatrix} \tag{4}$$

In order to express the system as a perturbed integrator chain,

- the design of a finite time fictitious controller $Z_2 = Z_{2f}$ which guarantees the finite time stabilization of the first subsystem of (3) (Z_1 dynamics) at the origin,
- the design of discontinuous control law $u(Z_1, Z_2)$ which guarantees first order sliding mode regime on the sliding manifold $\phi(Z)$ defined by $\phi(Z) = Z_2 - Z_{2f}$

Equation $\phi(Z) = 0$ describes the desired dynamics which satisfy the finite time stabilization of vector $[Z_1^T \; Z_2^T]^T$ to zero. The *global switching manifold* is defined as

$$\Phi \quad = \quad \{x \in X \mid \phi(Z_1, Z_2) = 0\} \tag{5}$$

on which, system (3) is forced to slide, via the discontinuous control u. In the following section, we will consider the finite time stabilization problem of the integrator chain, i.e. finding the fictitious control.

A. *Finite time stabilization of integrator chain*

Let us consider the first part of system (3), which is represented by the integrator chain defined as follows:

$$\begin{cases} \dot{z}_1 = z_2 \\ \quad \vdots \\ \dot{z}_{r-1} = z_r \end{cases} \tag{6}$$

The first step consists of finding a fictitious control law $z_r = Z_{2f}$ to stabilize system (6) in finite time.

We denote: $\lfloor z \rceil^{\alpha} = sign(z) * |z|^{\alpha}$, where α is positive real.

In this section, we will see how homogeneity can be used to achieve our objective.

Lemma 3.1 [1]. Suppose that, the time invariant system (7) below:

$$\dot{z} = f(z); f(0) = 0; z \in \mathbb{R}^{r-1} \tag{7}$$

is homogeneous of degree $k < 0$ with respect to the family of dilations δ_ε^p, $f(z)$ is continuous and $z = 0$ is its asymptotically stable equilibrium. Then the equilibrium of system (7) is globally finite-time stable.

1) Control law proposed by Hong [1]:
The application of homogeneity for finite time stabilization is not a recent approach. For example, let us consider the control law proposed in [1]. This law has been developed to stabilize an integrator chain in finite-time.

Theorem 1: [1]
Let $p_i, \beta_{i-1}, i = 1, ..., r-1$ and k, be constants satisfying given inequalities:

$$\begin{cases} p_1 & = & 1, ..., p_i = p_{i-1} + k, p_i > -k > 0, \\ & & i = 1, ..., r-1 \\ \beta_0 & = & p_2, (\beta_i + 1)p_{i+1} \geq (\beta_{i-1} + 1)p_i > 0, \\ & & i = 1, ..., r-3, \beta_{r-2} > 0, \end{cases} \tag{8}$$

Then there exist constants $l_i > 0, i = 1, 2, ..., r-1$ such that the control law $u(z) = u_{r-1}(z)$ renders the system (6) global-finite-time stable, where u_i, $i = 1, ..., r-1$ are defined as follows:

$$\begin{cases} u_0 & = & 0 \\ u_1 & = & -l_1 \lfloor \lfloor z_1 \rceil^{\beta_0} - 0 \rceil^{(p_1+k)/(p_1\beta_0)} \\ & \vdots & \\ u_{i+1} & = & -l_{i+1} \lfloor \lfloor z_{i+1} \rceil^{\beta_i} \\ & & - \lfloor u_i \rceil^{\beta_i} \rceil^{(p_{i+1}+k)/(p_{i+1}\beta_i)} \\ i & = & 0, ..., r-2. \end{cases} \tag{9}$$

2) New Control law:
The complexity in implementation and tuning of Hong's algorithm can be reduced by applying Tube Lemma.

Lemma 3.2 [23] (Tube Lemma): Consider the product space $X \times Y$, where Y is compact. If N is an open set of $X \times Y$ containing the slice $\{x_0\} \times Y$ of $X \times Y$, then N contains some tube $W \times Y$ about $\{x_0\} \times Y$, where W is a neighborhood of x_0 in X.

Lemma 3.3 [24]. Consider the system (7). Suppose f is homogeneous with respect to the family of dilations δ_ε^p and 0 is an attractive equilibrium under f. Then, 0 is a globally asymptotically stable equilibrium under f.

Lemma 3.4 [24]. Consider the system (7). Suppose f is homogeneous of degree k with respect to the family of dilations δ_ε^p and 0 is an asymptotically stable equilibrium under f. Then, for every $l > max\{-k, 0\}$, there exists a continuous, positive-definite function $V : \mathbb{R}^{r-1} \to \mathbb{R}$ that is homogeneous of degree l with respect to δ_ε^p, C^1 on $\mathbb{R}^{r-1}/\{0\}$, and such that $L_f V$ is continuous and negative definite.

Theorem 2: Consider the system (6), let $l_1, ..., l_{r-1} > 0$ be such that the polynomial $y^{r-1} + l_{r-1}(y^{r-2} + l_{r-2}(y^{r-3} + (... + l_2(y + l_1)...)))$ is Hurwitz.
There exist $k_m \in (-1/(r-1), 0)$ such that, for every $k \in (k_m, 0)$, the origin is globally finite-time stable for the system (6) under the feedback: $u_{r-1} = Z_{2f}(z_1, ..., z_{r-1})$ defined as follows:

$$\begin{cases} u_0 & = & 0 \\ u_1 & = & -l_1 \lfloor z_1 - 0 \rceil^{\alpha_1} \\ \vdots & & \\ u_{i+1} & = & -l_{i+1} \lfloor z_{i+1} - u_i \rceil^{\alpha_{i+1}} \\ i & = & 0, ..., r-2. \end{cases} \tag{10}$$

where $\alpha_i = (1 + i * k)/(1 + (i-1) * k), i = 1, ..., r-1$, $\alpha = \alpha_n$.
It can be seen that $u_{r-1} = Z_{2f}(z_1, ..., z_{r-1})$ can be written as below:
$u_{r-1} = -l_{r-1} \lfloor z_{r-1} + l_{n-1} \lfloor z_{r-2} + \lfloor ... + l_2 \lfloor z_2 + \lfloor z_1 \rceil^{\alpha_1} \rceil^{\alpha_2} ... \rceil^{\alpha_{r-2}} \rceil^{\alpha_{r-1}}$.

Proof: Let f_α denote the closed-loop vector field obtained by using the feedback (10) in (6). For each $\alpha > 0$, the vector field f_α is continuous and homogeneous of degree $k < 0$ with respect to the family of dilations $(p_1, ..., p_{r-1})$, where $p_i = 1 + (i-1) * k$, $i = 1, ..., r-1$.
Moreover, the vector field f_1 is linear with the Hurwitz characteristic polynomial $y^{r-1} + l_{r-1}(y^{r-2} + l_{r-2}(y^{r-3} + ... + l_2(y + l_1)))...))$. Therefore, by **Lemma 3.4**, there exists a positive-definite, radially unbounded, Lyapunov function $V : \mathbb{R}^{r-1} \to \mathbb{R}$ such that $L_{f_1} V$ is continuous and negative definite.
Let $\mathscr{A} = V^{-1}([0, 1])$ and $\mathscr{S} = bd\mathscr{A} = V^{-1}(\{1\})$. Then $\mathscr{A}$ and $\mathscr{S}$ are compact since V is proper and $0 \notin \mathscr{S}$ since V is positive definite.
Define $\varphi : (0, 1] \times \mathscr{S} \to \mathbb{R}$ by $\varphi(\alpha, z) = L_{f_\alpha} V(z)$. Then V is continuous and satisfies $\varphi(\alpha, z) < 0$ for all $z \in \mathscr{S}$, that is, $\varphi(\{1\} \times \mathscr{S}) \subset (-\infty, 0)$.
Since $\mathscr{S}$ is compact, by **tube lemma** there exists $\varepsilon > 0$ such that $\varphi((1-\varepsilon, 1] \times \mathscr{S}) \subset (-\infty, 0)$. It follows that for $\alpha \in (1-\varepsilon, 1]$, $L_{f_\alpha} V$ takes negative values on $\mathscr{S}$. Thus, $\mathscr{A}$ is strictly positively invariant under f_α for every $\alpha \in (1-\varepsilon, 1]$.
By **Lemma 3.3**, the origin is global asymptotic stable under f_α, for $\alpha \in (1-\varepsilon, 1]$. Finally, for $\alpha \in (1-\varepsilon, 1)$, by using **Lemma 3.1** the origin is globally finite-time stable. $\alpha = (1 + (r-1) * k)/(1 + (r-2) * k) \in (1-\varepsilon, 1)$ then $k \in (k_m, 0)$, where $k_m = -\frac{\varepsilon}{(r-2)*\varepsilon+1} < 0$.

■

B. Robust finite time controller design

Sliding mode control is used to converge the system's trajectories to the manifold (described in the previous section) in finite time. Once sliding mode is established, it rejects any parametric uncertainties and makes the control system robust.

Lemma 3.5 [15]. Consider the system (7). Suppose there are C^1 function $V(x)$ defined on a neighborhood $\hat{U} \subset \mathbb{R}^n$ of the origin, and real numbers $c > 0$ and $0 < \alpha < 1$, such that

1) $V(x)$ is positive definite on $\hat{U}$;
2) $\dot{V}(x) + cV^\alpha(x) \leqslant 0$, $\forall x \in \hat{U}$.

Then, the origin of system (7) is locally finite-time stable. If $\hat{U} = \mathbb{R}^n$ and $V(x)$ is also radially unbounded, the origin of system (7) is globally finite-time stable.
The sliding manifold ϕ, deduced from **Theorem 2**, is defined by $\phi_{r-1}(z_1, ..., z_r) = z_r - Z_{2f}$ and can be written in the

following form:

$$\begin{cases} \phi_0 &= z_1 \\ \phi_1 &= z_2 + l_1 * \lfloor \phi_0 \rceil^{\alpha_1} \\ &\vdots \\ \phi_i &= z_{i+1} + l_i * \lfloor \phi_{i-1} \rceil^{\alpha_i} \\ & i = 1, ..., r-1 \end{cases} \tag{11}$$

Where $Z_{2f} = -l_{r-1} * \lfloor \phi_{r-2} \rceil^{\alpha_{r-1}}$.

As an important aspect of our study is the comparison of our control law with that of Levant [2], we shall express the sliding manifold in the following form. Note that we have added the index 1 with the surface in this section. In the next section, as we introduce Levant's controller, we will identify his surface by the index 2.

$$\begin{cases} \phi_{0,1} &= z_1 \\ \\ N_{1,1} &= |\phi_{0,1}|^{\alpha_1} \\ \phi_{1,1} &= z_2 + l_1 * N_{1,1} * sign(\phi_{0,1}) \\ \vdots \\ N_{i,1} &= |\phi_{i-1,1}|^{\alpha_i} \\ \phi_{i,1} &= z_{i+1} + l_i * N_{i,1} * sign(\phi_{i-1,1}) \\ & i = 1, ..., r-1 \end{cases} \tag{12}$$

Theorem 3: Consider the nonlinear system (1) with a relative degree r with respect to the sliding variable $s(x,t)$. Suppose that hypotheses **Assumption 2** and **Assumption 3** are fulfilled and the system is minimum phase, with $\phi_{r-1,1}$ defined above:

$$\phi_{r-1,1}(z_1, ..., z_r) = z_r - Z_{2f} \tag{13}$$

then, the control input u defined by

$$u = -\mu sign(\phi_{r-1,1}(z_1, ..., z_r)) \tag{14}$$

$$with, \ \mu > \mu_{min} > 0 \tag{15}$$

leads to the establishment of r^{th} order sliding mode with respect to $\phi_{r-1,1}$ by attracting each trajectory in finite time.

Proof: In the design of a switching control function, the variable structure control u takes the form

$$u = -\mu sign(\phi_{r-1,1}(z_1, ..., z_r)) \tag{16}$$

and the gain μ is selected to satisfy the sliding mode condition [3] and **Lemma 3.5**,

$$\dot{\phi}_{r-1,1}.\phi_{r-1,1} < -\eta * |\phi_{r-1,1}| \tag{17}$$

The first derivative of $\phi_{r-1,1}$ is given by:

$$\dot{\phi}_{r-1,1} = \varphi + \gamma u - \dot{Z}_{2f} \tag{18}$$

By substitution (16) into (18) we get:

$$\dot{\phi}_{r-1,1} = -\mu * \gamma * sign(\phi_{r-1,1}) + \varphi - \dot{Z}_{2f} \tag{19}$$

therefore

$$\dot{\phi}_{r-1,1}.\phi_{r-1,1} = -\mu * \gamma * |\phi_{r-1,1}| + (\varphi - \dot{Z}_{2f}) * \phi_{r-1,1} \tag{20}$$

By choosing μ large enough the sliding mode condition (17) is satisfied.

The control law forces $\dot{\phi}_{r-1,1}$ and $\phi_{r-1,1}$ to zero in finite time, hence, Z_{2f} will become equal to z_r in finite time according to equation (13); and the system (3) will be reduced to one chain of integrator, stabilized in finite time by Z_{2f}. ∎

IV. COMPARISON CONTROL LAWS STUDY

As our controller is strikingly similar to the control algorithms presented by Levant [2], we have conducted a comparative study between the two algorithms.

A. Arbitrary-order sliding mode controller [2]

Let p be any positive number, $p \geq r$. Denote

$$\begin{cases} \phi_{0,2} &= z_1 \\ \\ N_{1,2} &= |z_1|^{(r-1)/r} \\ \phi_{1,2} &= z_2 + l_1 * N_{1,2} * sign(\phi_{0,2}) \\ &\vdots \\ N_{i,2} &= (|z_1|^{p/r} + |z_2|^{p/r-1} + ... + |z_i|^{p/r-i+1})^{(r-i)/p} \\ \phi_{i,2} &= z_{i+1} + l_i * N_{i,2} * sign(\phi_{i-1,2}) \\ & i = 1, ..., r-1 \end{cases} \tag{21}$$

Where $l_1, ..., l_{r-1}$ are positive numbers.

Theorem 4: [2]
Consider system (1) and let assumption **Assumption 1**, **Assumption 2**, **Assumption 3** have been fulfilled. Then with properly chosen positive parameters $l_1, ..., l_{r-1}$, μ the controller

$$u = -\mu sign(\phi_{r-1,2}(z_1, z_2, \cdots, z_r)) \tag{22}$$

leads to the establishment of an r-sliding mode $z_1 = 0$ attracting each trajectory in finite time. The convergence time is a locally bounded function of initial conditions.

A preliminary analysis of the previous control law shows that u forces the state $z_1, ..., z_r$ to reach the hyper-surface $\phi_{r-1,2} = 0$. On $\phi_{r-1,2} = 0$ the fictitious control law ($z_r = -l_{r-1} * N_{r-1,2} * sign(\phi_{r-1,2})$), attracts the state to reach the subsurface $(\phi_{r-2,2} = 0, \phi_{r-1,2} = 0)$, and so on.

On the other hand, the passage from $(\phi_{i+1,2} = 0)$ to $(\phi_{i,2} = 0, \phi_{i+1,2} = 0)$ is discontinuous and the subsurfaces do not have borders with previous surface as $sign(N_{i,2})$ is not equal to $sign(\phi_{i-1,2})$(except for i=1). For this reason, the states keep "jumping" on the two borders while converging to the origin where all surfaces and subsurface intersect.

B. Sliding Surfaces Comparison Study

The two controllers derived from **Theorem 3** and **Theorem 4** have similar structures with only one difference: $N_{i,1} \neq N_{i,2}$, for $i = 1, ..., r-1$. In this subsection, we study the effect of this difference on the sliding surfaces, **surface 1** ($\phi_{r-1,1} = 0$) and **surface 2** ($\phi_{r-1,2} = 0$).

For $r = 2$ the two surfaces are equivalent, as:

surface 1 is defined by:

$\phi_{1,1} = z_2 + l_1 * \lfloor z_1 \rceil^{\alpha_1} = 0$,($\alpha_1$ can be equal to 1/2)

and **surface 2** is defined by:

$\phi_{1,2} = z_2 + l_1 * \lfloor z_1 \rceil^{1/2} = 0$ Let's take the case $(r > 2)$:

It is evident that **surface 1** is continuous; in fact when $\phi_{i-1,1}$

vanishes to zero, $l_i * N_{i,1} * sign(\phi_{i-1,1})$ and $N_{i,1} = |\phi_{i-1,1}|^{\alpha_i}$ vanish as well, and **surface 1** remains continuous. On the other hand, when $\phi_{i-1,2}$ vanishes to zero, $l_i * N_{i,2} * sign(\phi_{i-1,2})$ will be $\pm l_i * N_{i,2}$
as $N_{i,2} \neq 0$ when $\phi_{i-1,2} = 0$, and therefore **surface 2** will be discontinuous. Hence the main difference between the two controllers is the discontinuity of the **surface 2**, which results in state jumping. This difference has been illustrated using simulation in the next section.

V. SIMULATION EXAMPLES

The performance of the two control laws presented in the previous sections has been evaluated through simulations. In the first simulation, we have studied stabilization of a chain of two integrators. The sliding surfaces have been presented in 3-D, to illustrate the evolution of the trajectories and the interaction of the surfaces. In the second simulation, an uncertain system has also been considered to verify the robustness and finite-time stabilization properties of the proposed control law.

A. Integrators chain example

In this simulation, we study the sliding surface resulted from the stabilization of a system of degree r = 3 ; In order to restrict the study to sliding surfaces, we suppose that the first order sliding mode is established $(z_3 = Z_{2f})$ and we study the fictitious control law Z_{2f} and the sliding surface $(\phi_2 = z_3 - Z_{2f} = 0)$.
Consider the below linear system (23):

$$\left\{ \begin{array}{lcl} \dot{z}_1 & = & z_2 \\ \dot{z}_2 & = & z_3 \end{array} \right. \tag{23}$$

System (23) is stabilized in finite time by $z_3 = Z_{2f,1}$ or $z_3 = Z_{2f,2}$:

$$\begin{array}{lcl} z_3 & = & Z_{2f,1} = -l_2 * \lfloor z_2 + l_1 * \lfloor z_1 \rceil^{\alpha_1} \rceil^{\alpha_2} \\ \phi_{2,1} & = & z_3 - Z_{2f,1} = z_3 + l_2 * \lfloor z_2 + l_1 * \lfloor z_1 \rceil^{\alpha_1} \rceil^{\alpha_2} = 0 \end{array} \tag{24}$$

where

$$l_1 = 1,\ l_2 = 1,\ \alpha_1 = 3/4,\ \alpha_2 = 2/3$$

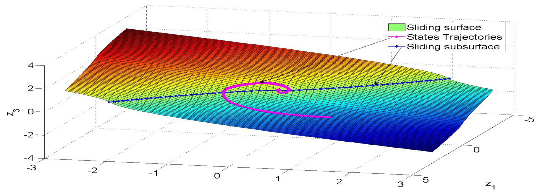

Fig. 1. **Surface 1**

$$\begin{array}{lcl} z_3 = Z_{2f,2} & = & -L_2 * \left| |z_2|^{p/r} + |z_1|^{p/2} \right|^{1/p} \\ & & * sign(z_2 + L_1 * \lfloor z_1 \rceil^{(r-1)/r}) \\ \phi_{2,1} = z_3 - Z_{2f,2} & = & z_3 + L_2 * \left| |z_2|^{p/r} + |z_1|^{p/2} \right|^{1/p} \\ & & * sign(z_2 + L_1 * \lfloor z_1 \rceil^{(r-1)/r}) \\ & & = 0 \end{array} \tag{25}$$

where

$$L_1 = 3,\ L_2 = 5,\ r = 3,\ p = 3$$

Contrary to the continuous **surface 1** and ($z_3 = Z_{2f,1}$), it

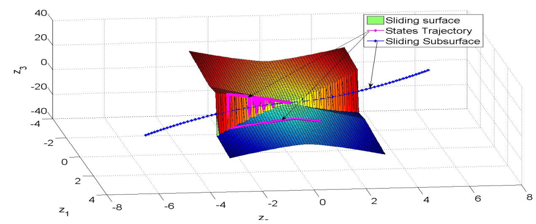

Fig. 2. **Surface 2**

can be seen in the figures that the **surface 2** and ($z_3 = Z_{2f,2}$) are discontinuous, and the subsurface ($\phi_{1,2} = 0, \phi_{2,2} = 0$) has no-border with the **surface 2** ($\phi_{2,2} = 0$). The trajectory has evolved discontinuously, seen as points jumping on the border. This state jumping is equivalent to the discontinuity of $z_3 = Z_{2f,2}$. The finite-time stability property of all the controllers can be verified by the remarkable fact that all surface and subsurface intersect at the origin.

B. Robust system simulation - an academic example

In order to verify the robustness property of the controllers, we have considered an academic kinematic model of a car [11] (see Fig.3) as the system to be stabilized. The system model is given by:

$$\begin{bmatrix} \dot{x}_1 \\ \dot{x}_2 \\ \dot{x}_3 \\ \dot{x}_4 \end{bmatrix} = \begin{bmatrix} w\cos(x_3) \\ w\sin(x_3) \\ {}^{w}/_{L}\tan(x_4) \\ 0 \end{bmatrix} + \begin{bmatrix} 0 \\ 0 \\ 0 \\ 1 \end{bmatrix} u$$

where x_1 and x_2 are the cartesian coordinates of the rear axle middle point, x_3 the orientation angle and x_4 the steering angle. u is the control input. w is the longitudinal velocity $(w = 10ms^{-1})$, and L the distance between the two axles $(L = 5m)$.
The velocity is supposed to be known with $\delta w = 5\%$ of error $(w = 100.5ms^{-1})$. The goal is to robustly steer the car from a given initial position to the trajectory $x_{2ref} = 10sin(0.05x_1) + 5$ in finite time; all the state variables are assumed to be measured in real time. We have defined the sliding variable

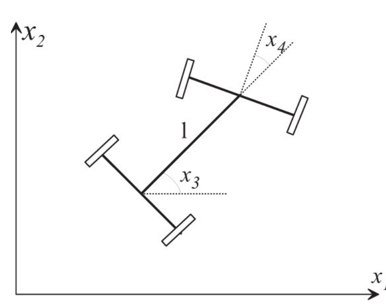

Fig. 3. Kinematic car model.

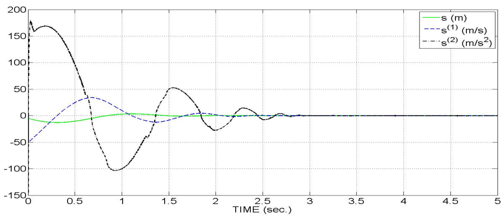

Fig. 4. $s(m)$, $\dot{s}$ (ms^{-1}) and $\ddot{s}$ (ms^{-2}) versus time (s).

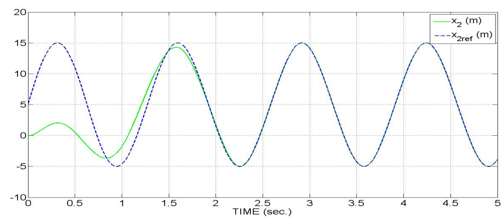

Fig. 5. x_2 (m) and x_{2ref} (m) versus time (s)

$s(x) = x_2 - x_{2ref}$: its relative degree with respect to the system is 3. A 3rd order sliding mode controller has been designed. The 3rd time derivative of s reads as $s^{(3)} = \varphi(\cdot) + \gamma(\cdot)u$, where

$$\begin{aligned}
\varphi(.) &= [\tfrac{1}{800} cos\left(\tfrac{x_1}{20}\right) * (cos(x_3))^2 \\
&\quad - \tfrac{1}{40L} sin\left(\tfrac{x_1}{20}\right) * \sin(x_3) * tan(x_4)] w^3 \cos(x_3) \\
&\quad + [-\tfrac{1}{20} sin\left(\tfrac{x_1}{20}\right) cos(x_3) sin(x_3) \\
\gamma(.) &= \tfrac{w^2}{L}\left[\tfrac{1}{2} cos\left(\tfrac{x_1}{20}\right) sin(x_3) + cos(x_3)\right] * \left[1 + tan^2(x_4)\right].
\end{aligned}$$

The 3^{rd} order SMC obtained is studied in 2 steps:

- The first step consists of defining the sliding surface, that is equivalent to the stabilization of a simple 2 integrator chain in finite time (by using the result of **theorem 2**).
- The second step consists of forcing the states to reach and remain on the sliding surface (by using **theorem 3**).

The robust control law can hence be expressed as:

$$\begin{aligned}
&u = -\mu * sign(\phi) with \\
&\phi = z_3 - u_2 \\
&where \\
&u_2 = -l_2 * \lfloor z_2 + l_1 * \lfloor z_1 \rceil^{\alpha_1} \rceil^{\alpha_2}
\end{aligned}$$

In our simulation, we have taken:

$$l_1 = 10,\ l_2 = 10,\ \alpha_1 = 3/4,\ \alpha_2 = 2/3,\ \mu = 5$$

Fig.4 displays the convergence of states (s, $\dot{s}$ and $\ddot{s}$) = (z_1, z_2 and z_3). Fig.5 displays the tracking of the desired trajectory by x_2 without chattering phenomena. These results show the applicability and robustness of the controller.

VI. CONCLUSIONS

In this paper, a new homogeneous higher order sliding mode controller has been proposed. The controller is able to steer to zero in finite time the output function of any uncertain smooth SISO dynamic system with known relative degree; The controller consists of two parts: a feedback controller, whose implementation is very easy, and a discontinuous part ensuring that system trajectories evolve on the proposed sliding surface. Simulation results show the effectiveness and robustness of the control law.

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