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System transformation of unstable systems induced by a shift-invariant subspace

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***Abstract*— Given an inner function, the orthogonal complement of the corresponding shift invariant subspace induces a system transformation for linear time-invariant systems, which is a generalization of the lifting technique for the sample-data control and Hambo-transform in the sense the inner function is arbitrary. This paper extends the transformation for systems with unstable eigenvalues, and derives a unified formula for transformation operators for both stable and anti-stable systems. A potential application is in the area of closed-loop system identification, where an unstable system is identified under the stabilizing feedback connection. The application to closed-loop system identification will be presented elsewhere.**

I. Introduction

Transformation is a versatile tool in the systems and control theory. The Fourier transform and the Laplace transform are essential in developing the theory. For example, it is a common knowledge that the Laplace transform of signals for a system described by a linear differential equation with constant coefficients leads to the notion of transfer functions.

A shift invariant subspace of the space of squarely integrable functions $L^2(0,\infty)$ is an important notion. For example, the lifting technique in sampled-data control [1], [12] uses the orthogonal complement to define a "lifted system," which becomes a fundamental tool to study the H^2 and H^∞ control problems. Another example is the Hambo transform induced by the generalized orthonormal basis functions [4], [5]. Actually, the two notions coincide when we work in an abstract way to define the system transform [8] using the orthogonal complement of a shift-invariant subspace corresponding to an inner function. Indeed, if the inner function is pure delay, it yields the lifting technique for sampled-data control, and if the inner function is rational, it yields the Hambo transform. Thus we can regard the transformation as a generalization of the lifting technique.

This extended version of the lifting technique has a number of applications. One such instance is the H^∞ control problem for a class of infinite dimensional systems. For such a class, so called Hamiltonian formula characterizing the minimal achievable H^∞ norm was derived in [13], [7], [9]. The approximation by a lower order model for of high order systems are studied using the transformation and filtered signals [3], [9]. Another important area is the system identification. It was shown in [10] that the transformation can be extended to stochastic systems where the signals are random processes and that standard subspace algorithms such as MOESP [11] can be employed.

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The purpose of the paper is to extend the transformation method for systems having unstable eigenvalues. Most of the works in the literature assumes that the system is stable [5], [10], with an exception of [9] where the transformation of the adjoint system of a stable system was considered. However, the formulae for stable and anti-stable systems are seemingly different.

This paper derives a unified formula for transformation operators for both stable and anti-stable systems. Moreover, we consider a feedback connection and prove that the transformation of the feedback system is the feedback of the the transformed systems. An application for such results is the closed-loop system identification problem, and it will be discussed in elsewhere [2].

II. Preliminaries

A. Signal spaces in time and frequency domains

Let $L^2(j\mathbb{R})$ be the space of square integrable functions of frequency $j\omega \in j\mathbb{R}$ with the inner product

$$\langle u, v\rangle = \frac{1}{2\pi}\int_{-\infty}^{\infty} \overline{v(j\omega)}u(j\omega)\, d\omega.$$

The space H^2 is the space of analytic functions in the right half plane with the norm

$$\|u\| = \sup_{\nu>0}\left(\frac{1}{2\pi}\int_{-\infty}^{\infty}|u(\nu+j\omega)|^2\ d\omega\right)^{1/2} < \infty. \qquad (1)$$

If $u \in H^2$, then non-tangential limits exists at almost every points on the imaginary axis, and the boundary value function is in $L^2(j\mathbb{R})$. With this relation, we identify H^2 as a subspace of $L^2(j\mathbb{R})$. It can be shown that the orthogonal complement of H^2 in $L^2(j\mathbb{R})$ is the space of analytic functions in the left half plane with the norm similarly defined as (1) except that $\nu < 0$, which is denoted as $H^2_\perp$.

The Fourier transform is the isomorphism between the signal spaces of time and frequency domains. The space $L^2(-\infty,\infty)$ of square integrable functions of time $-\infty < t < \infty$ is isomorphic to $L^2(j\mathbb{R})$ via the Fourier transform. Similarly, the spaces $L^2(0,\infty)$ and $L^2(-\infty,0)$ of square integrable functions of time $0 < t < \infty$ and $-\infty < t < 0$ are isomorphic to H^2 and $H^2_\perp$, respectively.

B. Multiplicative operator

A bounded function on the imaginary axis induces a multiplication operator on $L^2(j\mathbb{R})$ as a transfer function. The space of such bounded functions is L^∞ with the norm

$$\|h\|_\infty = \operatorname{ess\,sup}_\omega |h(j\omega)|.$$

For the sake of simplicity, we denote the multiplicative operator induced by a function $h \in L^\infty$ as h as well. It is easy to verify that the induced norm of the multiplicative operator $h : L^2(j\mathbb{R}) \to L^2(j\mathbb{R})$ is equal to $\|h\|_\infty$.

The space of bounded analytic functions in the right half plane is H^∞. If $h \in H^\infty$, then the multiplicative operator $h : L^2(j\mathbb{R}) \to L^2(j\mathbb{R})$ leaves H^2 invariant.

A function $\phi \in H^\infty$ is called inner if $|\phi(j\omega)| = 1$ for almost all ω. Let $\phi^\sim(s) = \overline{\phi(-\bar{s})}$ be the para-conjugate of ϕ. Then the statement $\phi(s)\phi^\sim(s) = 1$ is equivalent to ϕ is inner. Furthermore, ϕ as a multiplicative operator on $L^2(j\mathbb{R})$ is unitary.

C. Shift-invariant subspace and its orthogonal complement

If $\phi \in H^\infty$ is inner, then the space ϕH^2 is a closed subspace of H^2, called shift-invariant subspace. The orthogonal complement of ϕH^2 with respect to H^2 is denoted as $S = H^2 \ominus \phi H^2$, which plays an instrumental role in the subsequent discussion.

If ϕ is a rational inner function, say

$$\phi(s) = \frac{(p_1 - s)\cdots(p_r - s)}{(p_1 + s)\cdots(p_r + s)},$$

with distinct zeros, then S is spanned by

$$\left\{\frac{1}{p_1 + s}, \cdots, \frac{1}{p_r + s}\right\}.$$

If $\phi(s) = e^{-sh}$, $h > 0$, then the subspace S is nothing but the image of the Fourier transform of the squarely integrable functions supported on the interval $(0, h)$.

It is obvious that the spaces $L^2(j\mathbb{R})$, H^2, and $H^2_\perp$ have the following decompositions:

$$\begin{aligned} L^2(j\mathbb{R}) &= \oplus_{k=-\infty}^{\infty}\phi^k S, \\ H^2 = \oplus_{k=0}^{\infty}\phi^k S, &\quad H^2_\perp = \oplus_{k=-\infty}^{-1}\phi^k S, \end{aligned} \tag{2}$$

where $\phi^k = (\phi^\sim)^{-k}$ if $k < 0$.

From (2), any $u \in L^2(j\mathbb{R})$ has the expression

$$u = \sum_{k=-\infty}^{\infty} \phi^k u_k, \quad u_k \in S. \tag{3}$$

Furthermore, $\|u\|^2 = \sum_{k=-\infty}^{\infty} \|u_k\|^2$. In this sense, we can identify $L^2(j\mathbb{R})$ and $\ell^2(S)$.

If the signal u is vector-valued, we can apply the transformation component-wise. Thus (2) and (3) are valid if u_k is interpreted as an S-valued vector function.

III. Transformation of systems

A. Transformed system

Consider a linear system

$$\frac{d}{dt}x = Ax + Bu \tag{4}$$

$$y = Cx + Du, \tag{5}$$

where $A \in \mathbb{R}^{n\times n}$, and B, C, and D are matrices of compatible sizes. The system (4), (5) is denoted as (A, B, C, D) for simplicity. We assume that A does not have eigenvalues on the imaginary axis. Then the transfer function

$$h(s) = D + C\,(sI - A)^{-1}\,B \tag{6}$$

does not have poles on the imaginary axis, either. Hence $h \in L^\infty$, and it defines a multiplicative operator on $L^2(j\mathbb{R})$.

Because u and y are in $L^2(j\mathbb{R})$, the isomorphism between $L^2(j\mathbb{R})$ and $\ell^2(S)$ induces a bounded map h_D by the commutative diagram:

$$\begin{array}{ccc} L^2(j\mathbb{R}) & \xrightarrow{h} & L^2(j\mathbb{R}) \\ \downarrow & & \downarrow \\ \ell^2(S) & \xrightarrow{h_D} & \ell^2(S) \end{array} \tag{7}$$

The following theorem (Theorem 1) shows that the map h_D is shift-invariant and has a state space realization

$$\xi_{t+1} = \mathbf{A}\xi_t + \mathbf{B}u_t \tag{8}$$

$$y_t = \mathbf{C}\xi_t + \mathbf{D}u_t. \tag{9}$$

Theorem 1: Let $h : L^2(j\mathbb{R}) \to L^2(j\mathbb{R})$ be defined by the state space representation (4)-(5). Suppose that ϕ and $\phi^\sim$ are analytic at the spectrum of A. Then the map $h_D : \ell^2(S) \to \ell^2(S)$ defined by (7) has the realization (8)-(9), where the operators $\mathbf{A} : \mathbb{R}^n \to \mathbb{R}^n$, $\mathbf{B} : S \to \mathbb{R}^n$, $\mathbf{C} : \mathbb{R}^n \to S$, and $\mathbf{D} : S \to S$ are defined by

$$\mathbf{A}\xi = \phi^\sim(A)\xi, \tag{10}$$

$$\mathbf{B}u = \frac{1}{2\pi}\int_{-\infty}^{\infty} \overline{\Big(\phi^\sim(A)\,(j\omega I + A)^{-1}\,B} \;\overline{-\phi(j\omega)\,(j\omega I + A)^{-1}\,B\Big)}u(j\omega)d\omega, \tag{11}$$

$$(\mathbf{C}\xi)\,(s) = \Big(C\,(sI - A)^{-1} - \phi(s)C\,(sI - A)^{-1}\,\phi^\sim(A)\Big)\,\xi, \tag{12}$$

$$(\mathbf{D}u)\,(s) = h(s)u(s) - \phi(s)C\,(sI - A)^{-1}\,\mathbf{B}u. \tag{13}$$

Proof If the matrix A is stable, the theorem was proved in [9] except for the expression of the operator $\mathbf{B}$. From [9, Lemma 3],

$$\begin{aligned} \mathbf{B}u &= \int_{-\infty}^{0} \exp(-At)B\left(\mathcal{F}^{-1}\phi^\sim u\right)(t)dt \\ &= \frac{1}{2\pi}\int_{-\infty}^{\infty} \overline{(-j\omega I - A)^{-1}\,B}\phi^\sim(j\omega)u(j\omega)d\omega \\ &= \frac{1}{2\pi}\int_{-\infty}^{\infty} \overline{-\phi(j\omega)\,(j\omega I + A)^{-1}\,B}u(j\omega)d\omega \end{aligned}$$

where $\mathcal{F}^{-1}$ is the inverse Fourier transform. Note that $\phi^\sim(A)\,(sI + A)^{-1}\,B \in H^2_\perp$, and hence it is orthogonal to $u \in H^2$. This implies that the operator $\mathbf{B}$ is given by (11).

If the matrix A is anti-stable, the system $(-A, -B, C, D)$ is a stable system whose transfer function is $h(-s) = D - C\,(sI + A)^{-1}\,B$. Note that this corresponds to reversing the time axis. Let $\check{u}$ be the Fourier transform of the reversed signal of the inverse Fourier transform of $u \in L^2(j\mathbb{R})$, *i.e.*, $\check{u}(j\omega) = u(-j\omega)$. Express u as in (3). Then

$$\check{u} = \sum_{k=-\infty}^{\infty} \phi^k \check{u}_k, \quad \check{u}_k = \tau u_{-k+1} \tag{14}$$

where $\tau : S \to S$ is defined by

$$(\tau u)(s) = \phi(s)u(-s), \quad u \in S. \tag{15}$$

Notice that $\tau^2 = I$. Suppose that u and y are the input and the output of the system (A, B, C, D), respectively. Then $\check{u}$ and $\check{y}$ are the input and the output of $(-A, -B, C, D)$, respectively, where $\check{y}$ is defined similarly. Let $\mathbf{A}_s$, $\mathbf{B}_s$, $\mathbf{C}_s$, and $\mathbf{D}_s$ be the operators defined for the stable system $(-A, -B, C, D)$, or

$$\begin{aligned}
\mathbf{A}_s &= \phi^{\sim}(-A) = \phi(A) \\
\mathbf{B}_s u &= \frac{1}{2\pi}\int_{-\infty}^{\infty} \overline{\Big(-\phi^{\sim}(-A)(j\omega I - A)^{-1} B} \\
&\qquad \overline{+\phi(j\omega)(j\omega I - A)^{-1} B\Big)} u(j\omega) d\omega, \\
(\mathbf{C}_s\xi)(s) &= \Big(C(sI + A)^{-1} \\
&\qquad -\phi(s)C(sI + A)^{-1}\phi^{\sim}(-A)\Big)\xi, \\
(\mathbf{D}_s u)(s) &= h(-s)u(s) - \phi(s)C(sI + A)^{-1}\mathbf{B}_s u.
\end{aligned}$$

Thus the transformed system satisfies

$$\begin{aligned}
\zeta_{-t+2} &= \mathbf{A}_s\zeta_{-t+1} + \mathbf{B}_s\check{u}_{-t+1} \\
\check{y}_{-t+1} &= \mathbf{C}_s\zeta_{-t+1} + \mathbf{D}_s\check{u}_{-t+1}.
\end{aligned}$$

Putting $\xi_t = \zeta_{-t+2}$ and substituting (14), we have

$$\begin{aligned}
\xi_t &= \mathbf{A}_s\xi_{t+1} + \mathbf{B}_s\tau u_t \\
\tau y_t &= \mathbf{C}_s\xi_{t+1} + \mathbf{D}_s\tau u_t.
\end{aligned}$$

Since A is analytic at $\phi^{\sim}$, $\mathbf{A}_s$ is invertible. Hence the input u and the output y satisfy

$$\xi_{t+1} = \mathbf{A}_s^{-1}\xi_t - \mathbf{A}_s^{-1}\mathbf{B}_s\tau u_t \tag{16}$$

$$\begin{aligned}
y_t &= \tau\mathbf{C}_s\xi_{t+1} + \tau\mathbf{D}_s\tau u_t \qquad (17)\\
&= \tau\mathbf{C}_s\mathbf{A}_s^{-1}\xi_t + \left(\tau\mathbf{D}_s\tau - \tau\mathbf{C}_s\mathbf{A}_s^{-1}\mathbf{B}_s\tau\right)u_t.
\end{aligned}$$

From this, we have

$$\begin{aligned}
\mathbf{A}\xi &= \mathbf{A}_s^{-1}\xi = \phi^{\sim}(A)\xi \\
\mathbf{B}u &= -\mathbf{A}_s^{-1}\mathbf{B}_s\tau u \\
&= \frac{1}{2\pi}\int_{-\infty}^{\infty} -\phi^{\sim}(A)\overline{\Big(-\phi^{\sim}(-A)(j\omega I - A)^{-1} B} \\
&\qquad \overline{+\phi(j\omega)(j\omega I - A)^{-1} B\Big)}\phi(-j\omega)u(j\omega)d\omega \\
&= \frac{1}{2\pi}\int_{-\infty}^{\infty} \overline{\Big(\phi^{\sim}(A)(j\omega I + A)^{-1} B} \\
&\qquad \overline{-\phi(j\omega)(j\omega I + A)^{-1} B\Big)} u(j\omega) d\omega,
\end{aligned}$$

$$\begin{aligned}
(\mathbf{C}\xi) &= \left(\tau\mathbf{C}_s\mathbf{A}_s^{-1}\xi\right)(s) \\
&= \phi(s)\Big\{C(-sI + A)^{-1} \\
&\qquad -\phi(-s)C(-sI + A)^{-1}\phi^{\sim}(-A)\Big\}\phi^{\sim}(A)\xi \\
&= \Big(C(sI - A)^{-1} - \phi(s)C(sI - A)^{-1}\phi^{\sim}(A)\Big)\xi, \\
(\mathbf{D}u)(s) &= \left(\tau\mathbf{D}_s\tau u - \tau\mathbf{C}_s\mathbf{A}_s^{-1}\mathbf{B}_s\tau u\right)(s) \\
&= (\tau\mathbf{D}_s\tau u + \tau\mathbf{C}_s\mathbf{B}u)(s) \\
&= \phi(s)h(s)\phi(-s)u(s) \\
&\quad + \phi(s)\Big(C(-sI + A)^{-1} \\
&\qquad -\phi(-s)C(-sI + A)^{-1}\phi^{\sim}(-A)\Big)\mathbf{B}u \\
&= h(s)u(s) - \phi(s)C(sI - A)^{-1}u,
\end{aligned}$$

which turns out the same formulae as for a stable matrix. If the matrix A has both stable and anti-stable eigenvalues, then there is a non-singular matrix T such that

$$A = T\begin{bmatrix} A_s & 0 \\ 0 & A_a \end{bmatrix}T^{-1},$$

where A_s is stable and A_a is anti-stable. From the block diagonal structure, we conclude that the operators $\mathbf{A}$, $\mathbf{B}$, $\mathbf{C}$, and $\mathbf{D}$, are written exactly the same. Q.E.D.

Remark 1: Notice that the formulae for the operator $\mathbf{B}$ are seemingly different for stable and anti-stable systems in [9]. Theorem 1 uses the frequency domain, which proves useful for unifying the formulas for both stable and anti-stable systems.

B. Inverse

Consider the system (A, B, C, D) with the state space realization (4),(5) having the transfer function $h(s)$ as in (6). Let $(\mathbf{A}, \mathbf{B}, \mathbf{C}, \mathbf{D})$ be the operators of the transformed system.

Let K be a matrix of appropriate size. Then the operators of the transformed systems for the transfer functions $Kh(s)$, $h(s)K$, and $K + h(s)$ are easily derived as follows:

It is obvious that the transfer function $Kh(s)$ has a realization (A, B, KC, KD), and the corresponding operators $(\mathbf{A}, \mathbf{B}, K\mathbf{C}, K\mathbf{D})$. The transfer function $h(s)K$ has a realization (A, BK, C, DK), and the corresponding operators $(\mathbf{A}, \mathbf{B}K, \mathbf{C}, \mathbf{D}K)$. The transfer function $K + h(s)$ has a realization $(A, B, C, K + D)$, and the corresponding operators $(\mathbf{A}, \mathbf{B}, \mathbf{C}, K + \mathbf{D})$.

Less obvious is the inverse of a system. The following lemma shows that the inverse of the transformed system is the transformation of the inverse system.

Lemma 1: Consider the system (A, B, C, D) with the state space realization (4),(5) having the transfer function $h(s)$ as in (6). Assume that D is invertible, $h(s)^{-1}$ has the realization $(A_-, B_-, C_-, D_-) = \left(A - BD^{-1}C, BD^{-1}, -D^{-1}C, D^{-1}\right)$. Assume that A and A_- do not have eigenvalues on the imaginary axis. Let ϕ be an inner function, and $S = H^2 \ominus \phi H^2$. Assume that ϕ and $\phi^{\sim}$ are analytic at the spectra of A and A_-.

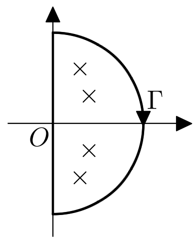

Fig. 1. The contour Γ which encircles the spectra of A and A_- in the right-half plane

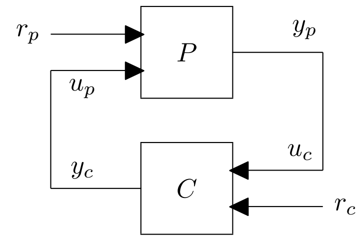

Fig. 2. Feedback connection

Let $(\mathbf{A}, \mathbf{B}, \mathbf{C}, \mathbf{D})$ and $(\mathbf{A}_-, \mathbf{B}_-, \mathbf{C}_-, \mathbf{D}_-)$ be the operators (10)-(13) for the systems for the systems (A, B, C, D) and (A_-, B_-, C_-, D_-), respectively. Then $\mathbf{D}$ is invertible. Furthermore $\mathbf{A}_- = \mathbf{A} - \mathbf{B}\mathbf{D}^{-1}\mathbf{C}$, $\mathbf{B}_- = \mathbf{B}\mathbf{D}^{-1}$, $\mathbf{C}_- = -\mathbf{D}^{-1}\mathbf{C}$, and $\mathbf{D}_- = \mathbf{D}^{-1}$.

Proof First, a straightforward calculation shows that

$$\begin{aligned}
&(\mathbf{D}_-\mathbf{D}u)(s) \\
&= h^{-1}(s)h(s)u(s) - \phi(s)h^{-1}(s)C\,(sI - A)^{-1}\,\mathbf{B}u \\
&\quad + \phi(s)D^{-1}C\,(sI - A_-)^{-1}\,\mathbf{B}_-\mathbf{D}u \\
&= u(s) - \phi(s)D^{-1}C\,(sI - A_-)\,(\mathbf{B}u - \mathbf{B}_-\mathbf{D}u)\,.
\end{aligned}$$

We shall prove $\mathbf{B} = \mathbf{B}_-\mathbf{D}$. Notice that

$$\begin{aligned}
&\mathbf{B}_-\mathbf{D}u \qquad (18)\\
&= \frac{1}{2\pi}\int_{-\infty}^{\infty} \overline{\Big\{\phi^{\sim}(A_-)\,(j\omega I + A_-)^{-1}\,BD^{-1}} \\
&\qquad \overline{-\phi(j\omega)\,(j\omega I + A_-)^{-1}\,BD^{-1}}\Big\} \\
&\qquad \times \Big\{\Big(D + C\,(j\omega I - A)^{-1}\,B\Big)\,u(j\omega) \\
&\qquad -\phi(j\omega)C\,(j\omega I - A)^{-1}\,\mathbf{B}u\Big\}\,d\omega \\
&= \frac{1}{2\pi}\int_{-\infty}^{\infty} \Big\{\phi(-j\omega)\,(j\omega I - A)^{-1}\,Bu(j\omega) \\
&\qquad -\phi^{\sim}(A_-)\,(j\omega I - A)^{-1}\,Bu(j\omega)\Big\}\,d\omega \\
&\quad + \frac{1}{2\pi j}\int_{\Gamma} \{(sI - A_-) - \phi^{\sim}(A_-)\phi(s)\,(sI - A_-) \\
&\qquad - (sI - A) + \phi^{\sim}(A_-)\phi(s)\,(sI - A)\}\,ds\mathbf{B}u.
\end{aligned}$$

where Γ is a closed contour in the right half plane that encircles clock-wise the anti-stable eigenvalues of A and A_- (see Fig. 1). Let E_- and E be the projection matrices on the anti-stable eigenspaces of A_- and A, respectively. Then, it follows that

$$\begin{aligned}
\frac{1}{2\pi j}\int_{\Gamma} (sI - A_-)^{-1}\,ds &= -E_-, \\
\frac{1}{2\pi j}\int_{\Gamma} \phi(s)\,(sI - A_-)^{-1}\,ds &= -\phi(A_-)E_-, \\
\frac{1}{2\pi j}\int_{\Gamma} (sI - A)^{-1}\,ds &= -E, \\
\frac{1}{2\pi j}\int_{\Gamma} \phi(s)\,(sI - A)^{-1}\,ds &= -\phi(A)E.
\end{aligned}$$

Hence form (11) and (18),

$$\begin{aligned}
&(\mathbf{B} - \mathbf{B}_-\mathbf{D})\,u \\
&= \frac{1}{2\pi}\int_{-\infty}^{\infty} \Big\{\phi^{\sim}(A_-)\,(j\omega I - A)^{-1}\,Bu(j\omega) \\
&\qquad -\phi^{\sim}(A)\,(j\omega I - A)^{-1}\,Bu(j\omega)\Big\}\,d\omega \\
&\quad + (I - \phi^{\sim}(A_-)\phi(A))\,E\mathbf{B}u.
\end{aligned}$$

If A is stable, then $(-j\omega I - A)^{-1}\,B \in H_{\perp}^2$ and $E = 0$. Thus $(\mathbf{B} - \mathbf{B}_-\mathbf{D})\,u = 0$. If A is anti-stable, then $E = I$ and

$$\mathbf{B}u = -\phi^{\sim}(A)\frac{1}{2\pi}\int_{-\infty}^{\infty} (j\omega I - A)^{-1}\,Bu(j\omega)d\omega.$$

If A has both stable and anti-stable eigenvalues, then the block diagonalization proves that $(\mathbf{B} - \mathbf{B}_-\mathbf{D})\,u = 0$. This proves $\mathbf{D}_-\mathbf{D} = I$. Hence $\mathbf{D}_- = \mathbf{D}^{-1}$ and $\mathbf{B}_- = \mathbf{B}\mathbf{D}^{-1}$. We can similarly prove $\mathbf{B}_-\mathbf{C} = \mathbf{A} - \mathbf{A}_-$ and $\mathbf{C}_- = -\mathbf{D}_-\mathbf{C}$. Q.E.D.

Remark 2: It should be noted that when the inner function is rational and the matrices A and A_- are stable the result was shown in [5]. Lemma 1 does not assume that the inner function is rational, and the system matrices may have unstable eigenvalues.

C. Feedback connection

Consider the feedback connection shown of a plant P and a controller C as shown in Fig. 2, where y_p and y_c are outputs, u_p and u_c are inputs, and r_p and r_c are exogenous inputs of the plant and the controller, respectively. Suppose that P and C are described by state-space realizations

$$\begin{aligned}
\frac{dx_p}{dt} &= A_p x_p + B_{p1}u_p + B_{p2}r_p \\
y_p &= C_p x_p + D_{p1}u_p + D_{p2}r_p \\
\frac{dx_c}{dt} &= A_c x_c + B_{c1}u_c + B_{c2}r_c \\
y_p &= C_c x_c + D_{c1}u_c + D_{c2}r_c.
\end{aligned}$$

Stack the variables

$$x_{\rm cl} = \begin{bmatrix} x_p \\ x_c \end{bmatrix}, \quad y_{\rm cl} = \begin{bmatrix} y_p \\ y_c \end{bmatrix}, \quad u_{\rm cl} = \begin{bmatrix} u_p \\ u_c \end{bmatrix}, \quad r_{\rm cl} = \begin{bmatrix} r_p \\ r_c \end{bmatrix},$$

and let

$$\begin{aligned}
A &= \begin{bmatrix} A_p & 0 \\ 0 & A_c \end{bmatrix}, \quad B_i = \begin{bmatrix} B_{pi} & 0 \\ 0 & B_{ci} \end{bmatrix},\ i = 1, 2 \\
C &= \begin{bmatrix} C_p & 0 \\ 0 & C_c \end{bmatrix}, \quad D_i = \begin{bmatrix} D_{pi} & 0 \\ 0 & D_{ci} \end{bmatrix},\ i = 1, 2.
\end{aligned}$$

Notice that the feedback connection imposes the relation

$$u_{\rm cl} = J y_{\rm cl}, \quad J = \begin{bmatrix} 0 & I \\ I & 0 \end{bmatrix}. \tag{19}$$

We assume that the feedback connection is well-posed and hence $I - JD$ is invertible. Hence the feedback connection in Fig. 2 when the input is $u_{\rm cl}$ and the output is $y_{\rm cl}$ has a state-space representation

$$\frac{dx_{\rm cl}}{dt} = A_{\rm cl} x_{\rm cl} + B_{\rm cl} r_{\rm cl} \tag{20}$$

$$y_{\rm cl} = C_{\rm cl} x_{\rm cl} + D_{\rm cl} r_{\rm cl}, \tag{21}$$

where

$$A_{\rm cl} = A + JB_1 (I - JD_1)^{-1} C \tag{22}$$

$$B_{\rm cl} = B_1 (I - JD_1)^{-1} JD_2 + B_2, \tag{23}$$

$$C_{\rm cl} = (I - D_1 J)^{-1} C, \quad D_{\rm cl} = (I - D_1 J)^{-1} D_2. \tag{24}$$

Let P and C have transformed system representations (8)-(9) using the operators $\left(\mathbf{A}_p, \begin{bmatrix}\mathbf{B}_{p1} & \mathbf{B}_{p2}\end{bmatrix}, \mathbf{C}_p, \mathbf{D}_p\right)$ and $\left(\mathbf{A}_c, \begin{bmatrix}\mathbf{B}_{c1} & \mathbf{B}_{c2}\end{bmatrix}, \mathbf{C}_c, \mathbf{D}_c\right)$, respectively. We would like to ask whether the transformed system of the feedback connection can be constructed from the transformed systems.

Let $(\mathbf{A}, \mathbf{B}_i, \mathbf{C}, \mathbf{D}_i)$ and $(\mathbf{A}_{\rm cl}, \mathbf{B}_{\rm cl}, \mathbf{C}_{\rm cl}, \mathbf{D}_{\rm cl},)$ be the operators of the transformed systems of (A, B_i, C, D_i) and $(A_{\rm cl}, B_{\rm cl}, C_{\rm cl}, D_{\rm cl})$, respectively. It is obvious that $\mathbf{A}$, $\mathbf{B}$, $\mathbf{C}$, and $\mathbf{D}$ satisfy

$$\mathbf{A} = \begin{bmatrix} \mathbf{A}_p & 0 \\ 0 & \mathbf{A}_c \end{bmatrix}, \quad \mathbf{B}_i = \begin{bmatrix} \mathbf{B}_{pi} & 0 \\ 0 & \mathbf{B}_{ci} \end{bmatrix}, \; i = 1, 2,$$

$$\mathbf{C} = \begin{bmatrix} \mathbf{C}_p & 0 \\ 0 & \mathbf{C}_c \end{bmatrix}, \quad \mathbf{D}_i = \begin{bmatrix} \mathbf{D}_{pi} & 0 \\ 0 & \mathbf{D}_{ci} \end{bmatrix}, \; i = 1, 2.$$

Theorem 2: Consider the feedback connection in Fig. 2. Assume that A_p, A_c, and $A_{\rm cl}$ do not have eigenvalues on the imaginary axis. Let ϕ be an inner function, and $S = H^2 \ominus \phi H^2$. Assume that ϕ and $\phi^{\sim}$ are analytic at the spectra of A_p, A_c, and $A_{\rm cl}$. Then the operators for the feedback system obey the following equations:

$$\mathbf{A}_{\rm cl} = \mathbf{A} + J\mathbf{B}_1 (I - J\mathbf{D}_1)^{-1} \mathbf{C}, \tag{25}$$

$$\mathbf{B}_{\rm cl} = \mathbf{B}_1 (I - J\mathbf{D}_1)^{-1} J\mathbf{D}_2 \mathbf{B}_2, \tag{26}$$

$$\mathbf{C}_{\rm cl} = (I - \mathbf{D}_1 J)^{-1} \mathbf{C}, \tag{27}$$

$$\mathbf{D}_{\rm cl} = (I - \mathbf{D}_1 J)^{-1} \mathbf{D}_2. \tag{28}$$

Proof The operators of the transformed system of the closed loop system is calculated by using Lemma 1. Details are omitted. Q.E.D.

If C is a stabilizing controller and the exogenous inputs are in H^2, then so are the inputs and the outputs. In this case, the transformed signals satisfy the following forward state equation even if the plant has unstable eigenvalues. More precisely, if $r_{\rm cl} \in H^2$, write

$$r_{\rm cl} = \sum_{k=0}^{\infty} \phi^k r_k, \quad r_k = \begin{bmatrix} r_{p,k} \\ r_{c,k} \end{bmatrix} \in S.$$

Then we have

$$\xi_{p,t+1} = \mathbf{A}_p \xi_{p,t} + \mathbf{B}_{p1} u_{p,t} + \mathbf{B}_{p2} r_{p,t}, \tag{29}$$

$$y_{p,t} = \mathbf{C}_p \xi_{p,t} + \mathbf{D}_{p1} u_{p,t} + \mathbf{D}_{p2} r_{p,t}, \tag{30}$$

$$\xi_{c,t+1} = \mathbf{A}_c \xi_{c,t} + \mathbf{B}_{c1} u_{c,t} + \mathbf{B}_{c2} r_{c,t}, \tag{31}$$

$$y_{c,t} = \mathbf{C}_c \xi_{c,t} + \mathbf{D}_{c1} u_{c,t} + \mathbf{D}_{c2} r_{c,t}, \tag{32}$$

with the feedback connection

$$\begin{bmatrix} u_{p,t} \\ u_{c,t} \end{bmatrix} = \begin{bmatrix} y_{c,t} \\ y_{p,t} \end{bmatrix}. \tag{33}$$

Remark 3: When ϕ is rational and A_p, A_c and $A_{\rm cl}$ are stable matrices, the results of this section was already proven in [5]. In this paper, we need not have to assume that ϕ is rational. Furthermore, we show that the assumptions on A_p and A_c are not necessary to obtain the result.

Remark 4: In [9], a stable system and an anti-stable system is connected in a special way to compute Schmidt pairs of a Hankel operator for a class of infinite dimensional systems. Theorem 2 considers the standard feedback connection.

D. Stochastic system

Consider the feedback system in Fig. 2 consisting of a plant and a controller having stochastic inputs. Describe the system by the following state equations:

$$dx_p = A_p x_p dt + B_{p1} d\zeta + B_{p2} dw, \tag{34}$$

$$d\eta = C_p x_p dt + D_{p1} d\zeta + D_{p2} dw, \tag{35}$$

$$dx_c = A_c x_c dt + B_c d\eta, \tag{36}$$

$$d\zeta = C_c x_c dt + D_c d\eta. \tag{37}$$

Define the signals

$$x_{\rm cl} = \begin{bmatrix} x_p \\ x_c \end{bmatrix}, \quad \eta_{\rm cl} = \begin{bmatrix} \eta_p \\ \eta_c \end{bmatrix}, \quad \zeta_{\rm cl} = \begin{bmatrix} \zeta_p \\ \zeta_c \end{bmatrix}.$$

Then the closed-loop system is described by

$$dx_{\rm cl} = A_{\rm cl} x_{\rm cl} dt + B_{\rm cl} d\omega \tag{38}$$

$$d\eta_{\rm cl} = C_{\rm cl} x_{\rm cl} dt + D_{\rm cl} d\omega. \tag{39}$$

where $A_{\rm cl}$, $B_{\rm cl}$, $C_{\rm cl}$, $D_{\rm cl}$ are exactly as in (22), (23), and (24).

Notice that if the controller stabilizes the feedback loop, then $A_{\rm cl}$ is stable. Thus the closed loop signals obey the results in [10]. In what follows, we will shall show that the stochastic signals in the transformed domain satisfy the discrete-time state-space equation even if the plant has unstable eigenvalues.

Let

$$w_{\delta,h}(t) = \begin{cases} \dfrac{w(t) - w(t-\delta)}{\delta}, & 0 < t \le h, \\ 0, & t > h. \end{cases}$$

Then $w_{\delta,h}$ is in $L^2(0,\infty)$ with probability 1. Let $y_{\delta,h}$ be the response of the system (20), (21) when the input $w_{\delta,h}$ is applied. If we consider the limit $\delta \to 0$, then the response of the system (38), (39) when the processes are terminated at time $t = h$ is recovered using the transformed system (29)-(33).

When ϕ is rational, then the space S is finite dimensional. Let

$$w_{\delta,h} = \sum_{k=0}^{\infty} \phi^k w_{\delta,h,k}, \quad y_{\delta,h} = \sum_{k=0}^{\infty} \phi^k y_{\delta,h,k}$$

We can show that as $\delta \to 0$ and $h \to \infty$ $w_{\delta,h,k}$ and $y_{\delta,h,k}$ are convergent sequences. Though the limits $\lim_{\delta\to 0, h\to\infty} w_{\delta,j,k}$ and $\lim_{\delta\to 0, h\to\infty} y_{\delta,j,k}$ are not squarely summable, they are the input and the output of the transformed system. Hence the transformed signals of (38), (39) satisfies the transformed system (29)-(33).

IV. Conclusions

This paper extended the results in [5], [10] to systems with unstable eigenvalues. It was shown that there is a unified formulae of transformation for stable and anti-stable systems, and that the transformed system can be described by a forward discrete-time system when the feedback system is stabilized even if the plant and/or the controller are unstable. The result can be applied to closed-loop system identification.

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