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Path following for a car-like robot using transverse feedback linearization and tangential dynamic extension

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***Abstract*— This article proposes a path following controller for the two input kinematic model of a car-like robot. A smooth dynamic feedback control law is designed to make the car's position follow a large class of curves with the desired speed along the curve. The controller guarantees the property of path invariance. The controller is designed by characterizing the path following manifold when one input is fixed. Once the path following manifold is found we apply dynamic extension to increase its dimension. We refer to this process as tangential dynamic extension. We then find a physically meaningful differentially flat output for the extended system which allows us to easily solve the path following problem.**

I. Introduction

The problem of generating accurate motion along a given path for a control system can be broadly classified as either path following problem or reference tracking problem. The authors in [1] highlighted the fundamental difference between path following and reference tracking. In path following the control objective is to make the output of the system approach and traverse a given path without *a priori* time parameterization associated to the motion along the path. In the tracking problem, the task of the controller is to make the output of the system approach and traverse a given path with a given time parameterization associated with the motion along the path. One of the main advantages of adopting the path following approach is that the path can be made an invariant set for the closed loop system. In the context of mobile robotics, this means that once the mobile robot is on the path, with appropriate orientation, it will never leave the path. In this paper we design a path following controller for the kinematic model of a car-like robot [2].

Tracking and stability of the car-like robot were analyzed in [3]. Path following for the mobile robot was studied in [4], [5], [6]. The approach in [4] is similar to the one followed in this paper. The main difference is that we take a set stabilization approach. In [7] it was shown that transverse feedback linearization can be used to solve the path following problem for the car-like robot. That solution ensures that the desired path is invariant for the closed-loop system but the translational velocity is fixed. The problem was reduced to the design of a controller for a single input system – only the steering control– but in doing so the robot motion along the path cannot be altered. To achieve the objective of path following while allowing control over the motion along the path we do not fix the translational velocity. We perform dynamic extension of the original system to achieve the desired goal. The resulting closed-loop system with dynamic controller is linear and controllable. Hence we are implicitly using the concept of differential flatness.

A large class of non-linear systems fall in the category of differentially flat systems. Roughly speaking, a nonlinear system is differentially flat if there exist a set of outputs (equal to the number of the inputs) such that all states and inputs can be uniquely determined from the desired output. Differentially flat systems were first introduced in [8] using differential algebra and later described using a Lie-Bäcklund transformation [9]. In [10] differential flatness was introduced under the setting of differential geometry. Finding a flat output involves finding a function that satisfies the conditions given in [11]. The search for a flat output can be simplified by noting that they often have strong geometric interpretations [12].

In this paper we choose a virtual output because it has very strong geometric meaning for the path following problem and show that it is a flat output. We use dynamic extension [13] of the original system to achieve the desired relative degree of the closed-loop system.

II. Path following for the car-like robot

Consider the kinematic model of a car-like robot, Figure 1,

$$\dot{x} = \begin{bmatrix} \cos x_3 & 0 \\ \sin x_3 & 0 \\ \frac{1}{\ell}\tan x_4 & 0 \\ 0 & 1 \end{bmatrix} \begin{bmatrix} v \\ \omega \end{bmatrix} \tag{1}$$

where $x \in \mathbb{R}^4$ is the state, the input $v \in \mathbb{R}$ is the translational speed and $\omega \in \mathbb{R}$ is the angular velocity of the steering angle. We take the car's position in the plane as the output of (1)

$$y = h(x) = \begin{bmatrix} x_1 & x_2 \end{bmatrix}^\top . \tag{2}$$

Suppose we are given a path to follow in the output space $\mathbb{R}^2$ of (1). Informally, our control objective is to design a control law that makes the output (2) of system (1) approach and move along the given path in a desired way. Suppose that the desired path is given as a regular parameterized curve

$$\begin{aligned} \sigma : \mathbb{D} &\to \mathbb{R}^2 \\ \lambda &\mapsto \begin{bmatrix} \sigma_1(\lambda) \\ \sigma_2(\lambda) \end{bmatrix}, \end{aligned} \tag{3}$$

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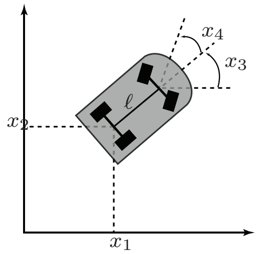

Fig. 1. The kinematic model of the car-like robot.

where $\sigma \in C^r$ with $r \geq 3$. Since σ is regular, without loss of generality, we can assume that it has a unit speed parameterization, i.e.,

$$(\forall\ \lambda \in \mathbb{D})\ \ \|\sigma'(\lambda)\| = 1.$$

Under this assumption, the curve σ is parameterized by its arc length. For closed curves with finite length L, this means that $\mathbb{D} = \mathbb{R} \bmod L$ and σ is L-periodic, i.e., for any $\lambda \in \mathbb{D}$, $\sigma(\lambda + L) = \sigma(L)$. When the curve is not closed $\mathbb{D} = \mathbb{R}$. We impose geometric restrictions on the class of curves, $\sigma(\cdot)$, considered [14].

Assumption 1: The path, $\sigma(\mathbb{D})$, is an embedded submanifold of $\mathbb{R}^2$ with dimension 1.

Assumption 2: There exists a smooth map $s : \mathbb{R}^2 \to \mathbb{R}^1$ such that 0 is a regular value of s and $\sigma(\mathbb{D}) = s^{-1}(0)$. Let $\gamma := s^{-1}(0)$.

Since the output (2) satisfies $\operatorname{rank}(dh_x) = 2$ for all $x \in \mathbb{R}^4$, the map $h : \mathbb{R}^4 \to \mathbb{R}^2$ is transversal [15] to γ and therefore, if Assumption 1 holds, the *lift* of γ to $\mathbb{R}^4$

$$\Gamma := (s \circ h)^{-1}(0) = \left\{ x \in \mathbb{R}^4 : s(h(x)) = 0 \right\}$$

is a three dimensional submanifold. Define $\alpha(x) := s \circ h(x)$.

Unlike previous work, we allow for dynamic control laws in this paper. Given a curve $\sigma(\mathbb{D})$ satisfying Assumptions 1 and 2, we seek a smooth control law of the form

$$\begin{aligned} \dot{\zeta} &= a(x,\zeta) + b(x,\zeta)u \\ \begin{bmatrix} v \\ \omega \end{bmatrix} &= c(x,\zeta) + d(x,\zeta)u. \end{aligned} \tag{4}$$

with $\zeta \in \mathbb{R}^q$ and $u = (u_1, u_2) \in \mathbb{R}^2$ and an open subset of initial conditions $U \times V \subset \mathbb{R}^4 \times \mathbb{R}^q$ such that $\gamma \subset h(U)$ and such that the closed-loop system satisfies

PF1 For each initial condition in $U \times V$, the output (2) along solutions of the closed-system (1), (4) asymptotically approaches the path.

PF2 The level set $s(y)$ is output invariant, i.e., if the system is initialized on the path with the velocity vector tangent to the curve the system remain on the path $\sigma(\mathbb{D})$ for all $t \geq 0$.

PF3 On the path, the car-like robot tracks a desired velocity or acceleration profile.

The dimension q of the controller state ζ is not fixed *a priori*. It will be determined based on analysis of the path following manifold which we discuss in the next section.

III. Dynamic extension

The path following manifold, denoted $\Gamma^\star$, associated with the curve γ is the maximal controlled invariant subset of the lift Γ. Physically it consists of all those motions of the car-like robot (1) for which the output signal (2) can be made to remain on the curve γ by suitable choice of control signal [16]. The path following manifold is the key object that allows one to treat the path following problem as a set stabilization problem. If the path following manifold can be made attractive and controlled invariant, then **PF1** and **PF2** will be satisfied.

When we apply the above definition to the car-like robot, or more generally, to any drift-less system, it is immediate that $\Gamma^\star = \Gamma$. This is because one can trivially make the entire set Γ controlled invariant by setting the translational speed v to zero. Specifically, in the case of the car, the equation

$$(\partial_{x_1}\alpha \cos x_3 + \partial_{x_2}\alpha \sin x_3)\, v = 0$$

can always be solved by choosing $v = 0$. From the point of view of mobile robots, this is not a useful characterization because such a controller causes the system to stop and hence not traverse the curve.

On the other hand, when $v \neq 0$ is fixed, the path following manifold can be characterized [7] using the steering input ω. In fact, in [7] it was shown that for the system (1) with v fixed, the function $\alpha = s \circ h$ yields a well-defined relative degree of 3 at each point on the path. This fact was used to apply transverse feedback linearization to stabilize the path following manifold and thereby solve the path following problem. The main deficiency with the solution presented in [7] is that **PF3** cannot be satisfied. In particular, since v is fixed the motion on the path is fixed. In this paper we present a solution to the path following problem that removes this deficiency.

Consider once again the model of a car like robot (1). The control objective is to make the output y approach and traverse the curve γ. Making $y \to \gamma$ is equivalent to making the state x of (1) approach the set Γ. Let $v = \mathrm{v} > 0$ be fixed. In [7] it was shown that for system (1), a path satisfying Assumptions 1 and 2, the path following manifold is given by

$$\begin{aligned} \Gamma^\star = \Big\{ x \in \mathbb{R}^4 : x = (\sigma(\lambda), \varphi(\lambda), \\ \arctan\left(\frac{\ell}{\mathrm{v}}\dot{\varphi}(\lambda)\right)\Big), \lambda \in \mathbb{D} \Big\} \end{aligned} \tag{5}$$

where $\varphi(\lambda) = \arg(\sigma_1' + j\sigma_2')$ is the angle $\sigma'(\lambda)$ makes with the y_1 axis.

Let $n^\star := \dim(\Gamma^\star)$. In this case $n^\star = 1$ and its co-dimension is $n - n^\star = 3$. Let $r^\star = 1$ denote the derivative of α at which the control input v appears. We use dynamic extension to generate a controller of the form (4) and thereby increase the dimension of the closed-loop system so that the dimension and co-dimension of $\Gamma^\star$ are equal. In other words, we delay the appearance of the input v in the derivatives of α so that ω and the delayed version of v appear in the same derivative. This effectively increases the dimension of the path following manifold; we call this approach tangential dynamic extension. This goal can be achieved if we increase the dimension of $\Gamma^\star$ by two which suggest we pick $q = n - n^\star - r^\star = 2$ in (4) so that the control law has two states $\zeta = (\zeta_1, \zeta_2)$.

Let $v = \mathrm{v} + \zeta_1$, where ζ_1 is the first state of our dynamics controller. In general [13] we are free to choose any dynamics for $\dot{\zeta}_1$ but we take the simplest possible structure for the control law (4) and let $\dot{\zeta}_1 = \zeta_2$. In order to finish defining the control law we let $\dot{\zeta}_2 = u_1$ where u_1 is a new, auxiliary input that we will use to indirectly change the translational velocity v. The structure of the control law so far is

$$\begin{aligned} \dot{\zeta}_1 &= \zeta_2 \\ \dot{\zeta}_2 &= u_1 \\ v &= \mathrm{v} + \zeta_1 \\ \omega &= u_2. \end{aligned} \tag{6}$$

For the extended system the path following manifold is given by

$$\Gamma^\star = \Big\{(x, \zeta) \in \mathbb{R}^4 \times \mathbb{R}^2 : x = (\sigma(\lambda), \varphi(\lambda), \arctan\left(\frac{\ell}{\mathrm{v}}\dot{\varphi}(\lambda)\right)\Big), \lambda \in \mathbb{D}\Big\}.$$

To simplify notation we will no longer distinguish between states of the system (x_1, x_2, x_3, x_4) and states of the controller (ζ_1, ζ_2). Let $x_5 := \zeta_1$, $x_6 := \zeta_2$. Therefore the system we study is given by

$$\begin{aligned} \dot{x} &= f(x) + g_1(x)u_1 + g_2(x)u_2 \\ &= \begin{bmatrix} (\mathrm{v} + x_5)\cos x_3 \\ (\mathrm{v} + x_5)\sin x_3 \\ \frac{(\mathrm{v}+x_5)}{\ell}\tan x_4 \\ 0 \\ x_6 \\ 0 \end{bmatrix} + \begin{bmatrix} 0 \\ 0 \\ 0 \\ 0 \\ 0 \\ 1 \end{bmatrix} u_1 + \begin{bmatrix} 0 \\ 0 \\ 0 \\ 1 \\ 0 \\ 0 \end{bmatrix} u_2 \end{aligned} \tag{7}$$

Our objective is to now design the control law $u = (u_1, u_2)$ to solve the the path following problem. Stabilizing the path following manifold in extended coordinates remains the key way to accomplish $\mathbf{PF1}$ and $\mathbf{PF2}$ in Section II. On the other hand since v is not fixed and because the path following manifold has dimension three, we expect to able to control the motion along the path in order to satisfy $\mathbf{PF3}$.

IV. Path following control design

In this paper we treat path following problem as a set stabilization problem and we follow the general approach of [16] for designing path following controllers, see also [14]. In order to satisfy $\mathbf{PF1}$ and $\mathbf{PF2}$ we first stabilize the path following manifold $\Gamma^\star$. Once the path manifold has been stabilized we use any remaining freedom in the control law to impose desired dynamics on the path.

We find a particular "virtual" output function for the system (7) and show that it yields a well defined relative degree. The benefit of using this physically meaningful output is that it facilitates control design. In this case the output yields a well-defined relative degree of $\{3, 3\}$ and hence system (7) is feedback linearizable.

Before implementing the above program we must introduce a projection operator in the output space of the car. This operator associates to each point y in the output space of (1) sufficiently close to the path γ a number in $\mathbb{D}$. Let $\gamma_\epsilon \subset \mathbb{R}^2$ denote a tubular neighbourhood of the curve γ. The tubular neighbourhood has the property that if $y \in \gamma_\epsilon$ then there exists a $y^\star \in \gamma$ that is closest to y. The tubular neighbourhood allows us to define the function

$$\begin{aligned} \varpi :& \gamma_\epsilon \to \mathbb{D} \\ & y \mapsto \arg\inf_{\lambda \in \mathbb{D}} \|y - \sigma(\lambda)\|. \end{aligned} \tag{8}$$

This function is as smooth as σ is which we assume to be at least C^3. Using the above map we define the "virtual" output function

$$\hat{y} = \begin{bmatrix} \pi(x) \\ \alpha(x) \end{bmatrix} = \begin{bmatrix} \varpi \circ h(x) \\ s \circ h(x) \end{bmatrix}. \tag{9}$$

We now show that as long as the car does not have zero translational speed, then this output yields a well-defined relative degree along the path.

Lemma 4.1: The dynamic extension of the car-like robot (7) with output (9) yields a well-defined vector relative degree of $\{3, 3\}$ at each point on $\Gamma^\star$ where $x_5 = \zeta_1 \neq -\mathrm{v}$.

Proof: Let $x^\star \in \Gamma$ be arbitrary. By definition of Γ the output $h(x^\star)$ is on the path γ. Let $\lambda^\star \in \mathbb{D}$ be such that $h(x^\star) = \sigma(\lambda^\star)$. By the definition of vector relative degree we must show that

$$L_{g_1}L_f^i\pi(x) = L_{g_2}L_f^i\pi(x) = L_{g_1}L_f^i\alpha(x) = L_{g_2}L_f^i\alpha(x) = 0$$

for $i \in \{0, 1\}$ in a neighbourhood of $x^\star$ and that the decoupling matrix

$$D(x^\star) = \begin{bmatrix} L_{g_1}L_f^2\pi(x^\star) & L_{g_2}L_f^2\pi(x^\star) \\ L_{g_1}L_f^2\alpha(x^\star) & L_{g_2}L_f^2\alpha(x^\star) \end{bmatrix} \tag{10}$$

is non-singular. Since

$$\frac{\partial\pi(x)}{\partial x_i} = \frac{\partial\alpha(x)}{\partial x_i} \equiv 0$$

for $i \in \{3, 4, 5, 6\}$, it is easy to check that $L_{g_j}L_f^i\pi(x) = L_{g_j}L_f^i\alpha(x) = 0$ for $i \in \{0, 1\}$, $j \in \{1, 2\}$.

To show that the decoupling matrix is full rank, it suffices to show that the determinant of $D(x^\star)$ is not zero. Direct calculations yield

$$\begin{aligned} L_{g_1}L_f^2\alpha &= \tfrac{(\mathrm{v}+x_5)^2}{\ell}(\partial_{x_2}\alpha\cos x_3 - \partial_{x_1}\alpha\sin x_3)\sec^2 x_4 \\ L_{g_2}L_f^2\alpha &= \partial_{x_1}\alpha\cos x_3 + \partial_{x_2}\alpha\sin x_3 \\ L_{g_1}L_f^2\pi &= \frac{(\mathrm{v}+x_5)^2}{\ell}(1 + \tan^2 x_4)(\sigma_2'\cos x_3 - \sigma_1'\sin x_3) \\ L_{g_2}L_f^2\pi &= \sigma_1'\cos x_3 + \sigma_2'\sin x_3 \end{aligned} \tag{11}$$

where $\sigma_i' = \frac{\partial \sigma_i}{\partial \lambda}\big|_{\lambda=\lambda^\star}$, $i \in \{1,2\}$. Hence

$$\det(D(x)) = \frac{(\mathrm{v}+x_5)^2}{\ell \cos^2 x_4}\left[\sigma_1' \partial_{x_2}\alpha - \sigma_2' \partial_{x_1}\alpha\right]. \qquad (12)$$

The only way for this determinant to vanish is if either (i) $\mathrm{v} = -x_5$ or (ii) $\sigma_1'\partial_{x_2}\alpha - \sigma_2'\partial_{x_1}\alpha = 0$. We argue that condition (ii) never occurs on the path because the vectors $\mathrm{col}(\partial_{x_1}\alpha, \partial_{x_2}\alpha)$ and σ' are orthogonal.

By the chain rule and the form of the output map (2)

$$\begin{bmatrix} \partial_{x_1}\alpha \\ \partial_{x_2}\alpha \end{bmatrix}_{x=x^\star} = \begin{bmatrix} \partial_{y_1}s \\ \partial_{y_2}s \end{bmatrix}_{y=h(x^\star)} = ds_{h(x^\star)}^\top.$$

By Assumption 2 the differential ds_y is non-zero when $y \in \gamma$. Thus the vector $ds_{h(x^\star)}^\top$ is a non-zero gradient vector and is orthogonal to the path at $h(x^\star)$. On the other hand the vector $\sigma'(\lambda^\star)$ is non-zero because σ is regular and also tangent to the curve. Hence $ds_{h(x^\star)}\sigma'(\lambda^\star) = 0$. If we rotate the vector $ds_{h(x^\star)}^\top$ by $\pi/2$ radians then the rotated vector and σ' will be linearly dependent. Let $R_{\frac{\pi}{2}}$ be a rotation by $\pi/2$. Then

$$R_{\frac{\pi}{2}} ds_{h(x^\star)}^\top = k(\sigma(\lambda^\star))\sigma'(\lambda^\star)$$

for some smooth, scalar-valued, non-zero function $k : \mathbb{R}^2 \to \mathbb{R}$. The function k is never equal to zero because the vector $ds_{h(x^\star)}^\top$ is never zero.

Returning to the expression for $\det(D(x))$, we have that

$$\begin{aligned}
\sigma_1'\partial_{x_2}\alpha - \sigma_2'\partial_{x_1}\alpha &= \left(R_{\frac{\pi}{2}} ds_{h(x^\star)}^\top\right)^\top \sigma'(\lambda^\star) \\
&= \left(k(\sigma(\lambda^\star))\sigma'(\lambda^\star)\right)^\top \sigma'(\lambda^\star) \\
&= k(\sigma(\lambda^\star))\|\sigma'(\lambda^\star)\|^2 \\
&= k(\sigma(\lambda^\star)).
\end{aligned}$$

We have shown for any $x^\star \in \Gamma$ with $x_5 \neq -\mathrm{v}$ that $\det(D(x^\star)) \neq 0$. Since $\Gamma^\star \subset \Gamma$, the lemma is proved. ■

An immediate consequence of Lemma 4.1 is that it allows us to define a local diffeomorphism using the function $\pi(x)$ and $\alpha(x)$ and their iterated Lie derivatives along the vector field $f(x)$.

Corollary 4.2: Let $x^\star \in \Gamma \backslash \{x \in \mathbb{R}^6 : x_5 + \mathrm{v} = 0\}$. There exists a neighbourhood $U \subset \mathbb{R}^6$ containing $x^\star$ such that the mapping $T : U \subset \mathbb{R}^6 \to T(U) \subset \mathbb{R}^6$, defined by

$$\begin{bmatrix} \eta_1 \\ \eta_2 \\ \eta_3 \\ \xi_1 \\ \xi_2 \\ \xi_3 \end{bmatrix} = T(x) = \begin{bmatrix} \pi(x) \\ L_f\pi(x) \\ L_f^2\pi(x) \\ \alpha(x) \\ L_f\alpha(x) \\ L_f^2\alpha(x) \end{bmatrix} \qquad (13)$$

is a diffeomorphism.

Using the coordinate transformation T from Corollary 4.2, in a neighbourhood of any point $x^\star \in \Gamma$ the system (7) in (η, ξ) coordinates reads

$$\begin{aligned}
\dot\eta_1 &= \eta_2 \\
\dot\eta_2 &= \eta_3 \\
\dot\eta_3 &= L_f^3\pi + L_{g_1}L_f^2\pi u_1 + L_{g_2}L_f^2\pi u_2\big|_{x=T^{-1}(\eta,\xi)} \\
\dot\xi_1 &= \xi_2 \\
\dot\xi_2 &= \xi_3 \\
\dot\xi_3 &= L_f^3\alpha + L_{g_1}L_f^2\alpha u_1 + L_{g_2}L_f^2\alpha u_2\big|_{x=T^{-1}(\eta,\xi)}
\end{aligned} \qquad (14)$$

This coordinate transformation is physically meaningful for path following applications. When $\xi = 0$ the system is restricted to evolve on the path following manifold. Thus stabilizing the ξ states is equivalent to getting the car on the desired path with heading velocity tangent to the path. We call the ξ-subsystem the transversal subsystem and the states ξ the transversal states. On the path following manifold the motion of the car-like robot on the path is governed by the η-dynamics. We call the η-subsystem the tangential subsystem and states η the tangential states. When the robot is on the path following manifold, i.e., $\xi = 0$ then η_1 determines the position of the robot on the path, η_2 represent velocity of the robot along the path and η_3 represent acceleration of the robot along the path.

Consider the regular feedback transformation

$$\begin{bmatrix} u_1 \\ u_2 \end{bmatrix} := D^{-1}(x)\left(\begin{bmatrix} -L_f^3\pi \\ -L_f^3\alpha \end{bmatrix} + \begin{bmatrix} v^{\parallel} \\ v^{\pitchfork} \end{bmatrix},\right) \qquad (15)$$

where $(v^{\parallel}, v^{\pitchfork})$ are auxiliary control inputs. By Lemma 4.1 this controller is well-defined in a neighbourhood of every $x^\star \in \Gamma \backslash \{x \in \mathbb{R}^6 : x_5 + \mathrm{v} = 0\}$. Thus in a neighbourhood of $x^\star$ the closed-loop system becomes

$$\begin{aligned}
\dot\eta_1 &= \eta_2 \\
\dot\eta_2 &= \eta_3 \\
\dot\eta_3 &= v^{\parallel} \\
\dot\xi_1 &= \xi_2 \\
\dot\xi_2 &= \xi_3 \\
\dot\xi_3 &= v^{\pitchfork}
\end{aligned} \qquad (16)$$

We refer to the control input $v^{\pitchfork}$ as the transversal input and $v^{\parallel}$ as the tangential input. The control law (15) has decoupled the transversal and tangential subsystems which makes designing $(v^{\parallel}, v^{\pitchfork})$ to solve the path following problem particularly easy. In summary, dynamic extension and transverse feedback linearization allow us to represent the system as a linear time invariant system (LTI) and use LTI controller design techniques to design the controller for system (16). Another way to state this is to say that the output (9) is a flat output for the car-like robot (1) [10], [17].

A. Transversal and tangential controller design

The objective of the transversal controller is to force the system to converge to the path. For that we need $\xi = 0$, i.e., we need to stabilize the origin of the transversal subsystem.

We use state feedback controller to control the transversal subsystem. The transversal controller is given by,

$$v^{\pitchfork}(\xi) = k_1\xi_1 + k_2\xi_2 + k_3\xi_3, \tag{17}$$

with $k_i < 0$, $i \in \{1,2,3\}$. This controller exponentially stabilizes $\xi = 0$ and hence makes the path following manifold attractive. Physically, since $\xi = 0$ is an equilibrium of the closed-loop transversal subsystem, if the robot is initialized on the path with the initial velocity tangent to the path, then it will remain on the path for all future time. Hence the property of path invariance is achieved.

In order to achieve the goal of controlling the speed of the robot on the curve a simple proportional feedback controller is used

$$v^{\parallel}(\eta) = k_4(\eta_1 - \eta_1^{ref}) + k_5(\eta_2 - \eta_2^{ref}) + k_6\eta_3, \tag{18}$$

where $k_i < 0$, $i \in \{4,5,6\}$. The parameter η_1^{ref} is the desired reference position on the path and η_2^{ref} is a desired reference velocity profile. Note however that whenever $x_5 = -\mathrm{v}$ the robot has no translational velocity. In that case the decoupling matrix loses rank and the control law (15) is not well-defined. Hence we cannot stabilize a particular point on the curve using this control law and henceforth we set $k_4 = 0$.

V. Implementation issues and simulation results

In order to implement the controller described in Section IV we must compute the coordinate transformation $(x_1, x_2, x_3, x_4, x_5, x_6) \mapsto (\xi_1, \xi_2, \xi_3, \eta_1, \eta_2, \eta_3)$ defined in (13), the feedback (15) with $D(x)$ defined in (10) and the transversal and tangential controllers (17), (18).

A. *Computation of transversal states*

We assume that we have a zero level set representation of the curve γ. Hence we know the function $\alpha(x) = s \circ h(x)$ and therefore ξ_1, ξ_2 and ξ_3 can be computed symbolically. Similarly the terms $L_f^3\alpha$, $L_{g_1}L_f^2\alpha$ and $L_{g_2}L_f^2\alpha$ can be computed easily to at least partially define the feedback (15).

B. *Computation of tangential states*

The states η_1, η_2 and η_3 are slightly more complicated to compute. In general a regular parameterized curve is not given with unit speed parameterization and finding a closed-form expression for the unit speed parameterization may be difficult or even impossible. Let $\widetilde{\sigma} : \mathbb{R} \to \mathbb{R}^2$ be the given C^r, $r \geq 3$ curve that satisfies Assumptions 1 and 2. We do not assume that $\widetilde{\sigma}$ has unit-speed parameterization. If $\widetilde{\sigma}$ models a closed curve then for some $T > 0$ it is true that for all $\lambda \in \mathbb{R}$ $\widetilde{\sigma}(\lambda + T) = \widetilde{\sigma}(\lambda)$.

Let $V \subseteq \mathbb{R}^2$ be a neighbourhood of $\mathbb{R}^2$ such that $V \cap \gamma$ contains a single connected component of γ. Since γ is a one-dimensional manifold, such a V exists. If γ is a closed curve then we can take V such that $\gamma \subset V$.

If γ is non-closed let $\mathcal{I} := (\lambda_s, \lambda_e) \subset \mathbb{R}$ be an interval of the real line such that $\gamma \cap V = \widetilde{\sigma}(\mathcal{I})$. Since we can always reparameterize $\widetilde{\sigma}$, without loss of generality we let $\lambda_s = 0$. If γ is closed we take $\mathcal{I} = [0, T)$. Let L denote the length of the portion of the curve in V. Now introduce a projection operator which is essentially the same as the map (8)

$$\lambda^\star = \widetilde{\varpi}(y) = \arg\inf_{\lambda \in \mathcal{I}} \|y - \widetilde{\sigma}(\lambda)\| \tag{19}$$

defined in γ_ϵ. To calculate the first tangential state we must find the unit-length parameter so we let

$$\eta_1 = g(\lambda^\star) := \int_0^{\lambda^\star} \left\| \frac{d\sigma}{d\lambda} \right\| du \tag{20}$$

so that $\eta_1 = g \circ \widetilde{\varpi} \circ h(x)$. To calculate η_2 we note

$$\begin{aligned}\eta_2 &= \frac{\partial(g \circ \widetilde{\varpi} \circ h)}{\partial x} \frac{dx}{dt} \\ &= \left(\frac{\partial g}{\partial \lambda}\right)\bigg|_{\lambda=\lambda^\star} \left(\frac{\partial \widetilde{\varpi}}{\partial y}\right)\bigg|_{y=h(x)} \begin{bmatrix} (v + x_5)\cos(x_3) \\ (v + x_5)\sin(x_3) \end{bmatrix}.\end{aligned}$$

Simple geometric arguments, similar to those used in the proof of Lemma 4.1, show that $\left.\frac{\partial \widetilde{\varpi}}{\partial y}\right|_y$ is given by

$$\frac{\partial \widetilde{\varpi}}{\partial y} = \frac{(\sigma'(\lambda^\star))^\top}{\|\sigma'(\lambda^\star)\|^2}. \tag{21}$$

Differentiating (20) one obtains

$$\eta_2 = \frac{(\sigma'(\lambda^\star))^\top}{\|\sigma'(\lambda^\star)\|} \begin{bmatrix} (v + x_5)\cos(x_3) \\ (v + x_5)\sin(x_3) \end{bmatrix}. \tag{22}$$

To simplify notation let

$$\Delta(x) := \frac{(\sigma'(\lambda^\star))^\top}{\|\sigma'(\lambda^\star)\|}, \ \Omega := \begin{bmatrix} (v + x_5)\cos(x_3) \\ (v + x_5)\sin(x_3) \end{bmatrix}.$$

To find η_3 we differentiate (22) and get $\eta_3 = \dot{\Delta}\Omega + \Delta\dot{\Omega}$. The term $\dot{\Omega}$ is easy to compute using the system dynamics (7). The term $\dot{\Delta} = \Delta'\dot{\lambda}$ can be found by noting that

$$\Delta' := \frac{\partial \Delta}{\partial \lambda} = \frac{(\sigma'')^\top \|\sigma'\|^2 - (\sigma')^\top \sum_{i=1}^2 \sigma_i'\sigma_i}{\|\sigma'\|^3} \tag{23}$$

and, using (20) and the chain rule,

$$\dot{\lambda} = \frac{1}{\|\sigma'\|^2}\eta_2. \tag{24}$$

This shows that the tangential state η_3 can be computed effectively using (21), (23), (24), Ω and $\dot{\Omega}$.

Finally, in order to implement the feedback transformation (15) we must find expressions for $L_f^3\pi$ and the first row of the decoupling matrix (10). The entries of the decoupling matrix are given by (11). Taking the time derivatives of η_3, tedious, yet easy, calculations yield an expression for $L_f^3\pi(x)$.

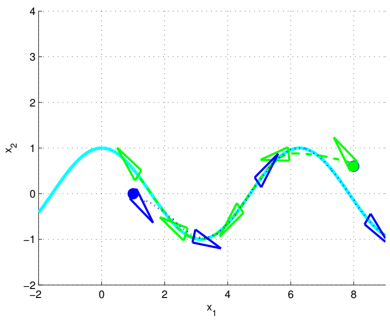

Fig. 2. Solid curve represents the desired path, dashed line represents the output trajectory of the closed-loop system.

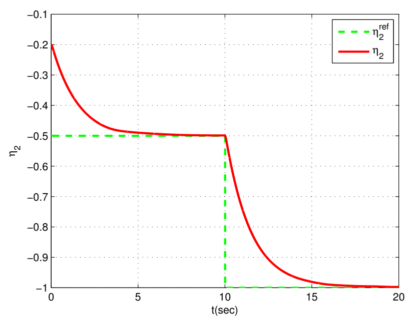

Fig. 3. Velocity along the path η_2 tracking the desired speed profile (25).

C. Simulation Results

We simulate the car-like robot (1) with dynamic controller (6) and feedback law (15), (17), (18) where (η, ξ) are defined in (13). Consider the curve $\sigma : \mathbb{R} \rightarrow \mathbb{R}^2$, $\lambda \mapsto \mathrm{col}(\lambda, \cos(\lambda))$ with implicit representation $\gamma = \{y \in \mathbb{R}^2 : s(y) = y_2 - \cos(y_1) = 0\}$. In the first simulation we solve the path following problem where the desired velocity profile along the curve is given by

$$\eta_2^{ref} = \begin{cases} -0.5 & 0 \leq t < 10s \\ -1 & t \geq 10s. \end{cases} \tag{25}$$

Figure 2 shows the position and orientation of the closed-loop system versus time. By choosing the transversal gains $\{k_1, k_2, k_3\}$ much larger than the tangential gains $\{k_5, k_6\}$ we can ensure that the exponential convergence of ξ to zero is much faster than the convergence to the desired profile along the curve. Figure 3 shows that the robot follows the desired speed profile.

In the second simulation we follow the same path shown in Figure 2 with a different velocity profile expressed as $\eta_2^{ref} = \frac{1}{5}\sin(\eta_1) + 2$. Here the motion along the path is not parameterized by time, but rather depends on the car's position along the path. Figure 4 shows that the robot follows the desired speed profile.

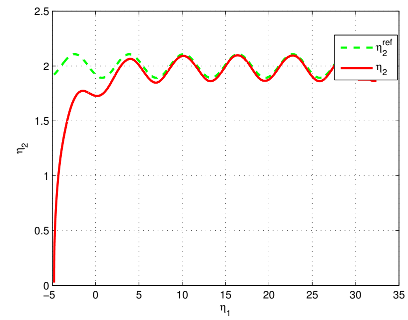

Fig. 4. Velocity along the path η_2 tracking the speed profile $\eta_2^{ref} = \frac{1}{5}\sin(\eta_1) + 2$.

REFERENCES

[1] A. P. Aguiar, J. P. Hespanha, and P. V. Kokotović, "Performance limitations in reference tracking and path following for nonlinear systems," *Automatica*, vol. 44, no. 3, pp. 598–610, 2008.

[2] A. D. Luca, G. Oriolo, and C. Samso, "Feedback control of a nonholonomic car-like robot," in *Robot Motion Planning and Control*, ser. Lecture Notes in Control and Information Sciences, J. Laumond, Ed. Springer Berlin, 1998, vol. 229, pp. 171–253.

[3] D. Wang and G. Xu, "Full-state tracking and internal dynamics of nonholonomic wheeled mobile robots," *IEEE/ASME Transactions on Mechatronics*, vol. 8, no. 2, pp. 203–214, 2003.

[4] L. Consolini, A. Piazzi, and M. Tosques, "Path following of car-like vehicles using dynamic inversion," *International Journal of Control*, pp. 1724–1738, 2003.

[5] G. Indiveri and M. L. Corradini, "Switching linear path following for bounded curvature car-like vehicles," in *5th IFAC/EURON Symposium on Intelligent Autonomous Vehicles*, july 2004.

[6] O. H. Dağci, U. Y. Ogras, and U. Özguner, "Path following controller design using sliding mode control theory," in *American Control Conference, 2003. Proceedings of the 2003*, vol. 1, June 2003, pp. 903 – 908 vol.1.

[7] C. Nielsen and M. Maggiore, "Maneuver regulation via transverse feedback linearization: Theory and examples," *Symposium on Nonlinear Control Systems (NOLCOS)*, September 2004.

[8] M. Fliess, J. Lévine, P. Martin, and P. Rouchon, "On differentially flat nonlinear systems," *Proceedings. IFAC-Symposium NOLCOS'92 Bordeaux*, pp. 408–412, 1992.

[9] ——, "A lie-backlund approach to equivalence and flatness of nonlinear systems," *Automatic Control, IEEE Transactions on*, vol. 44, no. 5, pp. 922 –937, May 1999.

[10] M. van Nieuwstadt, M. Rathinam, and R. M. Murray, "Differential flatness and absolute equivalence of nonlinear control systems," *SIAM J. Control and Optimization.*, vol. 36, pp. 1225–1239, July 1998.

[11] P. Rouchon, M. Fliess, J. Lvine, and P.Martin, "Flatness, motion planning and trailer systems," in *In: Proceedings. Conf. on Decision and Control*, 1993, pp. 2700–2705.

[12] O. J. Sørdalen, "Conversion of the kinematics of a car with n trailers into a chained form," in *ICRA (1)*, 1993, pp. 382–387.

[13] A. Isidori, *Nonlinear Control Systems*. Secaucus, NJ, U.S.A: Springer-Verlag New York, Inc., 1995.

[14] A. Hladio, C. Nielsen, and D. Wang, "Path following controller design for a class of mechanical systems," in *18th World Congress of the International Federation of Automatic Control*, Milano, Italy, August 2011, accepted.

[15] V. Guillemin and A. Pollack, *Differential Topology*. Englewood Cliffs NJ: Prentice-Hall, 1974.

[16] C. Nielsen, C. Fulford, and M. Maggiore, "Path following using transverse feedback linearization: Application to a maglev positioning system," *Automatica*, vol. 46, pp. 585–590, March 2010.

[17] P. Martin and P. Rouchon, "Feedback linearization and driftless systems," *Mathematics of Control, Signals, and Systems*, vol. 7, pp. 235–254, 1994.