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A Solution to the Initial Mean Consensus Problem via a Continuum Based Mean Field Control Approach

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Abstract— **This paper presents a continuum approach to the initial mean consensus problem via Mean Field (MF) stochastic control theory. In this problem formulation: (i) each agent has simple stochastic dynamics with inputs directly controlling its state's rate of change, and (ii) each agent seeks to minimize its individual cost function involving a mean field coupling to the states of all other agents. For this dynamic game problem, a set of coupled deterministic (Hamilton-Jacobi-Bellman and Fokker-Planck-Kolmogorov) equations is derived approximating the stochastic system of agents in the continuum (i.e., as the population size N goes to infinity). In a finite population system (analogous to the individual based approach): (i) the resulting MF control strategies possess an ε_N-Nash equilibrium property where ε_N goes to zero as the population size N approaches infinity, and (ii) these MF control strategies steer each individual's state toward the initial state population mean which is reached asymptotically as time goes to infinity. Hence, the system with decentralized MF control strategies reaches mean-consensus on the initial state population mean asymptotically (as time goes to infinity).**

I. INTRODUCTION

A *consensus process* is the process of dynamically reaching an agreement between the agents of a group on some common state properties such as position or velocity. The formulation of consensus systems is one of the important issues in the area of multi-agent control and coordination, and has been an active area of research in the systems and control community over the past few years (see [1] and the references therein, among many other papers).

There are two main classes of models for the consensus behaviour: (i) *individual based* (*Lagrangian*) models in the form of coupled Ordinary (Stochastic) Differential Equations (O(S)DEs), and (ii) *continuum based* (*Eulerian*) models in the form of Partial (integro-partial) Differential Equations (PDEs) in large population systems.

A variety of individual and continuum based consensus algorithms has been proposed in the past few years (see for example [2], [3], [4]). The key element of many of these algorithms is the use of local feedback by local communication (subject to the network topology) between agents to reach an agreement.

In this paper (similar to our previous works in [5], [6]) we aim to "synthesize" from the theory of optimal control the initial mean consensus behaviour of a set of agents rather than to analyze the behaviour resulting from ad-hoc feedback laws. The consensus formulation of this paper is motivated by many social, economic, and engineering models (see [6]).

In [5], [6] we synthesized the consensus behaviour as a dynamic game problem via individual based *stochastic Mean Field* (MF) *control* theory (see [7]). In this Dynamic Game Consensus Model (DGCM): (i) each agent has simple stochastic dynamics with inputs directly controlling its state's rate of change, and (ii) each agent seeks to minimize its individual cost function involving a mean field coupling to the states of all other agents.

Based on the MF (NCE) approach developed in [8], we derived an *individual based MF equation system* of the DGCM and explicitly compute its unique solution in [5], [6]. By applying the resulting MF control strategies, the system reaches initial mean consensus (i.e., consensus in the initial state population mean) asymptotically as time goes to infinity. Furthermore, these control laws possess an ε_N-Nash equilibrium property where ε_N goes to zero as the population size N goes to infinity.

This paper presents (based on the approach developed in [9] after [8]) a continuum (i.e., as the population size N goes to infinity) MF stochastic control approach to synthesize the initial mean consensus behaviour. The *continuum based MF equation system* of the DGCM consists of two coupled deterministic equations: (i) a nonlinear (backward in time) Hamilton-Jacobi-Bellman (HJB), and (ii) a nonlinear (forward in time) Fokker-Planck-Kolmogorov (FPK), which are also coupled to a (spatially averaged) cost coupling function approximating the aggregate effect of the agents in the infinite population limit. We study the stationary solutions and stability properties (based on the small perturbation analysis developed in [10]) of the continuum MF system of equations. Analogous to the individual based approach, we show (i) the ε_N-Nash equilibrium property of the resulting MF control laws, and (ii) the mean-consensus behaviour of the system by applying these MF control laws. Unlike [5], [6], the initial states for all the agents of the model in this paper are not necessarily assumed to be distributed according to a Gaussian distribution. However, the stationary solution of the system is distributed according to a Gaussian distribution.

The problem formulations and the results of this paper differ from those in [10] in the following respects: (i) in [10], as in the Lasry and Lions mean field games [11], for systems with finite population sizes a simplifying assumption was used stipulating that each agent's strategy depends only on its own driving Brownian motion, (ii) the ergodic individual cost

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functions of our multi-agent model is fundamentally different from the discounted logarithmic utility function considered in [10], and hence the analysis of the corresponding MF equation systems are different, and (iii) finally, the mean-consensus and Nash equilibrium properties of the MF control laws established in this paper have not been studied in [10].

In this paper the symbols ∂_t and ∂_z are respectively denote the partial derivative with respect to variables t and z, and ∂^2_{zz} denotes the second derivative with respect to z.

II. The Dynamic Game Consensus Model

Consider a system of N agents. The dynamics of the i^{th} agent is given by a controlled SDE:

$$dz_i(t) = u_i(t)dt + \sigma dw_i(t), \quad t \geq 0, \qquad 1 \leq i \leq N, \tag{1}$$

where $z_i(\cdot)$, $u_i(\cdot) \in \mathbb{R}$ are the state and control input of agent i, respectively; σ is a non-negative scalar; and $\{w_i(\cdot) : 1 \leq i \leq N\}$ denotes a sequence of mutually independent standard scalar Wiener processes on some *filtered probability space* $(\Omega, \mathscr{F}, \{\mathscr{F}_t\}_{t\geq 0}, \mathbb{P})$ where $\mathscr{F}_t$ is defined as the *natural filtration* given by σ-field $\sigma(z_i(\tau) : 1 \leq i \leq N, \tau < t)$. We assume that the initial states $\{z_i(0) : 1 \leq i \leq N\}$ are measurable on $\mathscr{F}_0$, mutually independent, and independent of Wiener processes $\{w_i : 1 \leq i \leq N\}$. It is important to note that the initial states for all the agents are not necessarily assumed to be distributed according to a Gaussian distribution.

Let the *admissible control* set of the i^{th} agent be $\mathscr{U}_i := \big\{u_i(\cdot) : u_i(t)$ is adapted to the sigma-field $\mathscr{F}_t,\ |z_i(T)|^2 = o(\sqrt{T}),\ \int_0^T (z_i(t))^2 dt = O(T),\ a.s.\big\}$. The objective of the i^{th} individual agent is to *almost surely* (a.s.) minimize its *ergodic* or *Long Run Average* (LRA) cost function given by

$$J_i^N(u_i, u_{-i}) := \limsup_{T\to\infty} \frac{1}{T} \int_0^T \Big((z_i - \frac{1}{N-1}\sum_{j\neq i}^N z_j)^2 + r u_i^2 \Big) dt, \tag{2}$$

where r is a positive scalar and $z^N_{-i}(\cdot) := \big(1/(N-1)\big)\sum_{j=1, j\neq i}^N z_j(\cdot)$ is called the *mean field* term. To indicate the dependence of J_i on u_i, $u_{-i} := (u_1, \cdots, u_{i-1}, u_{i+1}, \cdots, u_N)$ and the population size N, we write it as $J_i^N(u_i, u_{-i})$.

III. The Mean Field Control Methodology

We take the following steps to the DGCM (1)-(2) based on the MF control approach (developed in [9] after [8]):

1) **The continuum (infinite population) limit**: In this step a Nash equilibrium for the DGCM (1)-(2) in the continuum population limit (as N goes to infinity) is characterized by a "consistency relationship" between the individual strategies and the mass effect (i.e., the overall effect of the population on a given agent). This consistency relationship is described by a so-called MF equation system (see (11)-(13) below).
2) **ε_N-Nash equilibrium for the finite N model**: The distributed continuum based MF control law (derived from the MF equation system in Step 1) establishes an ε_N-Nash equilibrium (see Theorem 12) for the finite N population DGCM (1)-(2) where ε_N goes to zero asymptotically (as N approaches infinity).

A. Mean Field Approximation

In a large N population system, the *mean field* approach suggests that the cost-coupling function for a "generic" agent i $(1 \leq i \leq N)$ in (2),

$$c^N(z_i(\cdot), z_{-i}(\cdot)) := \Big(z_i(\cdot) - \frac{1}{N-1}\sum_{j\neq i}^N z_j(\cdot)\Big)^2,$$

be approximated by a deterministic function $c(z, \cdot)$ which only depends on $z = z_i$.

Replacing the function $c^N(z_i, z_{-i})$ with the deterministic function $c(z_i, \cdot)$ in the i^{th} agent's LRA cost function (2) reduces the DGCM (1)-(2) to a set of N independent optimal control problems.

Now we consider a "single agent" optimal control problem:

$$dz(t) = u(t)dt + \sigma dw(t), \quad t \geq 0, \tag{3}$$

$$\inf_{u\in\mathscr{U}} J(u) := \inf_{u\in\mathscr{U}} \limsup_{T\to\infty} \frac{1}{T}\int_0^T \big(c(z,t) + r u^2(t)\big) dt, \tag{4}$$

where $z(\cdot)$, $u(\cdot) \in \mathbb{R}$ are the state and control input, respectively; $w(\cdot)$ denotes a standard scalar Wiener process; $c(z, \cdot)$ is a known positive function; and $\mathscr{U}$ is the corresponding admissible control set of the generic agent.

An admissible control $u^o(\cdot) \in \mathscr{U}$ is called *a.s. optimal* if there exists a constant ρ^o such that

$$J(u^o) = \limsup_{T\to\infty} \frac{1}{T}\int_0^T \Big(c\big(z^o(t), t\big) + r\big(u^o(t)\big)^2 \Big) dt = \rho^o, \ a.s.,$$

where $z^o(\cdot)$ is the solution of (3) under $u^o(\cdot)$, and for any other admissible control $u(\cdot) \in \mathscr{U}$, we have a.s. $J(u) \geq \rho^o$.

The associated *Hamilton-Jacobian-Bellman* (HJB) equation of the optimal control problem (3)-(4) is given by (see [12] for the derivation)

$$\partial_t v(z,t) + \frac{\sigma^2}{2}\partial^2_{zz} v(z,t) + H\big(z, \partial_z v(z,t)\big) + c(z,t) = \rho^o, \tag{5}$$

where $v(z, \cdot)$ is the *relative value* function, ρ^o is the optimal cost and

$$H(z,p) := \min_{u\in\mathscr{U}} \{up + ru^2\}, \qquad z,\ p \in \mathbb{R},$$

is the *Hamiltonian*. For $x \in \mathbb{R}$ and $0 < t < \infty$, $v(x,t)$ is defined as

$$\inf_{u\in\mathscr{U}} \Big(\inf_{\tau\geq t} E\Big[\int_t^\tau \Big(c\big(z(s), s\big) + r\big(u(s)\big)^2 - \rho^o \Big) ds \big| z(t) = x \Big] \Big),$$

where the inner infimum is over all bounded stopping times with respect to the natural filtration $\{\mathscr{F}_t\}_{t\geq 0}$ (see [13]).

The solution of the optimal control problem (3)-(4) is

$$u^o(t) := H\big(z, \partial_z v(z,t)\big) = -\frac{1}{2r}\partial_z v(z,t), \qquad t \geq 0.$$

Substituting $u^o(\cdot)$ into the HJB equation (5) yields the (backward in time) nonlinear deterministic PDE:

$$\partial_t v(z,t) - \frac{1}{4r}\big(\partial_z v(z,t)\big)^2 + \frac{\sigma^2}{2}\partial^2_{zz} v(z,t) + c(z,t) = \rho^o. \tag{6}$$

We enunciate the following assumption:

(**A1**) We assume that the sequence $\{Ez_i(0) : 1 \le i \le N\}$ is a subset of a fixed compact set $\mathscr{A}$ independent of N, and has a compactly supported *probability density* $f_0(z)$ (which is not necessarily a Gaussian density). Let

$$f_N(x,t) := \frac{1}{N}\sum_{i=1}^{N} \delta\big(x - Ez_i(t)\big),$$

be the *empirical distribution density* associated with N agents where δ is the *Dirac delta*. We assume that $\{f_N(x,0) : N \ge 1\}$ converges weakly to f_0, *i.e.*, for any $\phi(x) \in C_b(\mathbb{R})$ (the space of bounded continuous functions on $\mathbb{R}$),

$$\lim_{N\to\infty}\int_B \phi(x) f_N(x,0)dx = \int_B \phi(x) f_0(x)dx,$$

for any subset $B \subset \mathscr{A}$. □

For any function $\phi(x) \in C_b$ on $\mathbb{R}$ we have

$$\int \phi(x) f_N(x,t)dx = \frac{1}{N}\sum_{i=1}^{N} \phi\big(Ez_i(t)\big).$$

Since the processes $\{z_i(\cdot) : 1 \le i \le N\}$ are *independent and identically distributed* (i.i.d.), by the *ergodic theorem* we have

$$\lim_{N\to\infty}\int \phi(x) f_N(x,t)dx = \int \phi(x) f^u(x,t)dx, \qquad a.s. \tag{7}$$

where $f^u(z,\cdot)$ is the density of the generic agent's state which evolves according to the SDE (3) with control law $u(\cdot) \in \mathscr{U}$.

The evolution of the population density $f^u(z,\cdot)$ satisfies the *Fokker-Planck-Kolmogorov* (FPK) equation

$$\partial_t f^u(z,t) + \partial_z\big(uf^u(z,t)\big) = \frac{\sigma^2}{2}\partial^2_{zz} f^u(z,t), \tag{8}$$

where $f^u(z,0) = f_0(z)$ is characterized by (**A1**).

Now by substituting the optimal control $u^o(\cdot)$ into its FPK equation (8) we get the (forward in time) nonlinear deterministic PDE

$$\partial_t f(z,t) - \frac{1}{2r}\partial_z\Big(\big(\partial_z v(z,t)\big) f(z,t)\Big) = \frac{\sigma^2}{2}\partial^2_{zz} f(z,t), \tag{9}$$

where $f(z,0) = f_0(z)$, and $v(z,\cdot)$ is the solution of the equation (6).

Finally, for a generic agent i the ergodic theorem in (7) suggests the approximation of $c^N(z_i, z^o_{-i})$ for a large N population system by

$$\bar{c}(z_i,\cdot) = \big(z_i - \int_{\mathbb{R}} z f(z,\cdot)dz\big)^2 = \big(\int_{\mathbb{R}} (z_i - z) f(z,\cdot)dz\big)^2, \tag{10}$$

where $f(z,\cdot)$ is the population density under the optimal control $u^o(\cdot)$ (i.e., $f(z,\cdot)$ is the solution of the equation (9)).

B. Mean Field Equation System

In this section we aim to construct the consistency relationship (between the individual strategies and the mass influence effect) in the stochastic MF control theory (based on the approach developed in [9] after [8]). The key idea is to prescribe a spatially averaged mass function $\bar{c}(z,\cdot)$ characterized by the property that it is reproduced as the average of all agents' states in the continuum of agents whenever each individual agent optimally tracks the same mass function $\bar{c}(z,\cdot)$.

Considering the continuum population limit (i.e., as N approaches ∞) of the DGCM (1)-(2) where $f(z,0) = f_0(z)$ is the initial population density and $\int_{\mathbb{R}} f(z,t)dz = 1$ for any $t \ge 0$, we obtain the following *continuum based mean field* (MF) equation system:

[MF-HJB]

$$\partial_t v(z,t) = \frac{1}{4r}\big(\partial_z v(z,t)\big)^2 - \bar{c}(z,t) + \rho^o - \frac{\sigma^2}{2}\partial^2_{zz} v(z,t), \tag{11}$$

[MF-FPK]

$$\partial_t f(z,t) = \frac{1}{2r}\partial_z\Big(\big(\partial_z v(z,t)\big) f(z,t)\Big) + \frac{\sigma^2}{2}\partial^2_{zz} f(z,t), \tag{12}$$

[MF-CC]

$$\bar{c}(z,t) = \Big(\int_{\mathbb{R}} (z - z') f(z',t)dz'\Big)^2, \tag{13}$$

(see the individual based version of this MF equation system in [5], [6]).

The system of equations (11)-(13) consists of: (i) the nonlinear (backward in time) *MF-HJB* equation (6) which describes the HJB equation of a generic agent's ergodic optimal problem (3)-(4) with cost coupling $\bar{c}(z,\cdot)$, (ii) the nonlinear (forward in time) *MF-FPK* equation (9) which describes the evolution of the population density with the optimal control law

$$u^o(t) := -\frac{1}{2r}\partial_z v(z,t), \qquad t \ge 0, \tag{14}$$

and (iii) the spatially averaged *MF-CC* (Cost-Coupling) (10) which is the aggregate effect of the agents in the infinite population limit.

IV. Analysis of the Mean Field Equation System

A. Gaussian Stationary Solution

In the stationary setting, the MF equation system (11)-(13) takes the form:

$$\frac{1}{4r}\big(\partial_z v_\infty(z)\big)^2 - \frac{\sigma^2}{2}\partial^2_{zz} v_\infty(z) = \bar{c}_\infty(z) - \rho^o, \tag{15}$$

$$\frac{1}{2r}\partial_z\Big(\big(\partial_z v_\infty(z)\big) f_\infty(z)\Big) = -\frac{\sigma^2}{2}\partial^2_{zz} f_\infty(z), \tag{16}$$

$$\bar{c}_\infty(z) = \Big(\int_{\mathbb{R}} (z - z') f_\infty(z')dz'\Big)^2. \tag{17}$$

Theorem 1: [12] For any arbitrary $\mu \in \mathbb{R}$, there exists the following solution of the stationary equation system (15)-(17):

$$v_\infty(z) = \sqrt{r}(z-\mu)^2, \quad \rho^o = \sigma^2\sqrt{r}, \tag{18}$$

$$f_\infty(z) = \frac{1}{\sqrt{2\pi s^2}}\exp\Big(-\frac{(z-\mu)^2}{2s^2}\Big), \quad s^2 := \frac{\sigma^2\sqrt{r}}{2}, \tag{19}$$

$$\bar{c}_\infty(z) = (z-\mu)^2, \tag{20}$$

where $v_\infty(z)$ is defined up to a constant. □

It is important to note that the stationary solution of the system $f_\infty(\cdot)$ is Gaussian even thought the initial states for all the agents are not necessarily assumed to be distributed according to a Gaussian distribution.

B. Stability Analysis

By taking the approach of [10] we study the small perturbation stability of the stationary solution (18)-(20) based on the linearization of the equation system (11)-(13). In this nonlinear equation system we let the perturbation of the solution be

$$v_\varepsilon(z,t) = v_\infty(z) + \varepsilon\ \tilde{v}(z,t), \tag{21}$$

$$f_\varepsilon(z,t) = f_\infty(z)\big(1+\varepsilon\ \tilde{f}(z,t)\big), \tag{22}$$

$$\bar{c}_\varepsilon(z,t) = \bar{c}_\infty(z) + \varepsilon\ \tilde{c}(z,t), \tag{23}$$

for $z \in \mathbb{R}$ and $t \geq 0$, where v_∞, f_∞ and $\bar{c}_\infty$ are defined in (18)-(20), and $\tilde{f}(z,0)$ and $\tilde{v}(z,0)$ are given and represent the perturbations on $f_\infty(z)$ and $v_\infty(z)$.

Remark 2: The reason why we take the relative perturbation form of the density function f in (22) is to employ the Hermite series expansion for the resulting linearized equation system (see below).

Since f is a probability density, we have

$$\int_{\mathbb{R}} \tilde{f}(z,t) f_\infty(z) dz = 0, \qquad t \geq 0, \tag{24}$$

$$\int_{\mathbb{R}} z f(z,0) dz = \mu + \varepsilon \int_{\mathbb{R}} z \tilde{f}(z,0) f_\infty(z) dz, \tag{25}$$

where μ is the mean of the Gaussian density function f_∞.

Proposition 3: ([12] after [10]) The linearization of the equation system (11)-(13) around the stationary solution (18)-(20) takes the form

$$\partial_t \tilde{v}(z,t) = \frac{1}{\sqrt{r}}(z-\mu)\partial_z \tilde{v}(z,t) - \frac{\sigma^2}{2}\partial^2_{zz}\tilde{v}(z,t) - \tilde{c}(z,t), \tag{26}$$

$$\begin{aligned}\partial_t \tilde{f}(z,t) = &-\frac{1}{\sqrt{r}}(z-\mu)\partial_z \tilde{f}(z,t) + \frac{\sigma^2}{2}\partial^2_{zz}\tilde{f}(z,t) \\ &- \frac{1}{\sigma^2 r}\Big(\frac{1}{\sqrt{r}}(z-\mu)\partial_z \tilde{v}(z,t) - \frac{\sigma^2}{2}\partial^2_{zz}\tilde{v}(z,t)\Big),\end{aligned} \tag{27}$$

$$\tilde{c}(z,t) = -2(z-\mu)\Big(\int_{\mathbb{R}} z\tilde{f}(z,t) f_\infty(z) dz\Big), \tag{28}$$

where $\tilde{f}(z,0)$ is given. □

For the analysis of the linearized equation system (26)-(28) we introduce the *Hermite polynomials* associated to the Hilbert space $L^2(\mathbb{R}, f_\infty(z)dz)$. In this space we have the inner product

$$(g,h) := \int_{\mathbb{R}} g(z)h(z) f_\infty(z) dz,$$

and the norm is given by $\|g\|_{L^2} := (g,g)^{1/2}$.

Definition 4: ([14]) We define the n^{th} Hermite polynomial, $n \in \mathbb{N}_0$, of the space $L^2(\mathbb{R}, f_\infty(z)dz)$ by

$$H_n(z) := (-1)^n s^{2n} \exp\Big(\frac{(z-\mu)^2}{2s^2}\Big)\frac{d^n}{dz^n}\exp\Big(\frac{-(z-\mu)^2}{2s^2}\Big),$$

where μ and s^2 are defined in Theorem 1. □

Lemma 5: ([12] after [10]) We have the following:

(a) The set of Hermite polynomials $\{H_n : n \in \mathbb{N}_0\}$ forms an orthogonal basis of the Hilbert space $L^2(\mathbb{R}, f_\infty(z)dz)$ such that

$$\big(H_m, H_n\big) = s^{2n} n!\ \delta(n,m), \tag{29}$$

where δ is the *Kronecker delta* function.

(b) The Hermite polynomials H_n are eigenfunctions of the operator

$$\mathscr{L} g(z) := \frac{1}{\sqrt{r}}(z-\mu)\partial_z g(z) - \frac{\sigma^2}{2}\partial^2_{zz} g(z), \tag{30}$$

such that $\mathscr{L} H_n = (1/\sqrt{r}) n H_n$ for any $n \in \mathbb{N}_0$. □

By using the operator $\mathscr{L}$ defined in (30) we can rewrite the equation system (26)-(28) as

$$\partial_t \tilde{v}(z,t) = \mathscr{L}\tilde{v}(z,t) - \tilde{c}(z,t), \tag{31}$$

$$\partial_t \tilde{f}(z,t) = -\frac{1}{\sigma^2 r}\mathscr{L}\tilde{v}(z,t) - \mathscr{L}\tilde{f}(z,t), \tag{32}$$

$$\tilde{c}(z,t) = -2(z-\mu)\Big(\int_{\mathbb{R}} z\tilde{f}(z,t) f_\infty(z) dz\Big), \tag{33}$$

where $\tilde{f}(z,0)$ is given.

Definition 6: [12] A stationary solution (v_∞, f_∞) of the nonlinear equation system (11)-(13) is linearly asymptotically stable if the solution $\tilde{f}$ of the linear equation system (26)-(28) with initial perturbation $\tilde{f}(z,0) \in L^2(f_\infty(z)dz)$ exists in $L^2(\mathbb{R}, f_\infty(z)dz)$ and $\lim_{t\to\infty} \|\tilde{f}(z,t)\|_{L^2} = 0$. □

Let $\tilde{f}(z,0) \equiv \sum_{n=0}^\infty k_n(0) H_n(z)$ and $\tilde{v}(z,0) \equiv \sum_{n=0}^\infty l_n(0) H_n(z)$ then since v and hence $\tilde{v}$ in (21) are defined up to a constant we choose $l_0(0) = 0$. On the other hand, by (24)

$$\int_{\mathbb{R}} \tilde{f}(z,0) f_\infty(z) dz = (H_0, \tilde{f}(z,0)) = k_0(0) = 0. \tag{34}$$

We enunciate the following assumption:

(**A2**) Assume that the initial perturbations $\tilde{f}(z,0)$ and $\tilde{v}(z,0)$ of the stationary solutions $f_\infty(z)$ and $v_\infty(z)$ are in the space $L^2(f_\infty(z)dz)$ and are such that

$$\tilde{f}(z,0) = \sum_{n=1}^\infty k_n(0) H_n(z), \qquad \tilde{v}(z,0) = \sum_{n=1}^\infty l_n(0) H_n(z),$$

for $z \in \mathbb{R}$. □

Theorem 7: Assume (**A1**) and (**A2**) hold. Then, we have the following:

(a) (Existence and uniqueness) There exists a well-defined unique, bounded and C^∞ (i.e., all of its partial derivatives exist) solution to the equation system (26)-(28) in the space $L^2(\mathbb{R}, f_\infty(z)dz)$ if $l_1(0) = -2\sqrt{r}s^2 k_1(0)$ and $l_n(0) = 0$ for all $n \geq 2$. This solution is given by

$$\tilde{v}(z,t) = -2\sqrt{r}s^2 k_1(0) H_1(z), \tag{35}$$

$$\tilde{f}(z,t) = k_1(0) H_1(z) + \sum_{n=2}^\infty \exp\Big(\frac{-nt}{\sqrt{r}}\Big) k_n(0) H_n(z), \tag{36}$$

$$\tilde{c}(z,t) = -2s^2 k_1(0) H_1(z), \tag{37}$$

for $t \geq 0$ and $z \in \mathbb{R}$.

(b) (Asymptotic stability) Under the unique, bounded and C^∞ solution (35)-(37), the stationary solution $(v_\infty, f_\infty, \bar{c}_\infty)$ of the nonlinear equation system (11)-(13) is linearly asymptotically stable if $k_1(0)$ and hence $l_1(0)$ are equal to zero. Then, the ε perturbed solutions (21)-(23) take the form

$$v(z,t) = v_\infty(z), \quad \bar{c}(z,t) = \bar{c}_\infty(z), \tag{38}$$

$$f(z,t) = f_\infty(z)\Big(1+\varepsilon\sum_{n=2}^{\infty}\exp\big(\frac{-nt}{\sqrt{r}}\big)k_n(0)H_n(z)\Big), \tag{39}$$

for $z \in \mathbb{R}$ and $t \geq 0$. Hence, the linearly asymptotically stable stationary equilibrium solution of the nonlinear equation system (11)-(13) is

$$v_\infty(z) = \sqrt{r}(z-\mu)^2, \quad \bar{c}_\infty(z) = (z-\mu)^2, \qquad z \in \mathbb{R},$$

$$f_\infty(z) = \frac{1}{\sqrt{2\pi s^2}}\exp\Big(-\frac{(z-\mu)^2}{2s^2}\Big), \qquad z \in \mathbb{R},$$

where $s^2 = \sigma^2\sqrt{r}/2$, and

$$\mu = \int_{\mathbb{R}} z f_0(z)dz, \tag{40}$$

is the initial state population mean.

Proof. See the appendix. □

Remark 8: Since $k_1(0) = \int_{\mathbb{R}} z\tilde{f}(z,0)f_\infty(z)dz$ and $l_1(0) = \int_{\mathbb{R}} z\tilde{v}(z,0)f_\infty(z)dz$, the interpretation of the assumption $k_1(0) = l_1(0) = 0$ in the above theorem is that the initial perturbations $\tilde{f}(z,0)$ and $\tilde{v}(z,0)$ are even functions in the space $L^2(\mathbb{R}, f_\infty(z)dz)$ (the same evenness assumption appears in Guéant's model [10]). Moreover, this assumption yields $\tilde{c}(z,t) = 0$ in (37). In other words, by the evenness assumption the initial perturbations ($\tilde{f}(z,0)$ and $\tilde{v}(z,0)$) have no effect on the cost perturbation $\tilde{c}(z,t)$.

V. Properties of Mean Field Control Laws

A. Mean-consensus

Definition 9: [5] *Mean-consensus* is said to be achieved asymptotically for a group of N agents if $\lim_{t\to\infty}|Ez_i(t) - Ez_j(t)| = 0$ for any i and j, $1 \leq i \neq j \leq N$. □

The unique (up to a constant) linearly asymptotically stable solution of the equation (11), $v(z,t)$, defined in (38) yields the following continuum based MF control law:

$$u^o(\cdot) = -\frac{1}{2r}\partial_z v(z,\cdot) = \frac{-1}{\sqrt{r}}(z(\cdot)-\mu),$$

by (14), where μ is the initial state population mean (40). Using this MF continuum based control law for a finite N population system (1)-(2) yields the control

$$u_i^o(\cdot) = -\frac{1}{2r}\partial_z v(z,\cdot)\big|_{z=z_i} = \frac{-1}{\sqrt{r}}(z_i(\cdot)-\mu), \tag{41}$$

for the i^{th} individual agent where $1 \leq i \leq N < \infty$.

Remark 10: The set of continuum based MF control laws (41) is the same as the set of individual based MF control laws derived by the LQG MF approach in [5], [6].

Applying the MF control laws (41) to the agents' dynamics (1) yields

$$dz_i^o(t) = \frac{-1}{\sqrt{r}}\big(z_i^o(t)-\mu\big)dt + \sigma dw_i(t), \ t \geq 0, \quad 1 \leq i \leq N. \tag{42}$$

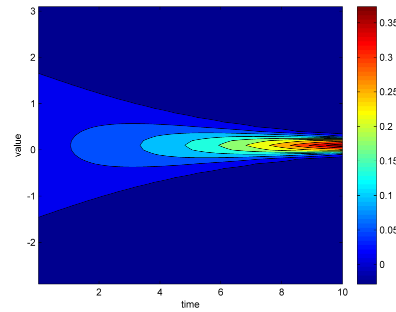

Fig. 1: The contour lines of population density functions

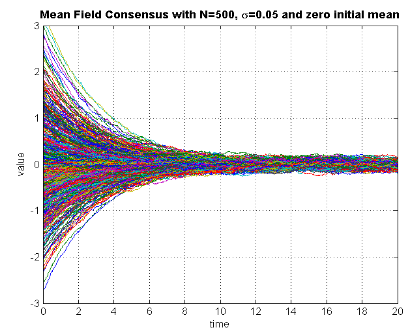

Fig. 2: Agent's individual state trajectories

The processes (42) have the solutions:

$$z_i^o(t) = \mu + e^{\frac{-t}{\sqrt{r}}}\big(z_i(0)-\mu\big) + \sigma\int_0^t e^{-\frac{(t-\tau)}{\sqrt{r}}}dw_i(\tau), \tag{43}$$

for $t \geq 0$ and $1 \leq i \leq N < \infty$. Now, we have the following theorem which is the analogous version of Theorem 3 in [5].

Theorem 11: [12] By applying the continuum based MF control laws (41) in a finite population DGCM (1)-(2), a mean-consensus is reached asymptotically (as time goes to infinity) with individual asymptotic variance $\sigma^2\sqrt{r}/2$. □

B. ε-Nash Equilibrium Property

Now we present the ε_N-Nash equilibrium property of the continuum based MF control laws (41) for a finite N population system (1)-(2) where ε_N goes to zero asymptotically (as the population size N approaches infinity).

Theorem 12: [12] The set of MF control laws $\{u_i^o \in \mathscr{U}_i : 1 \leq i \leq N\}$ in (41) generates an a.s. $O(\varepsilon_N)$-Nash equilibrium, i.e., for any fixed i, $1 \leq i \leq N$, we have

$$J_i^N(u_i^o, u_{-i}^o) - O(\varepsilon_N) \leq \inf_{u_i\in\mathscr{U}_i} J_i^N(u_i, u_{-i}^o) \leq J_i^N(u_i^o, u_{-i}^o), \quad a.s.$$

where $u_{-i}^o := (u_1^o, \cdots, u_{i-1}^o, u_{i+1}^o, \cdots, u_N^o)$. □

VI. Numerical Example

Consider a system (1)-(2) of 500 agents with $r = 10$ and $\sigma = 0.05$. The initial states of the agents are taking independently from a standard normal distribution, i.e., Gaussian distribution with mean zero and variance one. Fig. 1 shows the contour lines of the evolution of the population density functions. These density functions $f(z,t)$, $t>0$, is the solution of the MF-FPK equation (12) when $v(z,t)$ is the unique (up to a constant) linearly asymptotically stable solution of the nonlinear equation system (11). The state trajectories (43) of all the agents of the system are shown in Fig. 2. As shown in Figs. 1 and 2 the agents reach mean consensus in $\mu=0$ asymptotically (as time goes to infinity) with individual asymptotic variance $\sigma^2\sqrt{r}/2$.

APPENDIX

Proof of Theorem 7: (a) Note that at any time $t\geq 0$ the linearized cost $\tilde{c}(z,t)\in \mathrm{span}\big(H_1(z)\big)$ (see (A.1) below). Therefore, for $n\geq 2$ we can write the equation system (31)-(32) in the Hermite coordinates as the ODE

$$\frac{\partial}{\partial t}\begin{pmatrix} l_n(t) \\ k_n(t)\end{pmatrix}=\begin{pmatrix} \frac{n}{\sqrt{r}} & 0 \\ -\frac{n}{\sigma^2 r\sqrt{r}} & -\frac{n}{\sqrt{r}}\end{pmatrix}\begin{pmatrix} l_n(t) \\ k_n(t)\end{pmatrix}, \quad t\geq 0,$$

by Lemma 5-(b), where $l_n(0)$ and $k_n(0)$ are given. But, the only function in $L^2(\mathbb{R},f_\infty(z)dz)$ that satisfies

$$\partial_t l_n(t)=\frac{n}{\sqrt{r}}l_n(t), \qquad t\geq 0,$$

is the zero function. Hence, it is necessary that $l_n(0)$ is zero for $n\geq 2$ which implies that $l_n(t)$ is zero for all $n\geq 2$ and $t\geq 0$. We thus have

$$k_n(t)=\exp\big(\frac{-nt}{\sqrt{r}}\big)k_n(0), \qquad t\geq 0,$$

for any $n\geq 2$.

On the other hand, (33) yields

$$\tilde{c}(z,\cdot)=-2H_1(z)\big(z,\sum_{n=1}^{\infty}k_n(\cdot)H_n(z)\big)=-2s^2k_1(\cdot)H_1(z), \tag{A.1}$$

by (29) and since $z=\mu H_0(z)+H_1(z)$. Then the Hermite coordinates of the equation system (31)-(33) for $n=1$ satisfy the ODE (by Lemma 5-(b))

$$\frac{\partial}{\partial t}\begin{pmatrix} l_1(t) \\ k_1(t)\end{pmatrix}=\begin{pmatrix} \frac{1}{\sqrt{r}} & 2s^2 \\ -\frac{1}{2s^2 r} & -\frac{1}{\sqrt{r}}\end{pmatrix}\begin{pmatrix} l_1(t) \\ k_1(t)\end{pmatrix}, \tag{A.2}$$

for $t\geq 0$ where $l_1(0)$ and $k_1(0)$ are given, and $s^2=\sigma^2\sqrt{r}/2$. The matrix

$$A:=\begin{pmatrix} \frac{1}{\sqrt{r}} & 2s^2 \\ -\frac{1}{2s^2 r} & -\frac{1}{\sqrt{r}}\end{pmatrix},$$

has two zero eigenvalues and can be written in the Jordan normal form $A=PJP^{-1}$ with

$$J:=\begin{pmatrix} 0 & 1 \\ 0 & 0\end{pmatrix}, \quad P:=\begin{pmatrix} -2\sqrt{r}s^2 & 0 \\ 1 & -\sqrt{r}\end{pmatrix}.$$

Therefore, the unique and bounded solution of (A.2) is

$$\begin{pmatrix} l_1(t) \\ k_1(t)\end{pmatrix}=\exp(At)\begin{pmatrix} l_1(0) \\ k_1(0)\end{pmatrix}=Pe^{Jt}P^{-1}\begin{pmatrix} l_1(0) \\ k_1(0)\end{pmatrix}$$
$$=\begin{pmatrix} l_1(0)+t\left(\frac{l_1(0)}{\sqrt{r}}+2s^2k_1(0)\right) \\ k_1(0)-t\left(\frac{l_1(0)}{2rs^2}+\frac{k_1(0)}{\sqrt{r}}\right)\end{pmatrix}, \qquad t\geq 0.$$

Hence, its unique bounded solution in $L^2(\mathbb{R},f_\infty(z)dz)$ is given by

$$\begin{pmatrix} l_1(t) \\ k_1(t)\end{pmatrix}=\begin{pmatrix} l_1(0) \\ k_1(0)\end{pmatrix}, \qquad t\geq 0,$$

where $l_1(0)=-2\sqrt{r}s^2k_1(0)$. Then, it can be shown that for any fixed $t>0$, $(k_n(t))_n$ is in the space l^1 (i.e., the space of sequences whose series is absolutely convergent). Moreover, $\big(\tilde{f}(z,\cdot),\tilde{v}(z,\cdot),\tilde{c}_\infty(z,\cdot)\big)$ (defined in (35)-(37)) are bounded C^∞ functions in the Hilbert space $L^2(\mathbb{R},f_\infty(z)dz)$, and satisfy the equation system (26)-(28).

(b) By part (a) we get $\lim_{t\to\infty}k_n(t)=0$, for $n\geq 2$. Now by the Lebesgue Dominated Convergence theorem and (29), (36) yields $\lim_{t\to\infty}\|\tilde{f}(z,t)\|_{L^2}=k_1(0)s$, which is zero if $k_1(0)=0$ and by Part (a) $l_1(0)$ is equal to zero. Now since

$$\int_{\mathbb{R}}z\tilde{f}(z,0)f_\infty(z)dz=\Big(\mu H_0(z)+H_1(z),\sum_{n=2}^{\infty}k_n(0)H_n(z)\Big)=0,$$

by (29), the equality (25) yields (40). □

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