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On the Extension of the Hybrid Minimum Principle to Riemannian Manifolds

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Abstract— **This paper provides a geometrical derivation of the Hybrid Minimum Principle (HMP) for autonomous hybrid systems on Riemannian manifolds. The analysis is expressed in terms of extremal trajectories on the cotangent bundle of the manifold state space. In the case of autonomous hybrid systems, switching manifolds are defined as smooth embedded submanifolds of the state manifold. The HMP results are obtained in the case of time invariant switching manifolds on Riemannian manifolds.**

I. INTRODUCTION

The problem of hybrid systems optimal control (HSOC) in Euclidean spaces has been studied in many papers, see e.g [8], [12], [13], [14], [16], [17], [18]. In particular, [13] and [19] present an extension of the Minimum Principle to hybrid systems and [13] gives an iterative algorithm which is based upon the Hybrid Minimum Principle (HMP) necessary conditions for both autonomous and controlled switching systems. In general the previously cited papers consider HSOC problems with a priori given sequences of discrete transitions.

A geometric version of Pontryagin's Maximum Principle for a general class of state manifolds is given in [1], [4], [16]. In this paper, we employ the control needle variation method of [1] to analyze state variation propagation through switching manifolds and hence we obtain a Hybrid Minimum Principle for autonomous hybrid systems on Riemannian manifolds. It is shown that under appropriate hypotheses on the differentiability of the hybrid value function, the discontinuity of the adjoint variable at the optimal switching state and switching time is proportional to a differential form of the hybrid value function defined on the cotangent bundle of the state manifold. In the case of open control sets and Euclidean state spaces, this result for hybrid systems appeared in [13] without using the language of differential geometry. We note that the analysis in this paper is extended to the case of multiple autonomous switchings which has been treated in [13] for hybrid systems defined on Euclidean spaces.

Definition 2.1:

$$\mathbf{H} := \{H = Q \times \mathcal{M}, \mathbf{\Gamma}, A, I = \mathbf{\Sigma} \times U, F, \mathcal{N}\} \tag{1}$$

Where:
$Q = \{1, 2, 3, ..., |Q|\}$ is a finite set of *discrete states* and $\mathcal{M}$ is a smooth differentiable state manifold.
H is the *hybrid state space* of $\mathbf{H}$.
$\Gamma : H \times \Sigma \to Q$ is the time independent (partially defined) *discrete transition map*.
$A: \ Q \to 2^Q$ is a *set valued function* for which for a state $q \in Q$ all and only discrete controlled transitions in to the q dependent subset $A(q) \subset Q$ are allowed under Γ.
$\Sigma = \Sigma_u \cup \Sigma_c \cup \{id\}$ is a *finite set of distinct autonomous (i.e. uncontrolled) and controlled discrete event transition labels* extended with the identity element $\{id\}$ such that for $i \in Q$, $\sigma_{i,j} \in \Sigma$ only if $j \in A(i)$.
$U \subset R^u$ is a set of *admissible input control values*, where U is an open bounded set in R^u. The set of *admissible input control functions* is $\mathcal{U} := \mathcal{U}(U, L_\infty[0, T_*))$, the set of all bounded measurable functions on some interval $[0, T_*), T_* < \infty$, taking values in U.
$I := \Sigma \times U$ is a set of *system input values*.
F is a *indexed collection of vector fields* $\{f_j\}_{j \in Q}$ such that $f_j : \mathcal{M} \times U \to T\mathcal{M}$ is a vector field assigned to each control location such that f_j is continuous on $\mathcal{M} \times U$ and continuously differentiable on $\mathcal{M}$ for all $u \in U$.
$\mathcal{N} := \{\tilde{n}^k_\gamma : \gamma \in Q \times Q, k \in \mathbf{Z}_+\}$ is a collection of time independent manifold subcomponents such that for any ordered pair $\gamma = (p, q), \tilde{n}^k_\gamma$ is a smooth, i.e. C^∞, codimension 1 submanifold of $\mathcal{M}$, possibly with boundary $\partial\tilde{n}^k_\gamma$. By abuse of notation, in the case of embedded submanifolds (in $\mathcal{M}$), we describe the manifold subcomponents locally by $\tilde{n}^k_\gamma = \{x : \tilde{n}^k_\gamma(x) = 0\}$. A *switching manifold* $n_{p,q}$ is the union (over k) of a disjoint set of connected switching manifold components $n^k_{p,q} = \bigcup_{k_i; 1 \le i \le n(k)} \tilde{n}^{k_i}_{p,q}, \tilde{n}^{k_i}_{p,q} \in \mathcal{N}$, where

(i) $x \in \tilde{n}^{k_i}_\gamma$ is such that $x \in \tilde{n}^{k_i}_\gamma \bigcap \tilde{n}^{k_j}_\gamma, k_i \neq k_j$, if and only if $x \in \partial\tilde{n}^{k_i}_\gamma \bigcap \partial\tilde{n}^{k_j}_\gamma$.

(ii) If $\partial\tilde{n}^{k_i}_\gamma \bigcap \partial\tilde{n}^{k_j}_\gamma \neq \emptyset$ then $\partial\tilde{n}^{k_i}_\gamma \bigcap \partial\tilde{n}^{k_j}_\gamma$ is a piece-wise smooth codimention 2 submanifold of $\mathcal{M}$ (possibly with boundary).

(iii) (Local Finiteness Condition) For all $\gamma \in Q \times Q$, the family of switching manifolds subcomponents intersections are locally finite, i.e., for any $x \in \tilde{n}^{k_i}_\gamma \bigcap \tilde{n}^{k_j}_\gamma, k_i \neq k_j$ there exists a neighbourhood N_x of x meeting only a finite number of switching manifolds.
In this paper it is assumed that all phase transitions are autonomous, therefore there is no optimization over the

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discrete part of the hybrid control σ.

A1: The *initial state* $h_0 := (x(t_0), q_0) \in \mathbf{H}$ is such that $x_0 \notin \mathcal{N}$ for all $q_j \in Q$. It is assumed that for all p, q, whenever a trajectory governed by the controlled vector field f_p meets any given manifold $n_{p,q}$ transversally, there is an autonomous switching to the controlled vector field f_q, also transversal to $n_{p,q}$ and, conversely, any autonomous switching corresponds to a transversal intersection (for the definition of transversal intersection see [13]). ◆

A *hybrid system input* is a triple $I := (\tau, \sigma, u)$ defined on a half open interval $[t_0, T), T \leq \infty$, where $u \in \mathcal{U}$ and (τ, σ) is a hybrid switching sequence $(\tau, \sigma) = ((t_0, \sigma_0), (t_1, \sigma_1), (t_2, \sigma_2), \ldots)$ of pairs of increasing switching times and discrete input events where $\sigma_0 = id, \sigma_i \in \Sigma, i \geq 1$ and in general σ_i consists of both autonomous and controlled transition events.

A *hybrid state trajectory* is a triple (τ, q, x) consisting of a strictly increasing sequence of times $\tau = (t_0, t_1, t_2, \ldots)$, an associated sequence of discrete states $q = (q_0, q_1, q_2, \ldots)$, and a sequence $x(\cdot) = (x_{q_0}(\cdot), x_{q_1}(\cdot), x_{q_2}(\cdot), \ldots)$ of absolutely continuous functions $x_{q_j} : [t_j, t_{j+1}) \to \mathcal{M}$.

Let $\{l_j\}_{j \in Q}, l_j \in C^k(\mathcal{M} \times U; R_+), k \geq 1$ be a family of loss functions and $h \in C^k(\mathcal{M}; R_+), k \geq 1$, be a terminal cost with the associated initial time t_0, final time $t_f < \infty$, initial hybrid state $h_0 = (q_0, x_0)$, and the total number of switchings $\bar{L} < \infty$. Let
$S_L = ((t_0, \sigma_0), (t_1, \sigma_1), \ldots, (t_L, \sigma_L))$ be a hybrid switching sequence and let $u \in \mathcal{U}$ be a hybrid input function subject to ***A1***, where $L \leq \bar{L} < \infty$ is the number of switchings. We define the hybrid cost function as

$$J(t_0, t_f, h_0, u) := \sum_{i=0}^{L} \int_{t_i}^{t_{i+1}} l_{q_i}(x_{q_i}(s), u(s)) ds + h(x_{q_L}(t_f)), \quad u \in \mathcal{U}. \tag{2}$$

In the standard non-hybrid case, without loss of generality, the *Bolza problem* is equivalent to the *Mayer* problem (with possible state extension) which is given as follows:

$$J(t_0, t_f, h_0, u) := h(x_{q_L}(t_f)), \quad x_{q_L}(t_f) = \Phi^{(t_f, t_L)}_{f^u_{q_L}}(x(t_L)), \quad u \in \mathcal{U}. \tag{3}$$

where the flow Φ is defined below. In this paper we adopt:

A2: All pairs of states x_1, x_2 are mutually accessible via $\dot{x}_q(t) = f_q(x_q(t), u(t)), \quad q \in Q$. ◆

Definition 2.2: The *Hybrid Optimal Control Problem* (HOCP) is defined as the infimization of the hybrid cost over the hybrid input functions u.

$$J^o(t_0, t_f, h_0) = inf_{u \in \mathcal{U}} J(t_0, t_f, h_0, u). \tag{4}$$

The continuous dynamics of the hybrid system are specified as the following mapping:

$$(x_q, u) : [t_i, t_{i+1}) \to \mathcal{M} \times U, \tag{5}$$

which is an integral curve of f_q, satisfying

$$\dot{x}_q(t) = f_q(x_q(t), u(t)), \quad a.e.\, t \in [t_i, t_{i+1}).$$

f_q is the vector field defined on $\mathcal{M}$ for any given $u \in \mathcal{U}$ such that

$$f_q(., u) : \mathcal{M} \times [t_i, t_{i+1}) \to T\mathcal{M}, \tag{6}$$

$$u(t) \in U \subset R^u, u(.) \in L_\infty(U), \quad h_0 = (q_0, x_0), \quad i = 0, 1, \ldots, L,$$

$$x_{q_{i+1}}(t_{i+1}) = lim_{t \to t_{i+1}} x_q(t), \quad t_{L+1} = t_f < \infty. \tag{7}$$

In general, different control inputs result in different sets of discrete states of different cardinality. However, in this paper, we shall restrict the infimization to be over the class of control functions, $\mathcal{U}$, which generates an aprior given sequence of discrete transition events $\sigma_k, \quad k = 0, ..., L$

The time dependent flow associated to a differentiable time dependent vector field f_q is a map $\Phi_{f^u_q}$ as follows:

$$\Phi_{f^u_q} : [t_i, t_{i+1}) \times [t_i, t_{i+1}) \times \mathcal{M} \to \mathcal{M}, \quad (t, s, x) \to \Phi_{f^u_q}((t, s), x) \in \mathcal{M}. \tag{8}$$

By the definition above we have

$$\Phi^{(t,s)}_{f^u_q} : \mathcal{M} \to \mathcal{M}, \quad \Phi^{(s,s)}_{f^u_q}(x) = x, \tag{9}$$

$$\frac{d}{dt} \Phi^{(t,s)}_{f^u_q}(x)|_t = f_q(\Phi^{(t,s)}_{f^u_q}(x(t)), u(t)). \tag{10}$$

We associate $T\Phi^{(t,s)}_{f^u_q}(.)$ to $\Phi^{(t,s)}_{f^u_q}$ as the push-forward of $\Phi^{(t,s)}_{f^u_q}$ which is a generalization of the Jaccobian matrix of the smooth maps defined on Euclidean spaces, see [5].

$$T\Phi^{(t,s)}_{f^u_q} : T_x\mathcal{M} \to T_{\Phi^{(t,s)}_{f^u_q}(x)}\mathcal{M}. \tag{11}$$

The corresponding *tangent lift* of $f^u_q(.)$ is a time dependent vector field $f^{T,u}_q(.) \in TT\mathcal{M}$ defined on $T\mathcal{M}$ which is given as follows ($f^u_q(.)$ is used instead of $f_q(., u(t))$):

$$f^{T,u}_q(v_x) := \frac{d}{dt}|_{t=s} T_x \Phi^{(t,s)}_{f^u_q}(v_x), \quad v_x \in T_x\mathcal{M}, \tag{12}$$

where locally

$$f^{T,u}_q(x, v) = \left[f^{u,i}_q(x) \frac{\partial}{\partial x^i} + \left(\frac{\partial f^{u,i}_q}{\partial x^j} v^j\right) \frac{\partial}{\partial v^i} \right]^n_{i,j=1}, \tag{13}$$

and $T_x \Phi^{(t,s)}_{f^u_q}(.)$ is evaluated at $v_x \in T_x\mathcal{M}$, see [3]. For the sake of simplicity in the notations we use f_q instead of f^u_q.

Lemma 1: ([1]) Consider $f_q(x, u)$ as a time dependent vector field on $\mathcal{M}$ and $\Phi^{(t,s)}_{f_q}$ as the corresponding flow. The flow of $f^{T,u}_q$, denoted by $\Psi : I \times I \times T\mathcal{M} \to T\mathcal{M}, \quad I = [t_0, t_f]$, satisfies

$$\Psi(t, s, (x, v)) = (\Phi^{(t,s)}_{f_q}(x), T\Phi^{(t,s)}_{f_q}(v)) \in T\mathcal{M}, (x, v) \in T\mathcal{M}.$$

■

II. Pontryagin Maximum Principle for Optimal Control Systems

In general, a Bolza problem can be converted to a Mayer problem using an auxiliary state variable in the dynamics, see [1] and [13]. Here we review some results presented in [1], [2] and [4] which are essential for our proofs in this paper.

A. Elementary Control Perturbation and Tangent Perturbation

Consider the nominal control $u(.)$ and define the associated perturbed control as

$$u_\pi(t,\epsilon)=\begin{cases} u_1 & t_1-\epsilon\leq t\leq t_1,\\ u(t) & \text{elsewhere,}\end{cases} \tag{14}$$

where $u_1\in U, 0<\epsilon,$. Associated to $u_\pi(.,.)$ we have the corresponding state trajectory $x_\pi(t,\epsilon)$ on $\mathcal{M}$. It may be shown that under suitable hypotheses on the differentiability of x_π with respect to ϵ at the switching times, $lim_{\epsilon\rightarrow 0}x_\pi(t,\epsilon)=x(t)$ uniformly for $t_0\leq t\leq t_f$, see [?] and [2]. However in this paper we employ hypotheses on the differentiability of x_π before and after the switching times but must accommodate the fact that there may be a discontinuity of $\frac{dx_\pi(t)}{d\epsilon}|_{\epsilon=0}$ at a switching time t_i. Following (9), the flow resulting from the perturbed control is defined as:

$$\Phi^{(t,s),x}_{\pi,f_q}(\epsilon):[0,\tau]\rightarrow\mathcal{M},\quad x\in\mathcal{M},t,s\in[t_0,t_f],\tau\in R_+. \tag{15}$$

The following lemma gives the formula of the variation of $\Phi^{(t,s),x}_{\pi,f_q}(.)$ at $\epsilon=0^+$. We recall that the point $t_1\in(t_0,t_f)$ is called Lebesgue point of $u(.)$ if, ([4]):

$$\lim_{s\rightarrow t_1}\frac{1}{|s-t_1|}\int_{t_1}^{s}|u(\tau)-u(t_1)|d\tau=0. \tag{16}$$

For any $u\in L_\infty([t_0,T_*],U)$, u may be modified on a set of measure zero so that all points are Lebesgue points (see [10], page 158), in which case, necessarily, the value of any cost function is unchanged.

Lemma 2: ([1]) For the Lebesgue time t_1, the curve $\Phi^{(t,s),x}_{\pi,f_q}(\epsilon):[0,\tau]\rightarrow\mathcal{M}$ is differentiable at $\epsilon=0$ and the corresponding tangent vector $\frac{d}{d\epsilon}\Phi^{(t_1,s),x}_{\pi,f_q}|_{\epsilon=0}$ is

$$[f_q(x(t_1),u_1)-f_q(x(t_1),u(t_1))]\in T_x\mathcal{M}. \tag{17}$$

■

The tangent vector $[f_q(x(t_1),u_1)-f_q(x(t_1),u(t_1))]$ is called the *elementary perturbation vector* associated to the perturbed control u_π. The displacement of the elementary perturbation vectors at $x\in\mathcal{M}$ is explained by the push-forward of the flow corresponding to the vector field f_q. By definition $\Phi^{(t,s)}_{f_q}:\mathcal{M}\rightarrow\mathcal{M}$ therefore

$$T\Phi^{(t,s)}_{f_q}:T\mathcal{M}\rightarrow T\mathcal{M},\quad T\Phi^{(t,s)}_{f_q}(v_s)\in T_{\Phi^{(t,s)}_{f_q}(x)}\mathcal{M}, \tag{18}$$

where

$$v_s\in T_x\mathcal{M},\quad \Phi^{(s,s)}_{f_q}(x)=x. \tag{19}$$

B. Adjoint Processes

In the case $\mathcal{M}=R^n$ based on the differentiability of f_q, let us define the following variational differential system:

$$\dot{\lambda}^T(t)=-\lambda^T(t)\frac{\partial f_q}{\partial x}(x(t),u(t)),\quad t\in[t_0,t_f],x\in R^n. \tag{20}$$

As is shown in [2], the matrix solution φ of the system $\dot{\varphi}(t)=\frac{\partial f_q}{\partial x}(x(t),u(t))\varphi$ where $\varphi(0)=I$ gives the transformation between tangent vectors on the state trajectory $x(t)$ from time t_1 to t_2, that is to say considering v_1 as a tangent vector at $x(t_1)$, the push-forward of v_1 under the map $\Phi^{(t_2,t_1)}_{f_q}$ is

$$v_2=T\Phi^{(t_2,t_1)}_{f_q}(v_1)=\varphi(t_2-t_1)v_1, v_1\in T_{x(t_1)}R^n=R^n. \tag{21}$$

The vector $v(t)=\varphi(t)v(0)$ is the solution of the following differential equation, (see [2]):

$$\dot{v}(t)=\frac{\partial f_q}{\partial x}(x(t),u(t))v(t),\quad v(t)\in T_{x(t)}R^n=R^n. \tag{22}$$

The significance of (20), which plays a major role in the proof of the Maximum Principle for optimal control, is that along $x(t)$, $\lambda^T(t)v(t)$ remains constant, see [2]. For a general Riemannian manifold $\mathcal{M}$, the role of the adjoint process λ is played by a trajectory in the cotangent bundle of $\mathcal{M}$, i.e. $\lambda(t)\in T^*_{x(t)}\mathcal{M}$. Similar to the definition of the tangent lift we define the *cotangent lift* which corresponds to the variation of a differential form $\alpha\in T^*\mathcal{M}$ along $x(t)$, see [6]:

$$f_q^{T^*,u}(\alpha_x):=\frac{d}{dt}|_{t=s}T^*_x\Phi^{(t,s)^{-1}}_{f^u_q}(\alpha_x),\quad \alpha_x\in T^*_x\mathcal{M}, \tag{23}$$

where here $x=x(t)=\Phi^{(t,s)}_{f^u_q}(x(s))$. Similar to (13) in the local coordinates (x,p) of $T^*\mathcal{M}$, we have

$$f_q^{T^*,u}(x,p)=\left[f_q^{u,i}(x)\frac{\partial}{\partial x^i}-(\frac{\partial f_t^{u,i}}{\partial x^j}p^j)\frac{\partial}{\partial p^i}\right]^n_{i,j=1} \tag{24}$$

The mapping T^*_x is the pull back defined on the differential forms on the cotangent bundle of $\mathcal{M}$. The covector α_x is an element of $T^*_x\mathcal{M}$, see [6]. The following lemma gives the connection between the cotangent lift defined in (23) and its corresponding flow on $T^*\mathcal{M}$.

Lemma 3: ([1]) Consider $f_q(x(t),u(t))$ as a time dependent vector field on $\mathcal{M}$, then the flow $\Gamma:I\times I\times T^*\mathcal{M}\rightarrow T^*\mathcal{M}$, satisfies ($I=[t_0,t_f]$)

$$\Gamma(t,s,(x,p))=(\Phi^{(t,s)}_{f_q},(T^*_x\Phi^{(t,s)}_{f_q})^{-1}(p)),(x,p)\in T^*_x\mathcal{M}, \tag{25}$$

and Γ is the corresponding integral flow of $f_q^{T^*,u}$. ■

The mapping $(T^*_x\Phi^{(t,s)}_{f_q})^{-1}=T^*_x\Phi^{(-t,s)}_{f_q})$ is defined as a pull back of Φ^{-1} which its existence is guaranteed since $\Phi:\mathcal{M}\rightarrow\mathcal{M}$ is a diffeomorphism, see [1] . Equations (20, 22) in Euclidean spaces are generalized to Reimmanian manifolds by defining the tangent and cotangent lift in (12) and (23) .

For a given trajectory $\lambda(t)\in T^*\mathcal{M}$, its variation with respect to time, $\dot{\lambda}(t)$, is an element of $TT^*\mathcal{M}$. The vector field defined in (23) is the mapping $f_q^{T^*,u} : T^*\mathcal{M} \to TT^*\mathcal{M}$, which is the general description of (20) as the mapping from $\lambda(t) \in T^*\mathcal{M}$ to $\dot{\lambda}(t) \in TT^*\mathcal{M}$.

Proposition 1: ([1]) Let $f_q(.,u) : I \times \mathcal{M} \to \mathcal{M}$ be the time dependent vector field with the associated $f_q^{T,u}, f_q^{T^*,u}$. Along the state trajectory $x(t)$ which is the integral curve of $f_q(.,u)$ on $\mathcal{M}$ we have that

$$\langle \Gamma, \Psi \rangle : I \to R, \tag{26}$$

is a constant map, where Γ is the integral curve of $f_q^{T^*,u}$ in $T^*\mathcal{M}$ and Ψ is the integral curve of $f_q^{T,u}$ in $T\mathcal{M}$ and $I = [t_0, t_f]$.

■

Γ and Ψ play the role of $\lambda(.)$ and $v(.)$ appeared in the solution of (20) and (22). The variation of the elementary tangent perturbation in Lemma 2 is given in the following proposition.

Proposition 2: ([1]) Let $\Psi : [t_1, t_f] \to T\mathcal{M}$ be the integral curve of $f_q^{T,u}$ with the initial condition $\Psi(t_1) = [f_q(x(t_1), u_1) - f_q(x(t_1), u(t_1))] \in T_{x(t_1)}\mathcal{M}$, then

$$\frac{d}{d\epsilon}\Phi_{\pi,f_q}^{(t,t_1),x}|_{\epsilon=0} = \Psi(t), \quad t \in [t_1, t_f]. \tag{27}$$

■

By the result above and Lemma 1 we have

$$\frac{d}{d\epsilon}\Phi_{\pi,f_q}^{(t,t_1),x}|_{\epsilon=0,x(t)\in\mathcal{M}} = T\Phi_{f_q}^{(t,t_1)}([f_q(x(t_1),u_1) - f_q(x(t_1),u(t_1))]) \in T_{x(t)}\mathcal{M}. \tag{28}$$

For an optimal control problem restricted to non-hybrid cases on $\mathcal{M}$, evolving by the vector field $f_q(x(t), u(t)), q \in Q$ (q is fixed), the Hamiltonian function is defined as:

$$H : T^*\mathcal{M} \times U \to R, \tag{29}$$

$$H(x,p,u) = \langle p, f_q(x,u) \rangle,\ p \in T_x^*\mathcal{M}, f_q(x,u) \in T_x\mathcal{M}. \tag{30}$$

Employing the notation of Hamiltonian functions introduced in [4], the general Hamiltonian is a smooth function $H(u) = H \in C^\infty(T^*\mathcal{M})$ which is associated to a Hamiltonian vector field $\overrightarrow{H}$ as follows (see [4]).

Denote $\overrightarrow{H} \in TT^*\mathcal{M}$ as the associated Hamiltonian vector field to H by

$$\sigma_\lambda(., \overrightarrow{H}) = dH, \quad \lambda \in T^*\mathcal{M}, \tag{31}$$

where $\sigma_\lambda \in \Omega^2(T^*\mathcal{M})$ is the symplectic structure defined on $T^*\mathcal{M}$. The definition of Ω^k can be found in [5]. The Hamiltonian vector field satisfies the following equation, see [4]:

$$i_{\overrightarrow{H}}\sigma = -dH, \tag{32}$$

where $i_{\overrightarrow{H}}$ is the contraction mapping along the vector field $\overrightarrow{H}$, see [5], [11]. The Hamiltonian systems $\dot{\lambda}(t) = \overrightarrow{H}(\lambda(t)), \quad \lambda(t) = (x(t), p(t)) \in T^*\mathcal{M}$ is locally written as:

$$\begin{cases} \dot{x}(t) = \frac{\partial H}{\partial p^i} \\ \dot{p}(t) = -\frac{\partial H}{\partial x^i} \end{cases}, \lambda(t) \in T^*\mathcal{M}. \tag{33}$$

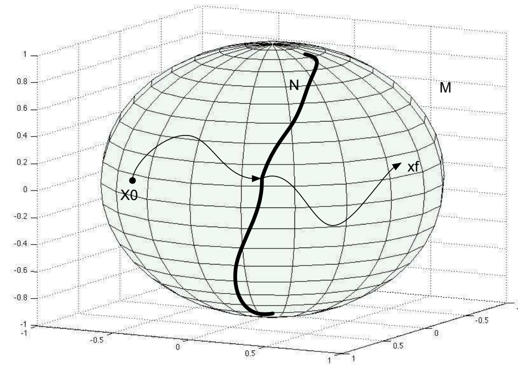

Fig. 1. Hybrid State Trajectory On An Sphere

III. Hybrid Minimum Principle for Autonomous Hybrid Systems

Here we consider a simple impulsive autonomous hybrid system consisting of one switching manifold. Consider a hybrid system with two distinct modes q_1, q_2 associated with the following dynamics:

$$\dot{x}_{q_i}(t) = f_{q_i}(x(t), u(t)), \quad a.e.\, t \in [t_i, t_{i+1}), i = 1, 2, \tag{34}$$

and

$$f_{q_i}(., u) : \mathcal{M} \times [t_i, t_{i+1}) \to T\mathcal{M}, i = 1, 2. \tag{35}$$

Here we assume the state trajectory of both phases are evolved on the same smooth differentiable n dimensional manifold $\mathcal{M}$. The switching manifold is an embedded $n-1$ dimensional submanifold $\mathcal{N}$ of $\mathcal{M}$. Similar to the proof in [13] we divide the proof into two different parts. First, the control needle variation is applied after the optimal switching time so there is no state variation propagation along the state trajectory before hitting the switching manifold.

Second, the needle variation is applied before the optimal switching time. Figure 1 shows an autonomous hybrid systems defined on an sphere with an embedded one dimensional switching manifold.

Recalling assumption ***A2*** on the accessibility of $\dot{x}(t) = f_{q_1}(x(t), u(t))$, let us define $v(x,t)$ for a hybrid system with one autonomous switching, i.e. $L = 1$, as follows:

$$v(x,t) = inf_{u\in\mathcal{U}} J(t_0, t_f, h_0, u), \quad x \in \mathcal{M}, t \in R, \tag{36}$$

where

$$\Phi_{f_{q_1}}^{(t_0,t)}(x_0) = x \in \mathcal{N} \subset \mathcal{M}. \tag{37}$$

In this paper we assume that the value function defined above is differentiable at the optimal switching state on the switching manifold.

Generalizing the HOCP results in the case $\mathcal{M} = R^n$ in [13] we have the following theorem which gives the HMP for autonomous hybrid systems with one autonomous switching which occurs on the switching manifold $\mathcal{N} \subset \mathcal{M}$.

$dv(x,t)$ is introduced here as the differential form of $v(.,t):\mathcal{M}\to R$ for a given $t\in R$ in the local coordinate of $x\in\mathcal{M}$.

Theorem 1: ([15]) Consider a hybrid system satisfying hypotheses ***A1*** and ***A2*** on a smooth Riemannian n dimensional state manifold $\mathcal{M}$ with the associated Riemannian metric g and an $n-1$ dimensional embedded switching submanifold $\mathcal{N}\subset\mathcal{M}$; then corresponding to the optimal control and optimal trajectory u^o, x^o, there exists a nontrivial $\lambda^o(t)\in T^*\mathcal{M}$ along the optimal state trajectory such that:

$$H_{q_i}(x^o(t),p^o(t),u^o(t))\le H_{q_i}(x^o(t),p^o(t),u_1),\quad \forall u_1\in U, t\in[t_0,t_f], i=1,2, \tag{38}$$

and the corresponding optimal adjoint variable $\lambda^o(t)\in T^*\mathcal{M}$ satisfies:

$$\dot{\lambda}^o(t)=\overrightarrow{H}_{q_i}(\lambda^o(t)),\quad t\in[t_0,t_f], i=1,2. \tag{39}$$

At the optimal switching state and switching time $x^o(t_s), t_s$ from q_1 to q_2 we have

$$p^o(t_s^-)=p^o(t_s^+)-\mu dv(x^o(t_s),t_s),\quad p^o(t_s^-),\lambda^o(t_s^+)\in T^*_{x(t_s)}\mathcal{M}, \tag{40}$$

$$x^o(t_0)=x_0,\ \ p^o(t_f)=dh(x^o(t_f))\in T^*_{x^o(t_f)}\mathcal{M},\quad dh=\sum_{i=1}^{n}\frac{\partial h}{\partial x^i}dx^i\in T^*_x\mathcal{M}, \tag{41}$$

$$H_{q_i}(x^o(t_s^-),p^o(t_s^-),u^o(t_s^-))= H_{q_{i+1}}(x^o(t_s^+),p^o(t_s^+),u^o(t_s^+)). \tag{42}$$

where $\mu\in R$ and

$$dv(x^o(t_s),t_s)=\Sigma_{j=1}^{n}\frac{\partial v(x^o(t_s),t_s)}{\partial x^j}dx^j\in T^*_{x^o(t_s)}\mathcal{M}, \tag{43}$$

■

IIn order to use the methods introduced in [4], [1], [2], we establish the following lemmas using the perturbed control $u_\pi(.)$ and the associated state variation at the final state $x^o(t_f)$.

Denote by $t_s(\epsilon)$ the switching time corresponding to $u_\pi(t,\epsilon)$ which is assumed to be differentiable from the right with respect to ϵ for all $u\in U$ then

Lemma 4: ([15]) For a HOCP defined on a Reimannian state manifold $\mathcal{M}$ with an associated metric g we have

$$\langle -dh(x^o(t_f)), v_\pi(t_f)\rangle\le 0,\quad \forall v_\pi(t_f)\in K_{t_f}, \tag{44}$$

where

$$K^1_{t_f}=\bigcup_{t_s\le t<t_f}\bigcup_{u\in U_t}T\Phi^{(t_f,t)}_{f_{q_2}}[f_{q_1}(x^o(t,u))-f_{q_1}(x^o(t),u^o(t))]\subset T_{x(t_f)}\mathcal{M},\ t\in[t_s,t_f], \tag{45}$$

and

$$\begin{aligned}K^2_{t_f}=&\bigcup_{t_0\le t<t_s}\bigcup_{u\in U_t}T\Phi^{(t_f,t_s)}_{f_{q_2}}\circ T\Phi^{(t_s,t)}_{f_{q_1}}\\ &\times[f_{q_1}(x^o(t,u))-f_{q_1}(x^o(t),u^o(t))]+\frac{dt_s(\epsilon)}{d\epsilon}\\ &\times T\Phi^{(t_f,t_s)}_{f_{q_2}}\big(f_{q_2}(x^o(t_s),u^o(t_s))\\ &-f_{q_1}(x^o(t_s^-),u^o(t_s^-))\big)\subset T_{x(t_f)}\mathcal{M},\ t\in[t_0,t_s),\end{aligned} \tag{46}$$

and

$$K_{t_f}=K^1_{t_f}\cup K^2_{t_f}. \tag{47}$$

■

Since $\mathcal{N}$ is an embedded submanifold of $\mathcal{M}$ there necessarily exists an embedding $i:\mathcal{N}\to\mathcal{M}$. The push-forward of the inclusion i is written as follows:

$$Ti: T_x\mathcal{N}\to T_x\mathcal{M}. \tag{48}$$

For any vector $X\in T_x\mathcal{N}$, the image vector $Ti(X)\in T_x\mathcal{M}$ is a tangent vector on $\mathcal{M}$. If we write the coordinate representation of X as $X=\Sigma_{j=1}^n X^j\frac{\partial}{\partial x^j}$ then $X\in T_x\mathcal{N}$ if and only if $X^j=0,\quad j>k$, where k is the dimension of $\mathcal{N}$, see [5], page 178. The following lemma gives the relation between the differential form $dv(x(t_s),t_s)=\Sigma_{j=1}^n\frac{\partial v(x(t_s),t_s)}{\partial x^j}dx^j\in T^*_{x(t_s)}\mathcal{M}$ and any tangent vector $X\in T_{x(t_s)}\mathcal{M}$ which is also a tangent vector in $T_{x(t_s)}\mathcal{N}$.

Lemma 5: ([15]) Consider the value function v for an autonomous HOCP with two different regimes separated by an embedded switching manifold $\mathcal{N}\subset\mathcal{M}$ of dimension k; at the optimal switching state $x^o(t_s)\in\mathcal{N}$ and switching time t_s we have

$$\langle dv(x^o(t_s),t_s), X\rangle=0,\quad \forall X\in Ti(T_{x(t_s)}\mathcal{N}). \tag{49}$$

■

IV. Simulation Results

The results presented in this paper are applied to hybrid systems defined on Riemannian manifolds. We employ the *Gradient Geodesic-HMP* (GG-HMP) as the optimization algorithm which is the extension to Rimannian manifolds of the HMP algorithm introduced in [13], see [7]. This is done by introducing a geodesic gradient flow algorithm on $\mathcal{N}$ and constructing an HMP algorithm along geodesics on $\mathcal{N}$.

Here we apply the GG-HMP algorithm to an HOCP defined on a torus with the following parametrization:

$$\begin{aligned}x(\zeta,w)&=(R+rcos(w))cos(\zeta),\\ y(\zeta,w)&=(R+rcos(w))sin(\zeta),\\ z(\zeta,w)&=rsin(w), w,\zeta\in[0,2\pi].\end{aligned} \tag{50}$$

where $R=1, r=0.5$. The hybrid system goes through each phase in numerical order and the dynamics are given in the local parametrization space of the torus T^2 as follows:

$$S_1\quad \begin{pmatrix}\dot{\zeta}\\ \dot{w}\end{pmatrix}=\begin{pmatrix}1.5 & 0\\ 0 & 1\end{pmatrix}\begin{pmatrix}\zeta\\ w\end{pmatrix}+\begin{pmatrix}1\\ 1\end{pmatrix}u, \tag{51}$$

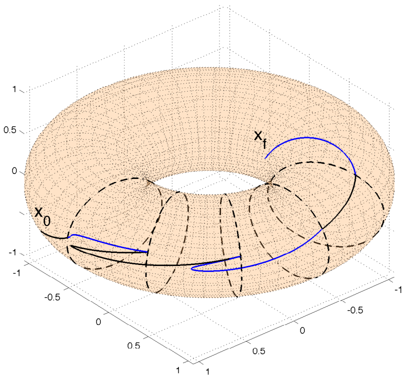

Fig. 2. Hybrid State Trajectory On the Torus

$$S_2 \quad \begin{pmatrix} \dot{\zeta} \\ \dot{w} \end{pmatrix} = \begin{pmatrix} 5 & 0 \\ 0 & 1 \end{pmatrix} \begin{pmatrix} \zeta \\ w \end{pmatrix} + \begin{pmatrix} 1 \\ 1 \end{pmatrix} u, \qquad (52)$$

$$S_3 \quad \begin{pmatrix} \dot{\zeta} \\ \dot{w} \end{pmatrix} = \begin{pmatrix} 3 & 0 \\ 0 & 4 \end{pmatrix} \begin{pmatrix} \zeta \\ w \end{pmatrix} + \begin{pmatrix} 1 \\ 1 \end{pmatrix} u, \qquad (53)$$

$$S_4 \quad \begin{pmatrix} \dot{\zeta} \\ \dot{w} \end{pmatrix} = \begin{pmatrix} 1 & 0 \\ 0 & 3 \end{pmatrix} \begin{pmatrix} \zeta \\ w \end{pmatrix} + \begin{pmatrix} 1 \\ 1 \end{pmatrix} u, \qquad (54)$$

$$S_5 \quad \begin{pmatrix} \dot{\zeta} \\ \dot{w} \end{pmatrix} = \begin{pmatrix} 1 & 0 \\ 0 & 2 \end{pmatrix} \begin{pmatrix} \zeta \\ w \end{pmatrix} + \begin{pmatrix} 1 \\ 1 \end{pmatrix} u, \qquad (55)$$

$$S_6 \quad \begin{pmatrix} \dot{\zeta} \\ \dot{w} \end{pmatrix} = \begin{pmatrix} 1 & 0 \\ 0 & 3 \end{pmatrix} \begin{pmatrix} \zeta \\ w \end{pmatrix} + \begin{pmatrix} 1 \\ 1 \end{pmatrix} u. \qquad (56)$$

We consider the induced metric from R^3 as the Riemannian metric on T^2. The switching submanifolds and the cost function are defined as follows:

$$\{0 \le w \le 2\pi, \zeta = 0, \frac{\pi}{6}, \frac{\pi}{3}, \frac{\pi}{2}, \frac{2\pi}{3}\}, \quad J = \int_0^8 u^2(t)dt, \ (57)$$

and the boundary conditions are given as:

$$x_0 = (1.4117, -0.4367, -0.1478) \in R^3, \qquad (58)$$
$$x_f = (-0.1478, -0.49980, 0.10130) \in R^3.$$

Figure 2 shows the state trajectory on the torus and Figure 3 depicts the adjoint variable with the discontinuity at the optimal switching times $\overline{t_s} = [1.2137, 2.6250, 4.0145, 5.2821, 6.6382]$.

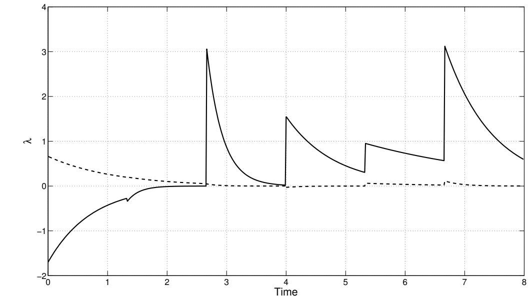

Fig. 3. Adjoint Process

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