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An Evolution Mean Field Equation System of Initial Mean Consensus Behaviour: A Stability Analysis

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Abstract— **The purpose of this paper is to study an evolution (i.e., forward in time) mean field equation system of a dynamic game initial mean consensus model. In this model: (i) each agent has simple stochastic dynamics with inputs directly controlling its state's rate of change, and (ii) each agent seeks to minimize its individual long run average cost function involving a mean field coupling to the states of all other agents. The Evolution Mean Field (EMF) equation system of the continuum (i.e., as the population size N goes to infinity) version of this model consists of two coupled (forward in time) deterministic PDEs which are also coupled to a (spatially averaged) cost coupling function. The stationary equilibrium of the EMF equation system yields a mean-consensus behaviour in the system. The small perturbation stability of the EMF equation system around this stationary equilibrium solution is established. Hence, the EMF equation system provides a forward in time process which asymptotically in time converges to the stationary equilibrium solution from any given initial condition in a infinitesimal neighborhood of that equilibrium.**

I. Introduction

A *consensus process* is the process of dynamically reaching an agreement between the agents of a group on some common state properties such as position or velocity. The formulation of consensus systems is one of the important issues in the area of multi-agent control and coordination, and has been an active area of research in the systems and control community over the past few years (see [1] and the references therein, among many other papers).

In [2], [3], [4] we synthesized the consensus behaviour as a dynamic game problem via *stochastic Mean Field* (MF) *control* (or *Nash Certainty Equivalence* (NCE)) theory (see [5]). In this Dynamic Game Consensus Model (DGCM): (i) each agent has simple stochastic dynamics with inputs directly controlling its state's rate of change, and (ii) each agent seeks to minimize its individual cost function involving a mean field coupling to the states of all other agents. This initial mean consensus formulation is motivated by many social, economic, and engineering models (see [3]).

Based on the MF (NCE) approach developed in [6], we derived an *individual based MF equation system* of the DGCM and explicitly computed its unique solution in [2], [3]. The resulting MF control strategies steer each individual's state toward the initial state population mean (i.e., initial mean consensus). Furthermore, these control laws possess an ε_N-Nash equilibrium property where ε_N goes to zero as the population size N goes to infinity.

In an analogous way and based on the approach developed in [7], the *continuum based MF equation system* of the DGCM is derived in [4]. Unlike [2], [3], the initial states for all the agents of the model in [4] are not necessarily assumed to be distributed according to a Gaussian distribution. The continuum (i.e., as the population size N goes to infinity) based MF equation system consists of two coupled deterministic equations: (i) a nonlinear (backward in time) Hamilton-Jacobi-Bellman (HJB), and (ii) a nonlinear (forward in time) Fokker-Planck-Kolmogorov (FPK), which are also coupled to a (spatially averaged) cost coupling function approximating the aggregate effect of the agents in the infinite population limit. The corresponding Gaussian stationary solution of the continuum MF equation system and its small perturbation stability analysis (based on the technique in [8]) are studied in [4].

The solution of the HJB equation is the relative value function which represents perturbations around the steady-state optimal cost rate with respect to an asymptotically stationary process. It turns out that this HJB equation in the MF system of equations has a larger class of stable perturbed solutions in forward time than in backward time [8].

In this paper we study an Evolution (i.e., forward in time) Mean Field (EMF) equation system of the continuum version DGCM where the initial states for all the agents are not necessarily assumed to be distributed according to a Gaussian distribution. The EMF equation system consists of two coupled (forward in time) deterministic PDEs which are also coupled to the cost coupling function. This forward in time mean field process has previously appeared in the study of "mean field games" models in [8], [9].

In this paper the linearized stability of the EMF equation system around the stationary equilibrium solution is shown. This stationary equilibrium yields a mean-consensus behaviour in the system. Hence, the EMF algorithm provides a forward in time process which asymptotically in time converges to the stationary equilibrium solution from any given initial condition in the infinitesimal neighborhood of that equilibrium.

The problem formulations and the results of this paper differ from those in [8] in the following respects: (i) in [8], as in the Lasry and Lions mean field games [10], for systems with finite population sizes a simplifying assumption was used stipulating that each agent's strategy depends only on its own driving Brownian motion, (ii) the ergodic individual cost

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functions of our multi-agent model is fundamentally different from the discounted logarithmic utility function considered in [8], and hence the analysis of the corresponding EMF equation systems are different, and (iii) finally, the consensus behaviour is not studied in [8].

In this paper the symbols ∂_t and ∂_z are respectively denote the partial derivative with respect to variables t and z, and ∂^2_{zz} denotes the second derivative with respect to z.

II. The Dynamic Game Consensus Model

Consider a system of N agents. The dynamics of the i^{th} agent is given by a controlled SDE:

$$dz_i(t) = u_i(t)dt + \sigma dw_i(t), \quad t \geq 0, \qquad 1 \leq i \leq N, \tag{1}$$

where $z_i(\cdot)$, $u_i(\cdot) \in \mathbb{R}$ are the state and control input of agent i, respectively; σ is a non-negative scalar; and $\{w_i(\cdot) : 1 \leq i \leq N\}$ denotes a sequence of mutually independent standard scalar Wiener processes on some *filtered probability space* $(\Omega, \mathscr{F}, \{\mathscr{F}_t\}_{t\geq 0}, \mathbb{P})$ where $\mathscr{F}_t$ is defined as the *natural filtration* given by σ-field $\sigma(z_i(\tau) : 1 \leq i \leq N, \tau < t)$. We assume that the initial states $\{z_i(0) : 1 \leq i \leq N\}$ are measurable on $\mathscr{F}_0$, mutually independent, and independent of Wiener processes $\{w_i : 1 \leq i \leq N\}$. It is important to note that the initial states for all the agents are not necessarily assumed to be distributed according to a Gaussian distribution.

Let the *admissible control* set of the i^{th} agent be $\mathscr{U}_i := \big\{u_i(\cdot) : u_i(t)$ is adapted to the sigma-field $\mathscr{F}_t$, $|z_i(T)|^2 = o(\sqrt{T})$, $\int_0^T (z_i(t))^2 dt = O(T)$, *a.s.*$\big\}$. The objective of the i^{th} individual agent is to *almost surely* (a.s.) minimize its *ergodic* or *Long Run Average* (LRA) cost function given by

$$J_i^N(u_i, u_{-i}) := \limsup_{T\to\infty} \frac{1}{T}\int_0^T \Big((z_i - \frac{1}{N-1}\sum_{j\neq i}^{N} z_j)^2 + r u_i^2\Big)dt, \tag{2}$$

where r is a positive scalar and $z_{-i}^N(\cdot) := \big(1/(N-1)\big)\sum_{j=1, j\neq i}^N z_j(\cdot)$ is called the *mean field* term. To indicate the dependence of J_i on u_i, $u_{-i} := (u_1, \cdots, u_{i-1}, u_{i+1}, \cdots, u_N)$ and the population size N, we write it as $J_i^N(u_i, u_{-i})$.

III. Previous Results

In this section we briefly summarize the main results of [4]. We take the following steps to the DGCM (1)-(2) based on the MF control approach (developed in [7] after [6]):

1) **The continuum (infinite population) limit**: In this step a Nash equilibrium for the DGCM (1)-(2) in the continuum population limit (as N goes to infinity) is characterized by a "consistency relationship" between the individual strategies and the mass effect (i.e., the overall effect of the population on a given agent). This consistency relationship is described by a so-called MF equation system (see (12)-(14) below).
2) **ε_N-Nash equilibrium for the finite N model**: The distributed continuum based MF control law (derived from the MF equation system in Step 1) establishes an ε_N-Nash equilibrium (see Theorem 4) for the finite N population DGCM (1)-(2) where ε_N goes to zero asymptotically (as N approaches infinity).

A. Mean Field Approximation

In a large N population system, the *mean field* approach suggests that the cost-coupling function for a "generic" agent i $(1 \leq i \leq N)$ in (2),

$$c^N(z_i(\cdot), z_{-i}(\cdot)) := \Big(z_i(\cdot) - \frac{1}{N-1}\sum_{j\neq i}^{N} z_j(\cdot)\Big)^2,$$

be approximated by a deterministic function $c(z, \cdot)$ which only depends on $z = z_i$.

Replacing the function $c^N(z_i, z_{-i})$ with the deterministic function $c(z_i, \cdot)$ in the i^{th} agent's LRA cost function (2) reduces the DGCM (1)-(2) to a set of N independent optimal control problems.

Now we consider a "single agent" optimal control problem:

$$dz(t) = u(t)dt + \sigma dw(t), \quad t \geq 0, \tag{3}$$

$$\inf_{u\in\mathscr{U}} J(u) := \inf_{u\in\mathscr{U}} \limsup_{T\to\infty} \frac{1}{T}\int_0^T \big(c(z,t) + r u^2(t)\big)dt, \tag{4}$$

where $z(\cdot)$, $u(\cdot) \in \mathbb{R}$ are the state and control input, respectively; $w(\cdot)$ denotes a standard scalar Wiener process; $c(z, \cdot)$ is a known positive function; and $\mathscr{U}$ is the corresponding admissible control set of the generic agent.

An admissible control $u^o(\cdot) \in \mathscr{U}$ is called *a.s. optimal* if there exists a constant ρ^o such that

$$J(u^o) = \limsup_{T\to\infty} \frac{1}{T}\int_0^T \Big(c\big(z^o(t), t\big) + r\big(u^o(t)\big)^2\Big)dt = \rho^o, \ a.s.,$$

where $z^o(\cdot)$ is the solution of (3) under $u^o(\cdot)$, and for any other admissible control $u(\cdot) \in \mathscr{U}$, we have a.s. $J(u) \geq \rho^o$.

The associated *Hamilton-Jacobian-Bellman* (HJB) equation of the optimal control problem (3)-(4) is given by (see [4] for the derivation)

$$\partial_t v(z,t) + \frac{\sigma^2}{2}\partial^2_{zz} v(z,t) + H\big(z, \partial_z v(z,t)\big) + c(z,t) = \rho^o, \tag{5}$$

where $v(z, \cdot)$ is the *relative value* function, ρ^o is the optimal cost and

$$H(z,p) := \min_{u\in\mathscr{U}} \{up + ru^2\}, \qquad z,\ p \in \mathbb{R},$$

is the *Hamiltonian*. For $x \in \mathbb{R}$ and $0 < t < \infty$, $v(x,t)$ is defined as

$$\inf_{u\in\mathscr{U}} \Big(\inf_{\tau\geq t} E\Big[\int_t^\tau \Big(c\big(z(s), s\big) + r\big(u(s)\big)^2 - \rho^o\Big)ds \big| z(t) = x\Big]\Big), \tag{6}$$

where the inner infimum is over all bounded stopping times with respect to the natural filtration $\{\mathscr{F}_t\}_{t\geq 0}$ (see [11]).

The solution of the optimal control problem (3)-(4) is

$$u^o(t) := H\big(z, \partial_z v(z,t)\big) = -\frac{1}{2r}\partial_z v(z,t), \qquad t \geq 0.$$

Substituting $u^o(\cdot)$ into the HJB equation (5) yields the (backward in time) nonlinear deterministic PDE:

$$\partial_t v(z,t) - \frac{1}{4r}\big(\partial_z v(z,t)\big)^2 + \frac{\sigma^2}{2}\partial^2_{zz} v(z,t) + c(z,t) = \rho^o. \tag{7}$$

We enunciate the following assumption:

(**A1**) We assume that the sequence $\{Ez_i(0) : 1 \le i \le N\}$ is a subset of a fixed compact set $\mathscr{A}$ independent of N, and has a compactly supported *probability density* $f_0(z)$ (which is not necessarily a Gaussian density). Let

$$f_N(x,t) := \frac{1}{N}\sum_{i=1}^{N} \delta\big(x - Ez_i(t)\big),$$

be the *empirical distribution density* associated with N agents where δ is the *Dirac delta*. We assume that $\{f_N(x,0) : N \ge 1\}$ converges weakly to f_0, *i.e.*, for any $\phi(x) \in C_b(\mathbb{R})$ (the space of bounded continuous functions on $\mathbb{R}$),

$$\lim_{N\to\infty} \int_B \phi(x) f_N(x,0)dx = \int_B \phi(x) f_0(x)dx,$$

for any subset $B \subset \mathscr{A}$. □

For any function $\phi(x) \in C_b$ on $\mathbb{R}$ we have

$$\int \phi(x) f_N(x,t)dx = \frac{1}{N}\sum_{i=1}^{N}\phi\big(Ez_i(t)\big).$$

Since the processes $\{z_i(\cdot) : 1 \le i \le N\}$ are *independent and identically distributed* (i.i.d.), by the *ergodic theorem* we have

$$\lim_{N\to\infty}\int \phi(x) f_N(x,t)dx = \int \phi(x) f^u(x,t)dx, \qquad a.s. \tag{8}$$

where $f^u(z,\cdot)$ is the density of the generic agent's state which evolves according to the SDE (3) with control law $u(\cdot) \in \mathscr{U}$.

The evolution of the population density $f^u(z,\cdot)$ satisfies the *Fokker-Planck-Kolmogorov* (FPK) equation

$$\partial_t f^u(z,t) + \partial_z\big(u f^u(z,t)\big) = \frac{\sigma^2}{2}\partial_{zz}^2 f^u(z,t), \tag{9}$$

where $f^u(z,0) = f_0(z)$ is characterized by (**A1**).

Now by substituting the optimal control $u^o(\cdot)$ into its FPK equation (9) we get the (forward in time) nonlinear deterministic PDE

$$\partial_t f(z,t) - \frac{1}{2r}\partial_z\Big(\big(\partial_z v(z,t)\big) f(z,t)\Big) = \frac{\sigma^2}{2}\partial_{zz}^2 f(z,t), \tag{10}$$

where $f(z,0) = f_0(z)$, and $v(z,\cdot)$ is the solution of the equation (7).

Finally, for a generic agent i the ergodic theorem in (8) suggests the approximation of $c^N(z_i, z_{-i}^o)$ for a large N population system by

$$\bar{c}(z_i,\cdot) = \big(z_i - \int_{\mathbb{R}} z f(z,\cdot)dz\big)^2 = \big(\int_{\mathbb{R}} (z_i - z) f(z,\cdot)dz\big)^2, \tag{11}$$

where $f(z,\cdot)$ is the population density under the optimal control $u^o(\cdot)$ (i.e., $f(z,\cdot)$ is the solution of the equation (10)).

B. Mean Field Equation System

In this section we aim to construct the "consistency relationship" (between the individual strategies and the mass influence effect) in the stochastic MF control theory (based on the approach developed in [7] after [6]). The key idea is to prescribe a spatially averaged mass function $\bar{c}(z,\cdot)$ characterized by the property that it is reproduced as the average of all agents' states in the continuum of agents whenever each individual agent optimally tracks the same mass function $\bar{c}(z,\cdot)$.

Considering the continuum population limit (i.e., as N approaches ∞) of the DGCM (1)-(2) where $f(z,0) = f_0(z)$ is the initial population density and $\int_{\mathbb{R}} f(z,t)dz = 1$ for any $t \ge 0$, we obtain the following *continuum based mean field* (MF) equation system:

[MF-HJB]

$$\partial_t v(z,t) = \frac{1}{4r}\big(\partial_z v(z,t)\big)^2 - \bar{c}(z,t) + \rho^o - \frac{\sigma^2}{2}\partial_{zz}^2 v(z,t), \tag{12}$$

[MF-FPK]

$$\partial_t f(z,t) = \frac{1}{2r}\partial_z\Big(\big(\partial_z v(z,t)\big) f(z,t)\Big) + \frac{\sigma^2}{2}\partial_{zz}^2 f(z,t), \tag{13}$$

[MF-CC]

$$\bar{c}(z,t) = \Big(\int_{\mathbb{R}} (z - z') f(z',t)dz'\Big)^2. \tag{14}$$

The system of equations (12)-(14) consists of: (i) the nonlinear (backward in time) *MF-HJB* equation (7) which describes the HJB equation of a generic agent's ergodic optimal problem (3)-(4) with cost coupling $\bar{c}(z,\cdot)$, (ii) the nonlinear (forward in time) *MF-FPK* equation (10) which describes the evolution of the population density with the optimal control law

$$u^o(\cdot) := -\frac{1}{2r}\partial_z v(z,\cdot), \tag{15}$$

and (iii) the spatially averaged *MF-CC* (Cost-Coupling) (11) which is the aggregate effect of the agents in the infinite population limit.

C. Gaussian Stationary Solution

In the stationary setting, the MF equation system (12)-(14) takes the form:

$$\frac{1}{4r}\big(\partial_z v_\infty(z)\big)^2 - \frac{\sigma^2}{2}\partial_{zz}^2 v_\infty(z) = \bar{c}_\infty(z) - \rho^o, \tag{16}$$

$$\frac{1}{2r}\partial_z\Big(\big(\partial_z v_\infty(z)\big) f_\infty(z)\Big) = -\frac{\sigma^2}{2}\partial_{zz}^2 f_\infty(z), \tag{17}$$

$$\bar{c}_\infty(z) = \Big(\int_{\mathbb{R}} (z - z') f_\infty(z')dz'\Big)^2. \tag{18}$$

Theorem 1: [4] For any arbitrary $\mu \in \mathbb{R}$, there exists the following solution of the stationary equation system (16)-(18):

$$v_\infty(z) = \sqrt{r}(z-\mu)^2, \quad \rho^o = \sigma^2\sqrt{r}, \tag{19}$$

$$f_\infty(z) = \frac{1}{\sqrt{2\pi s^2}}\exp\Big(-\frac{(z-\mu)^2}{2s^2}\Big), \quad s^2 := \frac{\sigma^2\sqrt{r}}{2}, \tag{20}$$

$$\bar{c}_\infty(z) = (z-\mu)^2, \tag{21}$$

where $v_\infty(z)$ is defined up to a constant. □

It is important to note that the stationary solution of the system $f_\infty(\cdot)$ is Gaussian even thought the initial states for all the agents are not necessarily assumed to be distributed according to a Gaussian distribution. The asymptotically linearized stability of the MF equation system (12)-(14) around the stationary solution (19)-(21) determines the solution of equation (12) uniquely as

$$v(z,t) = v_\infty(z) = \sqrt{r}(z-\mu)^2, \qquad t \geq 0, \quad (22)$$

where

$$\mu := \int_{\mathbb{R}} z f_0(z) dz, \quad (23)$$

is the initial state population mean (see [4]).

D. Mean-Consensus

Definition 2: [2] *Mean-consensus* is said to be achieved asymptotically for a group of N agents if $\lim_{t\to\infty} |Ez_i(t) - Ez_j(t)| = 0$ for any i and j, $1 \leq i \neq j \leq N$. □

Using (22) for a finite N population system (1)-(2) yields the set of control laws

$$u_i^o(\cdot) \equiv -\frac{1}{2r}\partial_z v(z,\cdot)\big|_{z=z_i} = \frac{-1}{\sqrt{r}}(z_i(\cdot)-\mu),\ 1 \leq i \leq N, \quad (24)$$

where μ is given in (23).

Applying the MF control laws (24) to the agents' dynamics (1) yields ($1 \leq i \leq N$)

$$dz_i^o(t) = \frac{-1}{\sqrt{r}}\big(z_i^o(t) - \mu\big)dt + \sigma dw_i(t), \qquad t \geq 0. \quad (25)$$

The processes (25) have the solutions:

$$z_i^o(t) = \mu + e^{\frac{-t}{\sqrt{r}}}\big(z_i(0)-\mu\big) + \sigma \int_0^t e^{-\frac{(t-\tau)}{\sqrt{r}}} dw_i(\tau),$$

for $t \geq 0$ and $1 \leq i \leq N < \infty$. We have the following theorem which is the analogous version of Theorem 3 in [2].

Theorem 3: [4] By applying the continuum based MF control laws (24) in a finite population DGCM (1)-(2), a mean-consensus is reached asymptotically (as time goes to infinity) with individual asymptotic variance $\sigma^2\sqrt{r}/2$. □

E. ε-Nash Equilibrium Property

Now we present the ε_N-Nash equilibrium property of the continuum based MF control laws (24) for a finite N population system (1)-(2) where ε_N goes to zero asymptotically (as the population size N approaches infinity).

Theorem 4: [4] The set of MF control laws $\{u_i^o \in \mathscr{U}_i : 1 \leq i \leq N\}$ in (24) generates an a.s. $O(\varepsilon_N)$-Nash equilibrium, i.e., for any fixed i, $1 \leq i \leq N$, we have

$$J_i^N(u_i^o, u_{-i}^o) - O(\varepsilon_N) \leq \inf_{u_i \in \mathscr{U}_i} J_i^N(u_i, u_{-i}^o) \leq J_i^N(u_i^o, u_{-i}^o), \quad a.s.$$

where $u_{-i}^o := (u_1^o, \cdots, u_{i-1}^o, u_{i+1}^o, \cdots, u_N^o)$. □

IV. The Evolution Mean Field Equation System

The relative value function $v(z,\cdot)$ defined in (6) represents perturbations around the steady-state optimal cost rate. It turns out that the corresponding HJB equation (12) in the MF system of equations has a larger class of stable perturbed solutions in forward time than in backward time (see [8]).

In this section we introduce an Evolution (i.e., forward in time) Mean Field (EMF) equation system (based on [8]) to exhibit a forward in time process which asymptotically (as time goes to infinity) converges to the stationary equilibrium solution (19)-(21) (where μ is given in (23)) from any given initial guess in a infinitesimal neighborhood of this equilibrium.

The EMF equation system is given by

$$\partial_t v(z,t) = \frac{-1}{4r}\big(\partial_z v(z,t)\big)^2 + \bar{c}(z,t) - \rho^o + \frac{\sigma^2}{2}\partial_{zz}^2 v(z,t), \quad (26)$$

$$\partial_t f(z,t) = \frac{1}{2r}\partial_z\Big(\big(\partial_z v(z,t)\big)f(z,t)\Big) + \frac{\sigma^2}{2}\partial_{zz}^2 f(z,t), \quad (27)$$

$$\bar{c}(z,t) = \Big(\int_{\mathbb{R}}(z-z')f(z',t)dz'\Big)^2, \quad (28)$$

for $t \geq 0$, where $v(z,0)$ and $f(z,0)$ are the given initial guess of the stationary equilibrium.

In the EMF equation system (26)-(28) the equations (27)-(28) are the same as (13)-(14) but the backward in time MF-HJB equation (12) is replaced by a forward in time equation (26). It is important to note that the stationary solution of the EMF equation system (26)-(28) is the same as that of the MF equation system (12)-(14) (see Theorem 1).

A. Stability Analysis

By taking the approach of [8]) we study the small perturbation stability of the EMF equation system (26)-(28) around the stationary equilibrium solution (19)-(21) (where μ is given in (23)). In the nonlinear EMF equation system (26)-(28) we let the perturbation of the solution be

$$v_\varepsilon(z,t) = v_\infty(z) + \varepsilon\ \tilde{v}(z,t), \quad (29)$$

$$f_\varepsilon(z,t) = f_\infty(z)\big(1 + \varepsilon\ \tilde{f}(z,t)\big), \quad (30)$$

$$\bar{c}_\varepsilon(z,t) = \bar{c}_\infty(z) + \varepsilon\ \tilde{c}(z,t), \quad (31)$$

for $z \in \mathbb{R}$ and $t \geq 0$, where v_∞, f_∞ and $\bar{c}_\infty$ are defined in (19)-(21), and $\tilde{f}(z,0)$ and $\tilde{v}(z,0)$ are given and represent the perturbations on $f_\infty(z)$ and $v_\infty(z)$.

Remark 5: The reason why we take the relative perturbation form of the density function f in (30) is to employ the Hermite series expansion for the resulting linearized equation system (see below).

By (20) and since f is a probability density, (30) implies that

$$\int_{\mathbb{R}} \tilde{f}(z,t) f_\infty(z) dz = 0, \qquad t \geq 0. \quad (32)$$

On the other hand, we have $\mu \equiv \int_{\mathbb{R}} z f_0(z)dz = \int_{\mathbb{R}} z f_\infty(z)dz$ (by (23) and (20)) and therefore (30) yields

$$\int_{\mathbb{R}} z \tilde{f}(z,0) f_\infty(z) dz = 0. \quad (33)$$

Proposition 6: The linearization of the EMF equation system (26)-(28) around the stationary equilibrium solution (19)-(21) takes the form

$$\partial_t \tilde{v}(z,t) = \frac{-(z-\mu)}{\sqrt{r}}\partial_z \tilde{v}(z,t) + \frac{\sigma^2}{2}\partial_{zz}^2 \tilde{v}(z,t) + \tilde{c}(z,t), \quad (34)$$

$$\partial_t \tilde{f}(z,t) = \frac{-(z-\mu)}{\sqrt{r}}\partial_z \tilde{f}(z,t) + \frac{\sigma^2}{2}\partial_{zz}^2 \tilde{f}(z,t) - \frac{1}{\sigma^2 r}\Big(\frac{(z-\mu)}{\sqrt{r}}\partial_z \tilde{v}(z,t) - \frac{\sigma^2}{2}\partial_{zz}^2 \tilde{v}(z,t)\Big), \quad (35)$$

$$\tilde{c}(z,t) = -2(z-\mu)\Big(\int_{\mathbb{R}} z \tilde{f}(z,t) f_\infty(z) dz\Big), \quad (36)$$

where $\tilde{f}(z,0)$ and $\tilde{v}(z,0)$ are given.

Proof. See the appendix. □

For the analysis of the linearized equation system (34)-(36) we introduce the *Hermite polynomials* associated to the Hilbert space $L^2(\mathbb{R}, f_\infty(z)dz)$. In this space we have the inner product $(g,h) := \int_{\mathbb{R}} g(z)h(z)f_\infty(z)dz$, and the norm is given by $\|g\|_{L^2} := (g,g)^{1/2}$.

Definition 7: ([12]) We define the n^{th} Hermite polynomial, $n \in \mathbb{N}_0$, of the space $L^2(\mathbb{R}, f_\infty(z)dz)$ by

$$H_n(z) := (-1)^n s^{2n} \exp\Big(\frac{(z-\mu)^2}{2s^2}\Big)\frac{d^n}{dz^n}\exp\Big(\frac{-(z-\mu)^2}{2s^2}\Big),$$

where μ and s^2 are defined in (23) and Theorem 1. □

Lemma 8: ([4] after [8]) We have the following:

(a) The set of Hermite polynomials $\{H_n : n \in \mathbb{N}_0\}$ forms an orthogonal basis of the Hilbert space $L^2(\mathbb{R}, f_\infty(z)dz)$ such that

$$\big(H_m, H_n\big) = s^{2n} n!\ \delta(n,m), \tag{37}$$

where δ is the *Kronecker delta* function.

(b) The Hermite polynomials H_n are eigenfunctions of the operator

$$\mathscr{L}g(z) := \frac{1}{\sqrt{r}}(z-\mu)\partial_z g(z) - \frac{\sigma^2}{2}\partial^2_{zz} g(z), \tag{38}$$

such that $\mathscr{L}H_n = (1/\sqrt{r})nH_n$ for any $n \in \mathbb{N}_0$. □

By using the operator $\mathscr{L}$ defined in (38) we can rewrite the equation system (34)-(36) as

$$\partial_t \tilde{v}(z,t) = -\mathscr{L}\tilde{v}(z,t) + \tilde{c}(z,t), \tag{39}$$

$$\partial_t \tilde{f}(z,t) = -\frac{1}{\sigma^2 r}\mathscr{L}\tilde{v}(z,t) - \mathscr{L}\tilde{f}(z,t), \tag{40}$$

$$\tilde{c}(z,t) = -2(z-\mu)\Big(\int_{\mathbb{R}} z\tilde{f}(z,t)f_\infty(z)dz\Big), \tag{41}$$

where $\tilde{f}(z,0)$ and $\tilde{v}(z,0)$ are given.

Definition 9: [4] A stationary solution (v_∞, f_∞) of the nonlinear equation system (12)-(14) (where μ is given in (23)) is linearly asymptotically stable if the solution $\tilde{f}$ of the linear equation system (34)-(36) with initial perturbation $\tilde{f}(z,0) \in L^2(f_\infty(z)dz)$ exists in $L^2(\mathbb{R}, f_\infty(z)dz)$ and $\lim_{t\to\infty}\|\tilde{f}(z,t)\|_{L^2} = 0$. □

Let $\tilde{f}(z,0) \equiv \sum_{n=0}^{\infty} k_n(0)H_n(z)$ and $\tilde{v}(z,0) \equiv \sum_{n=0}^{\infty} l_n(0)H_n(z)$ then since v and hence $\tilde{v}$ in (29) are defined up to a constant we choose $l_0(0) = 0$. On the other hand, (32) and (33) respectively yield

$$\int_{\mathbb{R}} \tilde{f}(z,0)f_\infty(z)dz = (H_0, \tilde{f}(z,0)) = k_0(0) = 0,$$

and

$$\int_{\mathbb{R}} z\tilde{f}(z,0)f_\infty(z)dz = (\mu H_0 + H_1, \tilde{f}(z,0)) = k_1(0) = 0.$$

We enunciate the following assumption:

(**A2**) Assume that the initial perturbations $\tilde{f}(z,0)$ and $\tilde{v}(z,0)$ of the stationary solutions $f_\infty(z)$ and $v_\infty(z)$ are in the space $L^2(f_\infty(z)dz)$ and are such that

$$\tilde{f}(z,0) = \sum_{n=2}^{\infty} k_n(0)H_n(z), \qquad \tilde{v}(z,0) = \sum_{n=1}^{\infty} l_n(0)H_n(z),$$

for $z \in \mathbb{R}$. □

Theorem 10: Assume (**A1**) and (**A2**) hold. Then, we have the following:

(a) (Existence and uniqueness) There exists a well-defined unique, bounded and C^∞ (i.e., all of its partial derivatives exist) solution to the equation system (34)-(36) in the space $L^2(\mathbb{R}, f_\infty(z)dz)$. The solution is

$$\tilde{v}(z,t) = \frac{l_1(0)}{2}\Big(1 + \exp\big(\frac{-2t}{\sqrt{r}}\big)\Big)H_1(z) + \sum_{n=2}^{\infty} l_n(0)\exp\big(\frac{-nt}{\sqrt{r}}\big)H_n(z), \tag{42}$$

$$\tilde{f}(z,t) = \frac{-l_1(0)}{4\sqrt{r}s^2}\Big(1 - \exp\big(\frac{-2t}{\sqrt{r}}\big)\Big)H_1(z) + \sum_{n=2}^{\infty}\Big(k_n(0) - \frac{nt}{\sigma^2 r\sqrt{r}} l_n(0)\Big)\exp\big(\frac{-nt}{\sqrt{r}}\big)H_n(z), \tag{43}$$

$$\tilde{c}(z,t) = \frac{-l_1(0)}{4\sqrt{r}}\Big(1 - \exp\big(\frac{-2t}{\sqrt{r}}\big)\Big), \tag{44}$$

for $t \geq 0$ and $z \in \mathbb{R}$.

(b) (Asymptotic linearized stability) Under the unique, bounded and C^∞ solution (42)-(44), the stationary equilibrium solution $(v_\infty, f_\infty, \bar{c}_\infty)$ (where μ is given in (23)) of the EMF equation system (26)-(28) is linearly asymptotically stable if $l_1(0) = 0$.

Proof. See the appendix. □

Remark 11: Since $l_1(0) = \int_{\mathbb{R}} z\tilde{v}(z,0)f_\infty(z)dz$, the interpretation of the assumption $l_1(0) = 0$ in the above theorem is that the initial perturbation $\tilde{v}(z,0)$ is an even function in the space $L^2(\mathbb{R}, f_\infty(z)dz)$ (the same evenness assumption appears in Guéant's model [8]). Moreover, this assumption yields $\tilde{c}(z,t) = 0$ in (44). In other words, by the evenness assumption the initial perturbations ($\tilde{f}(z,0)$ and $\tilde{v}(z,0)$) have no effect on the cost perturbation $\tilde{c}(z,t)$.

The ε perturbed asymptotically stable solution of the equation (26), $v(z,\cdot) = v_\infty(z) + \varepsilon\tilde{v}(z,\cdot)$ where $\tilde{v}(z,\cdot)$ is defined in (42) with $l_1(0) = 0$, yields the control law:

$$u^o(z,\cdot) = -\frac{1}{\sqrt{r}}(z-\mu) + \varepsilon\sum_{n=2}^{\infty} n l_n(0)\exp\big(\frac{-nt}{\sqrt{r}}\big)H_{n-1}(z),$$

by (15) where we use the fact that $\partial_z H_n(z) = nH_{n-1}(z)$. Hence, the resulting control law of the (forward in time) EMF equation system (26)-(28): (i) gives the same asymptotic steady-state performance as (24) which is derived from the (backward/forward) MF equation system (12)-(14), and (ii) has a larger class of stable perturbed solutions than the control law (24) in the transient state.

APPENDIX

Proof of Proposition 6: For (34) and (35) we follow the approach of Proposition 8 in [8]. By substituting (29) and (31) into the equation (26) we get

$$\varepsilon\partial_t\tilde{v}(z,t) = \frac{-1}{4r}\big(\partial_z v_\infty(z)\big)^2 - \frac{\varepsilon}{\sqrt{r}}(z-\mu)\partial_z\tilde{v}(z,t) + \bar{c}_\infty(z) + \varepsilon\tilde{c}(z,t) - \rho^o + \frac{\sigma^2}{2}\partial^2_{zz}v_\infty(z) + \varepsilon\frac{\sigma^2}{2}\partial^2_{zz}\tilde{v}(z,t) + O(\varepsilon^2),$$

where we use $\partial_z v_\infty(z) = 2\sqrt{r}(z-\mu)$ by (19). Since v_∞, $\bar{c}_\infty$ and ρ^o are the solutions of the stationary equation (16), the terms of first order in ε in the above equation yield (34).

By substituting (29) and (30) into the MF-FPK equation (26) we get

$$\begin{aligned}\varepsilon\ \partial_t \tilde{f}(z,t) f_\infty(z) &= \frac{1}{2r}\partial_z\Big(\big(\partial_z v_\infty(z)\big)f_\infty(z)\Big) + \\ &+ \varepsilon\frac{1}{2r}\partial_z\Big(\big(\partial_z v_\infty(z)\big)\big(\tilde{f}(z,t)f_\infty(z)\big) + \big(\partial_z \tilde{v}(z,t)\big)f_\infty(z)\Big) \\ &+ \frac{\sigma^2}{2}\partial^2_{zz} f_\infty(z) + \varepsilon\frac{\sigma^2}{2}\partial^2_{zz}\big(\tilde{f}(z,t)f_\infty(z)\big) + O(\varepsilon^2). \qquad \text{(A.1)}\end{aligned}$$

By (17) and the properties of the Gaussian form of f_∞:

$$\partial_z f_\infty(z) = \frac{-(z-\mu)}{s^2} f_\infty(z),\ \partial^2_{zz} f_\infty(z) = \big(\frac{(z-\mu)^2}{s^4} - \frac{1}{s^2}\big) f_\infty(z),$$

where $s^2 = \sigma^2\sqrt{r}/2$ is defined in (20), the terms of first order in ε in the perturbed equation (A.1) yield (35).

Finally, by substituting (30) into equation (14) we get

$$\begin{aligned}\bar{c}(z,t) &= \Big(\int_{\mathbb{R}}(z-z')f(z',t)dz'\Big)^2 \\ &= \Big((z-\mu) + \varepsilon\int_{\mathbb{R}}(z-z')\tilde{f}(z',t)f_\infty(z')dz'\Big)^2 = (z-\mu)^2 \\ &\quad + 2\varepsilon(z-\mu)\Big(\int_{\mathbb{R}}(z-z')\tilde{f}(z',t)f_\infty(z')dz'\Big) + O(\varepsilon^2) \\ &= \bar{c}_\infty(z) + \varepsilon\ \tilde{c}(z,t) + O(\varepsilon^2).\end{aligned}$$

This together with (32) results in (36). □

Proof of Theorem 10: (a) Note that at any time $t \geq 0$ the linearized cost $\tilde{c}(z,t) \in \text{span}\big(H_1(z)\big)$ (see (A.2) below). Therefore, for $n \geq 2$ we can write the equation system (39)-(40) in the Hermite coordinates as the ODE

$$\frac{\partial}{\partial t}\begin{pmatrix} l_n(t) \\ k_n(t)\end{pmatrix} = \begin{pmatrix} -\frac{n}{\sqrt{r}} & 0 \\ -\frac{n}{\sigma^2 r\sqrt{r}} & -\frac{n}{\sqrt{r}}\end{pmatrix}\begin{pmatrix} l_n(t) \\ k_n(t)\end{pmatrix}, \quad t \geq 0,$$

by Lemma 8-(b), where $l_n(0)$ and $k_n(0)$ are given. The unique and bounded solution of the above equation is

$$l_n(t) = l_n(0)\exp\big(\frac{-nt}{\sqrt{r}}\big), \quad t \geq 0, \qquad n \geq 2,$$
$$k_n(t) = \Big(k_n(0) - \frac{nt}{\sigma^2 r\sqrt{r}} l_n(0)\Big)\exp\big(\frac{-nt}{\sqrt{r}}\big), \quad t \geq 0, \quad n \geq 2,$$

On the other hand, (41) yields

$$\tilde{c}(z,\cdot) = -2H_1(z)\big(z, \sum_{n=1}^{\infty} k_n(\cdot)H_n(z)\big) = -2s^2 k_1(\cdot)H_1(z), \qquad \text{(A.2)}$$

by (37) and since $z = \mu H_0(z) + H_1(z)$. Then the Hermite coordinates of the equation system (39)-(41) for $n=1$ satisfy the ODE (by Lemma 8-(b))

$$\frac{\partial}{\partial t}\begin{pmatrix} l_1(t) \\ k_1(t)\end{pmatrix} = \begin{pmatrix} -\frac{1}{\sqrt{r}} & -2s^2 \\ -\frac{1}{2s^2 r} & -\frac{1}{\sqrt{r}}\end{pmatrix}\begin{pmatrix} l_1(t) \\ k_1(t)\end{pmatrix}, \qquad \text{(A.3)}$$

for $t \geq 0$ where $l_1(0)$ and $k_1(0) = 0$ are given and $s^2 = \sigma^2\sqrt{r}/2$. The matrix

$$A := \begin{pmatrix} -\frac{1}{\sqrt{r}} & -2s^2 \\ -\frac{1}{2s^2 r} & -\frac{1}{\sqrt{r}}\end{pmatrix},$$

has one zero and one negative eigenvalue, and can be written in the Jordan normal form $A = PJP^{-1}$ with

$$J := \begin{pmatrix} 0 & 0 \\ 0 & \frac{-2}{\sqrt{r}}\end{pmatrix}, \quad P := \begin{pmatrix} -2\sqrt{r}s^2 & 2\sqrt{r}s^2 \\ 1 & 1\end{pmatrix}.$$

Therefore, the solution of (A.3) is

$$\begin{pmatrix} l_1(t) \\ k_1(t)\end{pmatrix} = \begin{pmatrix} \frac{l_1(0)}{2}\Big(1 + \exp\big(\frac{-2t}{\sqrt{r}}\big)\Big) \\ \frac{-l_1(0)}{4\sqrt{r}s^2}\Big(1 - \exp\big(\frac{-2t}{\sqrt{r}}\big)\Big)\end{pmatrix}, \qquad t \geq 0.$$

It can be shown that for any fixed $t > 0$, $(k_n(t))_n$ and $(l_n(t))_n$ are in the space l^1 (i.e., the space of sequences whose series is absolutely convergent). Moreover, $\tilde{f}(z,\cdot) = \sum_{n=2}^{\infty} k_n(\cdot)H_n(z)$ and $\tilde{v}(z,\cdot) = \sum_{n=1}^{\infty} l_n(\cdot)H_n(z)$ are well-defined, bounded, C^∞ functions in the Hilbert space $L^2(\mathbb{R}, f_\infty(z)dz)$, and satisfy the equation system (34)-(36). This completes the proof of Part (a).

(b) By part (a) we get

$$\lim_{t\to\infty} k_1(t) = -l_1(0)/(4\sqrt{r}s^2), \quad \lim_{t\to\infty} k_n(t) = 0, \quad n \geq 2.$$

Now by the Lebesgue Dominated Convergence theorem and (37), (43) yields $\lim_{t\to\infty}\|\tilde{f}(z,t)\|_{L^2} = l_1(0)/(4\sqrt{r}s)$, which is zero if $l_1(0) = 0$. Hence, based on Definition 9 the stationary equilibrium solution $(v_\infty, f_\infty, \bar{c}_\infty)$ (where μ is given in (23)) is linearly asymptotically stable if $l_1(0)$ is equal to zero. □

REFERENCES

[1] W. Ren, R. W. Beard, and E. M. Atkins, "A survey of consensus problems in multi-agent coordination," in *Proc. ACC*, Portland, OR, Jun. 2005, pp. 1859–1864.

[2] M. Nourian, P. E. Caines, R. P. Malhamé, and M. Huang, "A solution to the consensus problem via stochastic mean field control," in *2nd IFAC NecSys Workshop*, Annecy, France, Sep. 2010 (http://www.cim.mcgill.ca/~arman/mnourian/necsys10.pdf).

[3] ——, "Nash, social and centralized solutions to consensus problems via mean field control theory," under review in *IEEE Trans. Autom. Cont.*, Feb. 2011 (http://www.cim.mcgill.ca/~arman/mnourian/mfcon11.pdf).

[4] M. Nourian, P. E. Caines, and R. P. Malhamé, "A game theoretic synthesis of consensus behaviour: A continuum approach," Technical Report, McGill University, March, 2011 (A preliminary version appeared in the *50th IEEE CDC/ECC*, Orlando, FL., Dec. 2011) http://www.cim.mcgill.ca/~arman/mnourian/cont20march.pdf.

[5] P. E. Caines, "Bode lecture: mean field stochastic control," *48th IEEE CDC, Shanghai, China*, Dec., 2009. [Online]. Available: http://www.ieeecss-oll.org/video/mean-field-stochastic-control

[6] M. Huang, P. E. Caines, and R. P. Malhamé, "Large-population cost-coupled LQG problems with nonuniform agents: Individual-mass behavior and decentralized ε-Nash equilibria," *IEEE Trans. Autom. Cont.*, vol. 52, no. 9, pp. 1560–1571, 2007.

[7] H. Yin, P. G. Mehta, S. P. Meyn, and U. V. Shanbhag, "Synchronization of coupled oscillators is a game," *conditionally accepted in IEEE Trans. Autom. Cont., Dec. 2010.*

[8] O. Guéant, "A reference case for mean field games models," *J. Math. Pures & App.*, vol. 92, no. 3, pp. 276–294, 2009.

[9] Y. Achdou and I. Capuzzo-Dolcetta, "Mean field games: Numerical methods," *SIAM J. Numeric. Analy.*, vol. 48, no. 3, pp. 1136–1162, 2010.

[10] J.-M. Lasry and P.-L. Lions, "Mean field games," *Jap. J. Math.*, vol. 2, no. 1, pp. 229–260, March 2007.

[11] V. S. Borkar, "Ergodic control of diffusion processes," *Proc. Int. Congress of Math.*, pp. 1299–1310, 2006.

[12] M. Abramowitz and I. A. Stegun, *Handbook of Mathematical Functions with Formulas, Graphs, and Mathematical Tables*. New York: Dover, 1964.