

STABILITY OF INTERCONNECTED THERMODYNAMIC SYSTEMS

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Abstract—In this paper, a formulation of thermodynamic systems in terms of contact geometry is proposed and a systematic approach to the description of interconnected thermodynamic systems is developed. The paper presents a number of results on the stability of interconnected thermodynamic systems.

I. INTRODUCTION

Since the early seventies, there has been extensive research in geometric foundations of thermodynamic systems initiated by the pioneering work of Hermann [1]. Particular attention has been paid to the dynamical description of the evolution of thermodynamic systems (see, e.g., [2], [3], [4], [5]). However, to the best of our knowledge, there has not been any systematic investigation of the properties of interconnected thermodynamic systems. Certain results on interconnected systems have been presented in [6], [7], albeit without analysis of the interconnection structure.

In our previous paper [8], a systematic approach to the description of interconnected thermodynamic systems was developed. The key feature of this description is the notion of interconnection constraints which constrain its evolution and the state space. However, there are always degrees of freedom characterizing the evolution of the composite system. These degrees of freedom correspond to those controls (forces) which are not specified through the interconnection constraints.

Our goal in this paper is to specify a minimal set of conditions on the thermodynamic controls such that the resulting evolution of a composite system conforms with physical observations. In particular, we are interested in the stability analysis of interconnected thermodynamic systems.

This paper is organized as follows: in Section II, the basic facts from equilibrium thermodynamics are presented. We review contact geometry and its relation to equilibrium thermodynamics and define the evolutionary equations of a controlled system on the equilibrium manifold. Section III is devoted to the modelling of an interconnected thermodynamic system. Finally, Section III shows the stability analysis of two particular interconnected structures.

II. THERMODYNAMIC ESSENTIALS

A. Properties of a Physical Thermodynamic System

A (physical) thermodynamic system is a physical substance separated from its environment which interacts with that environment through energy and material exchange.

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In the following, we consider a special class of thermodynamic systems satisfying the following physical assumptions:

- P1.** The system consists of a single phase, single component substance.
- P2.** The system (substance) does not undergo any chemical transformations [9].

In an equilibrium state, a thermodynamic system can be completely described by the *internal energy* U , the *entropy* S , the *volume* V , and the *mole number* N . All these are *extensive* parameters, i.e. they satisfy the *additivity property*. An extensive parameter X characterizing a composite system is equal to the sum of parameters X_i characterizing the respective subsystems.

We restrict attention to the special class of thermodynamic processes occurring in thermodynamic systems which are called *quasi-static (QS) processes* and defined as a sequence of equilibrium states. This represents an idealization which never occurs in practice. However, we assume that

- P3.** The rate of change of the thermodynamic parameters of the environment is sufficiently small that there are no gradients in intensive variables within the system.

P1, **P2** and **P3** lead us to the formal description of a thermodynamic system based on the internal energy function

$$U = U(S, V, N). \quad (1)$$

Definition 1: An *equilibrium state* is defined as an element of the graph $\mathcal{E} = (S, V, N, U(S, V, N))$ of the energy function U , and the locus of all equilibrium states is referred to as the *equilibrium energy manifold*.

As a result, a quasi-static process can be defined as a map $\psi : \mathbb{T} \rightarrow \mathcal{E}$, where $T = [t_0, t_f] \subset \mathbb{R}_{\geq 0}$. In the following we will identify \mathbb{T} with the positive semi-axis: $\mathbb{T} = \mathbb{R}_{\geq 0}$.

The behaviour of a homogeneous *closed thermodynamic system* (no material exchange, i.e., $N = \text{const}$) is governed by the *first* and the *second law of thermodynamics*. The first law states that the change of energy in a quasi-static process can be written as

$$dU = W + Q, \quad (2)$$

where W and Q are the differential forms describing the work done on the system, and the amount of heat transferred to the system. W and Q cannot be written as exact differentials. However, for the case of the work associated with a change in volume, it follows from first principles that $W = -pdV$, where p is the *internal pressure*. The second term is addressed by the second law. It says that there exists an integrating factor $\frac{1}{T}$ such that $\frac{Q}{T} = dS$, where dS is

an exact differential. T is referred to as the *temperature*. Therefore, (2) can be rewritten in differential form as

$$dU = TdS - pdV. \quad (3)$$

Furthermore, the set of all meaningful thermodynamic processes is characterized by the following requirement:

P4. In a closed system, any trajectory $\gamma_{12} \in \Gamma$ satisfies the following inequality (known as the *Clausius inequality*):

$$\int_{\gamma_{12}} \frac{Q}{T} \leq \int_{\gamma_{12}} dS, \quad (4)$$

which implies $\int_{\gamma_{12}} dS \geq 0$ for each *adiabatic* trajectory, that is to say for each trajectory not accompanied by heat transfer between the system and its environment (i.e., $Q = 0$).

By definition, a *reversible* trajectory $\bar{\gamma}_{12} \in \bar{\Gamma} \subset \Gamma$ is one for which $\int_{\bar{\gamma}_{34}} dS = 0$ for any $\bar{\gamma}_{34} \subset \bar{\gamma}_{12}$. Thus, for any reversible trajectory $\bar{\gamma}_{12}$: $\int_{\bar{\gamma}_{34}} \frac{Q}{T} = 0$ for any $\bar{\gamma}_{34} \subset \bar{\gamma}_{12}$.

One consequence of the Clausius inequality is that the total entropy of an adiabatic system strictly increases in an irreversible process and remains constant in a reversible one (see [10] for a detailed discussion).

In the region of state space where no phase changes occur, the function $U(\cdot)$ is assumed to be sufficiently smooth that all required partial derivatives exist. Therefore, generalizing (3) we can write the differential of U as

$$dU = \frac{\partial U}{\partial S}(S, V, N)dS + \frac{\partial U}{\partial V}(S, V, N)dV + \frac{\partial U}{\partial N}(S, V, N)dN, \quad (5)$$

where the partial derivatives $\frac{\partial U}{\partial S} = T$, $\frac{\partial U}{\partial V} = -p$, and $\frac{\partial U}{\partial N} = \mu$ are the *temperature*, the *internal pressure* (note the minus sign), and the *chemical potential*, respectively. The temperature, the internal pressure and the chemical potential are *intensive parameters*, i.e. they are invariant with respect to the partitioning of the system. Note that all the intensive and extensive variables take on only positive values.

B. Geometry of the Equilibrium Manifold \mathcal{E}

Let \mathcal{M} be a smooth $(2n + 1)$ -dimensional manifold equipped by a special 1-form $\omega \in T^*\mathcal{M}$ which satisfy the condition $\omega \wedge (d\omega)^n \neq 0$. Such 1-forms are called *maximally non-integrable*, or *contact forms* [11]. Correspondingly, the pair (\mathcal{M}, ω) is called the *contact manifold*. An integral manifold of the contact form ω has the least possible dimension which is equal to n for a $(2n + 1)$ -dimensional manifold \mathcal{M} . These submanifolds are called *Legendre (sub)manifolds*.

In the following we will consider a specific contact form:

Definition 2: Let $(x^0, x^1, \dots, x^n, p_1, \dots, p_n)$ be the local coordinates on \mathcal{M} . The *fundamental thermodynamic contact 1-form* is defined as

$$\omega = dx^0 - p_i dx^i, \quad 1 \leq i \leq n. \quad (6)$$

One can check that this 1-form indeed satisfies the maximal non-integrability condition. Note that here and throughout the paper we adopt the Einstein summation convention: the

terms are summed over all indices which appear both in a lower and an upper position.

The following lemma provides a way to characterize Legendre manifolds.

Lemma 3 ([12], [13]): Let $N = \{1, \dots, n\}$ be the set of indices. Given the contact form (6), a disjoint partitioning $I, J \subset N$, $I \cap J = \emptyset$, $I \cup J = N$ with n_I and n_J components, and a smooth function $\zeta(x^I, p_J)$, the following equations define a Legendre manifold on (\mathcal{M}, ω) :

$$p_I = \frac{\partial \zeta}{\partial x^I}, \quad x^J = -\frac{\partial \zeta}{\partial p_J}, \quad x^0 = \zeta - p_J \frac{\partial \zeta}{\partial p_J}. \quad (7)$$

Conversely, every Legendre manifold is defined in a neighbourhood of every point by these formulae for at least one choice of the subset I .

The function ζ is called a *generating function* of the Legendre manifold \mathcal{L} .

It turns out [8] that the equilibrium manifold \mathcal{E} in an $(n + 1)$ -dimensional space can be described as a Legendre manifold on a $(2n + 1)$ -dimensional contact manifold. To show this we consider the Legendre manifold \mathcal{L}_U which is characterized in the following way: let in (7), $n = 3$, $J = \emptyset$, and $\zeta(x^1, \dots, x^3)$ be the state equation $U(S, V, N)$. The variables (x^1, x^2, x^3) are identified with (S, V, N) . Using the formulae given by Lemma 3, we obtain the following description:

$$x^0 = U(x), \quad p_i = \frac{\partial U}{\partial x^i}(x), \quad (8)$$

where the remaining variables x and p are $x^0 = U$, and $(p_1, p_2, p_3) = (T, -p, \mu)$. Thus, the Legendre manifold \mathcal{L}_U of 1-form (6) with generating function (1) can be seen as an immersion of the equilibrium energy manifold \mathcal{E} into the $(2n + 1)$ -dimensional contact manifold \mathcal{M} .

The manifold \mathcal{L}_U will be referred to as the *Legendre equilibrium energy manifold*.

C. Equations of Motion on the Equilibrium Manifold

The following theorem characterizes the generic vector field on the equilibrium energy manifold \mathcal{L}_U .

Theorem 4 ([8]): Let \mathcal{L}_U be the Legendre equilibrium energy manifold given by (7) with the generating function $U = U(x^1, \dots, x^n)$. Then the generic smooth vector field X describing the evolution of the system on the manifold \mathcal{L}_U is

$$X = p_i \Lambda^{[i]} \frac{\partial}{\partial x^0} + \Lambda^{[i]} \frac{\partial}{\partial x^i} + \Lambda^{[i]} \frac{\partial^2 U}{\partial x^i \partial x^j} \frac{\partial}{\partial p_j}, \quad (9)$$

where $\Lambda^{[i]}$ are smooth functions.

The smooth functions $\Lambda^{[i]}$ in (9) are referred to as *thermodynamic forces*.

For X (9), the corresponding differential equations have the following form:

$$\begin{cases} \dot{x}^0 = p_i \Lambda^{[i]} \\ \dot{x}^i = \Lambda^{[i]} \\ \dot{p}_i = \frac{\partial^2 U}{\partial x^i \partial x^j} \Lambda^{[j]}. \end{cases}$$

D. Static Stability of a Thermodynamic System

The property of the static stability should not be confused with the notion of the stability of a dynamical system.

Definition 5: An equilibrium point $\bar{x} = (\bar{S}, \bar{V}, \bar{N}) \in \mathcal{L}_U$ is said to be infinitesimally statically stable if there exists a neighbourhood $N_\epsilon(\bar{x}) \subset \mathcal{M}$, $\epsilon > 0$ such that for any perturbation $\delta x \in N_\epsilon(\bar{x})$, $\delta x \rightarrow \bar{x}$ as $t \rightarrow \infty$.

It is shown [14, Ch. 8] that the equilibrium state x is statically stable if the Hessian D^2U is positively definite at \bar{x} . In particular, for a closed system this is equivalent to the following conditions:

$$\frac{\partial^2 U}{\partial S^2} > 0, \quad \frac{\partial^2 U}{\partial V^2} > 0, \quad \frac{\partial^2 U}{\partial S^2} \frac{\partial^2 U}{\partial V^2} - \left(\frac{\partial^2 U}{\partial S \partial V} \right)^2 > 0. \quad (10)$$

Moreover, it is shown that the static stability property is a necessary condition for the absence of phase transitions. That is to say, assumption **P1** implies (10). In the following, we do not consider in detail the nature of the perturbation δx since assumption **P3** rules out this situation. However, the property of a system to be statically stable has many important consequences which go beyond the local perturbation analysis. In Sec. IV we will see that the above conditions play a role in the stability analysis of an interconnected thermodynamic system.

III. MODELS OF INTERCONNECTED THERMODYNAMIC SYSTEMS

A. Composite Thermodynamic System

Let us consider a structure (a composite system) Σ consisting of k thermodynamic systems Σ_i , $i = 1, \dots, k$ characterized by the state functions $U^i = U^i(S^i, V^i, N^i)$.

Let (\mathcal{M}, ω) be a $2n + 1$ -dimensional contact manifold, where $n = k \cdot n_s$, $n_s = 3$ and ω be the thermodynamic contact 1-form (6). The total energy

$$U_\Sigma(x_\Sigma) = U^1(x^1, \dots, x^p) + \dots + U^k(x^{(n-p+1)}, \dots, x^n) \quad (11)$$

as the generating function, and the thermodynamic variables

$$x_\Sigma = (S^1, V^1, N^1, \dots, S^k, V^k, N^k)$$

define the composite thermodynamic system Σ . The corresponding Legendre manifold \mathcal{L}_Σ is described as in (7) and the generic dynamics along \mathcal{L}_Σ are defined in the same way as in Sec. II-C.

B. Extended Thermodynamic 1-form

If a composite system contains movable elements, its energy function U_Σ must be augmented by a kinetic energy term:

$$U_{total} = U_\Sigma + U_k = U_\Sigma + \sum_{i=1}^{n_k} \frac{(\rho^i)^2}{2m_i}, \quad (12)$$

where ρ^i is the linear moment of the i -th movable element, and m_i - its mass. The velocity σ of the movable element is a conjugated variable with respect to the moment ρ : $\sigma = \frac{dU_k}{d\rho}$. For a rotating body the same result can be obtained

by substituting ρ and σ with the angular moment l and the angular speed ϕ .

In the same way, the thermodynamic contact 1-form must be augmented in the following way:

$$\omega_t = \omega - p_{n+j} dx^{n+j}, \quad 1 \leq j \leq n_k,$$

where $p_{n+j} = \sigma_j$, and $x^{n+j} = \rho^j$.

C. Geometric Interconnection Constraints

In a real system there are always heat, work, or matter flows between the subsystems. These flows, together with the constructive and geometrical restrictions form the interconnection structure of the composite system.

The interconnection structure can be described by a set of geometric constraints imposed on the system. Now we will state this formally.

Definition 6: The m equations

$$\gamma_j : \mathcal{M} \rightarrow \mathbb{R}, \quad 1 \leq j \leq m, \quad m < n$$

such that $\text{rank} \left(\frac{\partial \gamma}{\partial (x,p)} \right) = m$ are called the *geometric interconnection constraints*. The submanifold $\Gamma = \{q \in \mathcal{M} | \gamma(q) = 0\}$ is called the *constraint manifold*.

The dynamics of the constrained thermodynamic system has to be confined to the intersection $\tilde{\mathcal{L}}_\Sigma = \mathcal{L}_\Sigma \cap \Gamma$. We note that both \mathcal{L}_Σ and Γ can be seen as integral manifolds of corresponding exterior systems: $I_\Sigma = \{d\phi_i\}_{i=0, \dots, n}$ with $\phi_0 = x^0 - U_\Sigma(x)$, $\phi_i = p_i - \frac{\partial U_\Sigma}{\partial x^i}(x)$ generates \mathcal{L}_Σ , and $I_\Gamma = \{d\gamma_j\}_{j=1, \dots, m}$ generates Γ .

Below, we describe the most common types of constraints. To put the problem into physical context we shall specifically use thermodynamic notation instead of differential geometric one.

All geometric interconnection constraints can be divided in two groups: global and local ones.

1) *Global Constraints:* Global constraints are imposed on the whole composite system. In the following, we will assume the overall system to be isolated from its environment. In this case the sum of all extensive variables except entropy is constant, i.e., $\sum_{i=1}^k V^i = V_\Sigma$, $\sum_{i=1}^k N^i = N_\Sigma$.

In an isolated system there is no net work or heat transfer. Therefore, according to the first law, the total energy of an isolated system remains constant: $U_{total} = \text{const}$.

Note that the requirement on the system to be isolated is not overly restrictive. If there is any transfer between the system and its environment, one can extend the composite system by including the environment as a new subsystem.

2) *Local Constraints:* Local constraints characterize a group of subsystems whereas the group may consist of 1 subsystem. Consider the following several examples.

- 1) The i -th and the j -th subsystem are connected through a matter-conducting membrane and isolated from the remaining subsystems: $N^i + N^j = \text{const}$.
- 2) The i -th subsystem undergoes a reversible transformation: $S^i = \text{const}$. One particular example of a reversible process is a quasi-static compression of a gas in a cylinder. We recall that any process accompanied by heat transfer is irreversible.

D. Kinematic Constraints

The second class of constraints consists of kinematic constraints, i.e., the constraints on both the thermodynamic variables and their velocities. This class is subdivided into holonomic and nonholonomic constraints [15], [16]. The holonomic constraints may be integrated and expressed as constraints on the thermodynamic variables. Therefore we will consider them as a special case of geometric constraints. The nonholonomic constraints, in turn, do not restrict the possible position of the system on the respective Legendre manifold but only the system's dynamics.

Below we describe one particularly important case of non-holonomic constraints describing the interrelation between the thermal and mechanical components of the total system's energy.

Consider the full differential of the total energy U_{total} (12). It can be written as:

$$dU_{total} = dU_{rev} + dU_{irr},$$

where dU_{rev} is the reversible (mechanical) component of the total energy change and dU_{irr} is the irreversible (thermal) component. These two components contain mechanical or thermal terms: $dU_{rev} = -p_i dV^i + \sigma d\rho$ and $dU_{irr} = T_i dS^i$, where ρ is defined in Subsection III-B. Thus we can introduce two fictitious energy variables: mechanical (reversible) energy U_{rev} and thermal (irreversible) energy U_{irr} such that $U_{total} = U_{rev} + U_{irr}$. In an isolated composite system (i.e., $U_{total} = const$) without dissipation there is no transformation between these two energies. Therefore we can formulate the following two constraints:

$$\begin{aligned} -p_i \dot{V}^i + \sigma \dot{\rho} &= 0, \\ T_i \dot{S}^i &= 0. \end{aligned} \quad (13)$$

For an isolated system with dissipation the constraints (13) take the following form:

$$\begin{aligned} -p_i \dot{V}^i + \sigma \dot{\rho} &= -\phi(\rho), \\ T_i \dot{S}^i &= \phi(\rho), \end{aligned} \quad (14)$$

where $\phi(\rho)$ is the rate of dissipation of mechanical energy as a function of ρ . Note that one of these two equations is redundant since it can be substituted by the geometric constraint $U_{total} = const$.

IV. STABILITY ANALYSIS

In this section a particular case of a two-compartmental composite thermodynamical system under different interconnection constraints is considered. We present the respective dynamics equations and briefly outline the derivation procedure. For the details, the reader is referred to the recent authors' results [8]. In this paper, our main goal is study the dynamic stability of the resulting composite thermodynamic systems.

To keep equations moderately sized we assume that there is no matter transfer within the system. Therefore, all respective energy functions are considered to be the functions of S and V only.

All results in this section are illustrated by the particular thermodynamic system where the subsystems have energy functions of the following form:

$$U^i = (V^i)^{-\frac{1}{c}} \exp\left(\frac{S^i}{cN^i R}\right), \quad i = 1, 2, \quad (15)$$

where $R = 8.314$ is the gas constant and $c = 5/2$ is the dimensionless heat capacity at constant volume. The energy functions (15) correspond to the state equation of a diatomic ideal gas.

Finally, before we proceed with the stability analysis, we remark that the Clausius inequality (see **P4**) can be expressed as the following inequality:

$$\dot{S}_\Sigma = \sum_{i=1}^n \dot{S}^i \geq 0. \quad (16)$$

A. A 2-compartment System with Heat Linkage

Let us first consider a structure shown in Fig. 1. It consists of a closed cylinder containing two thermodynamic systems, Σ_1 and Σ_2 , separated by a fixed partition wall. All walls are assumed to be rigid and impermeable to matter. The cylinder walls are thermally isolated except for the partition wall. Therefore, there is a heat flow between the systems Σ_1 and Σ_2 . Each system Σ_i is characterized by the set of thermodynamic variables (S^i, V^i, T_i, p_i) .

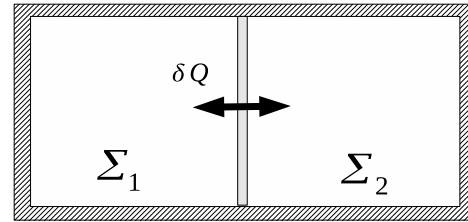


Fig. 1. A composite thermodynamic system. δQ denotes the heat exchange.

Now the interconnection constraints can be formulated. The volumes of the subsystems remain constant and so $V^i = const$, $i = 1, 2$. Since the composite system is isolated from its environment, the overall energy is constant: $U_\Sigma = const$.

The 1-forms $d\gamma_i$ corresponding to the interconnection constraints described above are

$$d\gamma_1 = dV^1, \quad d\gamma_2 = dV^2, \quad d\gamma_3 = dU_\Sigma,$$

whence the following set of algebraic equations is obtained:

$$\Lambda_V^{[1]} = 0, \quad \Lambda_V^{[2]} = 0, \quad T_1 \Lambda_S^{[1]} + T_2 \Lambda_S^{[2]} = 0.$$

Hence, one can express the thermodynamic control $\Lambda_S^{[2]}$ in terms of the thermodynamic control $\Lambda_S^{[1]}$: $\Lambda_S^{[2]} = -\frac{T_1}{T_2} \Lambda_S^{[1]}$. The system of differential equations governing the evolution

of the composite system is:

$$\begin{cases} \dot{S}^1 = \Lambda_S^{[1]} \\ \dot{T}_1 = c_{SS}^1 \Lambda_S^{[1]} \\ \dot{p}_1 = -c_{VS}^1 \Lambda_S^{[1]} \\ \dot{S}^2 = -\frac{T_1}{T_2} \Lambda_S^{[1]} \\ \dot{T}_2 = -c_{SS}^2 \frac{T_1}{T_2} \Lambda_S^{[1]} \\ \dot{p}_2 = c_{VS}^2 \frac{T_1}{T_2} \Lambda_S^{[1]}, \end{cases} \quad (17)$$

where $c_{AB}^i = \frac{\partial^2 U_\Sigma}{\partial A^i \partial B^i}$, $i = 1, k$, $A, B \in \{S, V, N\}$. Note that the equations with zero right-hand sides are not shown.

While the model (17) is very useful for simulation purposes, it turns out to be inadequate for the stability analysis. The reason for this is the fact that the evolution of a thermodynamic system occurs on the restricted state Legendre manifold $\bar{\mathcal{L}}_\Sigma$. Therefore, the system's evolution has to be described by a system of differential-algebraic equations where the algebraic equations come from (8).

Thus, one can rewrite the equations (17) as

$$\begin{cases} \dot{S}^1 = \Lambda_S^{[1]} \\ \dot{S}^2 = -\frac{T_1}{T_2} \Lambda_S^{[1]} \\ T_1 = \frac{\partial U_\Sigma}{\partial S^1}(S^1) \Big|_{V^1=\bar{V}^1}, \quad p_1 = -\frac{\partial U_\Sigma}{\partial V^1}(S^1) \Big|_{V^1=\bar{V}^1}, \\ T_2 = \frac{\partial U_\Sigma}{\partial S^2}(S^2) \Big|_{V^2=\bar{V}^2}, \quad p_2 = -\frac{\partial U_\Sigma}{\partial V^2}(S^2) \Big|_{V^2=\bar{V}^2}, \end{cases} \quad (18)$$

where \bar{V}^1 and \bar{V}^2 are the fixed values of compartment volumes. Furthermore, from the definition of U_Σ it follows that T_1 and p_1 do not depend on S^2 and T_2 and p_2 do not depend on S^1 .

Lemma 7: Let $\Lambda_S^{[1]}$ be a continuous function of $(T_2(S^2) - T_1(S^1))$ such that

- i. $(T_2 - T_1) \cdot \Lambda_S^{[1]}(T_2 - T_1) > 0$,
- ii. $[\Lambda_S^{[1]}(T_2 - T_1) = 0] \Rightarrow [(T_2 - T_1) = 0]$.

Then the system (18) has the one-dimensional asymptotically stable limit set

$$\Psi = \{(S^1, S^2) \in \mathbb{R}_{\geq 0}^2 \mid T_2(S^2) - T_1(S^1) = 0\}.$$

Proof The time derivative of the total entropy \dot{S}_Σ has the following form:

$$\dot{S}_\Sigma = \Lambda_S^{[1]} - \frac{T_1}{T_2} \Lambda_S^{[1]} = \frac{T_2 - T_1}{T_2} \Lambda_S^{[1]} > 0,$$

which is positive in virtue of (i). Therefore the condition of entropy growth (16) holds.

Condition (ii.) guarantees that the equilibrium set of the system (18) is

$$\Psi = \{(S^1, S^2) \in \mathbb{R}_{\geq 0}^2 \mid T_2(S^2) - T_1(S^1) = 0\}.$$

To analyze the stability property of the equilibrium set Ψ we define a LaSalle function $V = \frac{1}{2}(T_2(S^2) - T_1(S^1))^2$. Its time derivative is

$$\begin{aligned} \dot{V} &= (T_2(S^2) - T_1(S^1))(-c_{SS}^2 \frac{T_1}{T_2} \Lambda_S^{[1]} - c_{SS}^1 \Lambda_S^{[1]}) = \\ &= -(T_2 - T_1)(c_{SS}^2 \frac{T_1}{T_2} + c_{SS}^1) \Lambda_S^{[1]}. \end{aligned}$$

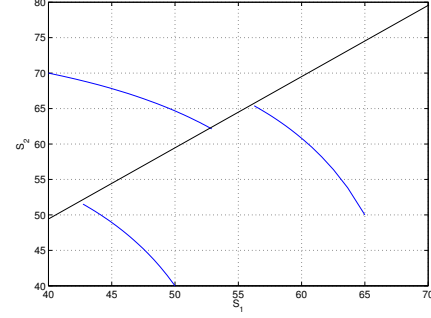


Fig. 2. System's trajectories for different initial conditions and the subset of equilibrium states for an ideal gas system. The parameters are $N^1 = N^2 = 1$, $V^1 = 1$, $V^2 = 3$.

Since the partial derivatives c_{SS}^i , $i = 1, 2$ are strictly positive (10) and due to (i.) we have $\dot{V} < 0$ off of Ψ . ■

The last part of the proof shows an interplay between the static and the dynamic thermodynamic stability. The conditions $c_{SS}^i > 0$ come from the static stability, whereas the stability of the equilibrium set Ψ has been formulated for the dynamical system (18).

Fig. 2 shows the evolution of a particular system for different initial values of (S^1, S^2) with state trajectories asymptotically converging to a one-dimensional subset.

Furthermore, one can see from equations (18) that, as long as the conditions of Lemma 7 hold, the choice of the form of thermodynamic control $\Lambda_S^{[1]}$ does not influence the equilibrium state eventually attained by the system. The equilibrium state is solely determined by the initial values of entropies $(S^1(t_0), S^2(t_0))$.

B. A 2-compartment System with Work Linkage

Now consider the same system as in Sec. IV-A but with a frictionless movable piston of mass m instead of the partition wall. In this case, both the total energy and the thermodynamic contact 1-form must be augmented as described in Sec. III-B. Furthermore, the set of interconnection constraints will change. In particular, the 1-forms $d\gamma_i$ corresponding to the new interconnection constraints are

$$d\gamma_1 = dV^1 + dV^2, \quad d\gamma_2 = dS^1, \quad d\gamma_3 = dS^2, \quad d\gamma_4 = dU_t,$$

and the corresponding algebraic equations are

$$\Lambda_V^{[1]} + \Lambda_V^{[2]} = 0, \quad \Lambda_S^{[i]} = 0, \quad -p_i \Lambda_V^{[i]} + \frac{1}{m} \rho \Lambda_\rho = 0,$$

where Λ_ρ is the thermodynamic control associated with ρ .

Furthermore, there is one nonholonomic constraints which relates ρ to V^1 :

$$\rho = \frac{m}{A} \dot{V}^1 = \frac{m}{A} \Lambda_V^{[1]}, \quad (19)$$

where A is the sectional area of the cylinder. It can be seen that this constraint cannot be integrated to yield any equivalent expression in terms of thermodynamic variables. Therefore, this constraint restricts the dynamics of the system while not restricting the set of admissible thermodynamic variables.

The geometric constraints taken together with kinematic ones allow us to express Λ_ρ :

$$\Lambda_\rho = -(p_2 - p_1)A. \quad (20)$$

Hence, the differential equations describing the dynamics of the composite system:

$$\begin{cases} \dot{T}_1 = c_{SV}^1 \frac{A}{m} \rho \\ \dot{V}^1 = \frac{A}{m} \rho \\ \dot{p}_1 = -c_{VV}^1 \frac{A}{m} \rho \\ \dot{T}_2 = -c_{SV}^2 \frac{A}{m} \rho \\ \dot{V}^2 = -\frac{A}{m} \rho \\ \dot{p}_2 = c_{VV}^2 \frac{A}{m} \rho \\ \dot{\rho} = -A(p_2 - p_1) \\ \dot{\sigma} = -\frac{A}{m}(p_2 - p_1). \end{cases} \quad (21)$$

Using the same approach as in Sec. IV-A, we rewrite (21) as a system of differential-algebraic equations

$$\begin{cases} \dot{V}^1 = \frac{A}{m} \rho \\ \dot{\rho} = -A(p_2(V^2) - p_1(V^1)) \\ p_1 = -\frac{\partial U_\Sigma}{\partial V^1}(V^1)|_{S^1=\bar{S}^1}, \quad T_1 = \frac{\partial U_\Sigma}{\partial S^1}(V^1)|_{S^1=\bar{S}^1}, \\ p_2 = -\frac{\partial U_\Sigma}{\partial V^2}(V^2)|_{S^2=\bar{S}^2}, \quad T_2 = \frac{\partial U_\Sigma}{\partial S^2}(V^2)|_{S^2=\bar{S}^2}, \\ V^2 = V_\Sigma - V^1, \quad \sigma = \frac{1}{m} \rho \end{cases} \quad (22)$$

where \bar{S}^1 and \bar{S}^2 are the fixed values of the respective entropies, and $V_\Sigma = V^1 + V^2$ is the total volume of the cylinder.

Lemma 8: The system (22) has a unique (non-asymptotically) stable equilibrium at the point $(V_{eq}^1, \rho_{eq}) = (V^*, 0)$, where V^* is the solution to the equation $p_2(V_\Sigma - V^*) - p_1(V^*) = 0$.

Proof The solution V^* to the equation $p_2(V_\Sigma - V^*) - p_1(V^*) = 0$ is unique as is shown by the global constant-sign behaviour of the first derivative:

$$\frac{d}{dV^*} (p_2(V_\Sigma - V^*) - p_1(V^*)) = -(c_{VV}^2 + c_{VV}^1) < 0,$$

where the sign of c_{VV}^i , $i=1,2$, is determined according to (10).

Performing the change of variables $v = V^1 - V^*$ one can rewrite (22) as

$$\begin{cases} \dot{v} = \frac{A}{m} \rho \\ \dot{\rho} = -A(\bar{p}_2(v) - \bar{p}_1(v)), \\ \bar{p}_1(v) = p(V^* + v), \quad \bar{p}_2(v) = p(V_\Sigma - V^* - v) \end{cases} \quad (23)$$

with an equilibrium point at the origin.

Let us choose the following test function:

$$G = \frac{\rho^2}{2m} + \int_0^v (\bar{p}_2(x) - \bar{p}_1(x)) dx.$$

Its total time derivative is

$$\frac{d}{dt} G = -\frac{A}{m} \rho (\bar{p}_2(v) - \bar{p}_1(v)) + \frac{A}{m} \rho (\bar{p}_2(v) - \bar{p}_1(v)) = 0.$$

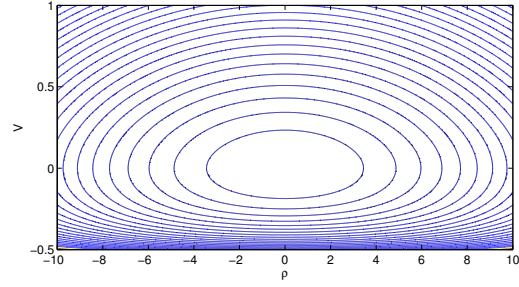


Fig. 3. Level curves of the first integral G of the system 23. The parameters are $N^1 = N^2 = 1$, $S^1 = 100$, $S^2 = 150$, $V_\Sigma = 4$.

Therefore we conclude that $G(\bar{V}, \rho)$ is the Lyapunov function for the system (23). ■

In Fig. 3, a number of level curves of $G(\bar{V}, \rho)$ for a particular thermodynamic system are shown.

V. CONCLUSIONS

We have discussed the modelling of interconnected thermodynamic systems and analyzed the stability of the resulting systems. In the subsequent paper it will be shown that a system of the type above with heat and work linkage exhibits asymptotically stable behavior with respect to a certain (well-defined) attraction set.

REFERENCES

- [1] R. Hermann, *Geometry, Physics and Systems*. M. Dekker, 1973.
- [2] R. Mrugała, J. D. Nulton, J. C. Schön, and P. Salamon, "Contact structure in thermodynamic theory," *Reports on Mathematical Physics*, vol. 29, no. 1, pp. 109–121, 1991.
- [3] H. C. Öttinger and M. Grmela, "Dynamics and thermodynamics of complex fluids. II. Illustrations of a general formalism," *Physical Review E*, vol. 56, no. 6, pp. 6633–6655, 1997.
- [4] D. Eberard, B. M. Maschke, and A. J. van der Schaft, "An extension of hamiltonian systems to the thermodynamic phase space: towards a geometry of nonreversible processes," *Reports on Mathematical Physics*, vol. 60, no. 2, pp. 175–198, 2007.
- [5] A. Favache, V. S. dos Santos Martins, D. Dochain, and B. M. Maschke, "Some properties of conservative port contact systems," *IEEE Transactions on Automatic Control*, vol. 54, no. 10, pp. 2341–2350, 2009.
- [6] H. C. Öttinger, *Beyond Equilibrium Thermodynamics*. Wiley, 2005.
- [7] A. Favache, D. Dochain, and B. Maschke, "A geometric perspective to open irreversible thermodynamic systems: Generic, matrix and port-contact systems," in *Proc. 48th IEEE Conference on Decision and Control*, 2009, pp. 8387–8392.
- [8] D. Gromov and P. E. Caines, "Interconnection of thermodynamic control systems," in *Proc. IFAC World Congress 2011*, 2011, paper TuC14.5.
- [9] A. D. McNaught and A. Wilkinson, Eds., *IUPAC. Compendium of Chemical Terminology*, 2nd ed. Blackwell Scientific Publications, 1997, XML on-line corrected version: <http://goldbook.iupac.org>.
- [10] D. Kondepudi and I. Prigogine, *Modern Thermodynamics: From Heat Engines to Dissipative Structures*. Wiley, 1998.
- [11] H. Geiges, *An Introduction to Contact Topology*. Cambridge University Press, 2008.
- [12] V. I. Arnold, *Mathematical Methods of Classical Mechanics*, 2nd ed. Springer, 1989.
- [13] A. Kushner, V. Lychagin, and V. Rubtsov, *Contact Geometry and Nonlinear Differential Equations*. Cambridge University Press, 2007.
- [14] H. B. Callen, *Thermodynamics and an Introduction to Thermostatistics*, 2nd ed. Wiley, 1985.
- [15] J. I. Neimark and N. A. Fufaev, *Dynamics of Nonholonomic Systems*. AMS, 1972, Translations of Mathematical Monographs.
- [16] A. M. Bloch, *Nonholonomic Mechanics and Control*. Springer, 2003.