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Robust Tracking by Reduced-order Disturbance Observer: Linear Case

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***Abstract*— The disturbance observer (DOB) is one of the widely-used control methods for robust tracking under plant uncertainties and external disturbances. Typically it is implemented by two (so-called) Q-filters as well as an inverse model for the nominal plant. Observing that these two Q-filters have identical dynamics, we propose a reduced-order implementation of the conventional DOB configuration. Moreover, we analyze this newly proposed configuration and claim that the robust stability condition, which has been found for the conventional configuration, still holds for the reduced-order case, and the robust transient performance can also be maintained as before with a saturation function introduced in the feedback loop.**

I. INTRODUCTION

The disturbance observer (DOB), originally introduced in [1], is known as one of powerful tools for robust control [2] and has been successfully applied to many practical problems in various fields such as servo systems, robotics, optical disc drives, and automotive vehicles. See [3]–[9] and references therein.

Its conventional configuration is depicted in Fig. 1 (the shaded part); the input that is applied to the uncertain plant $P(s)$ is compared (after passing through the low-pass filter $Q_B(s)$) with the signal $\hat{u}_p$ which is generated by the inverse dynamics of nominal plant $P_n^{-1}(s)$ (after filtered by $Q_A(s)$). Thus one may expect that $\bar{u} - \hat{u}_p$ is similar to the external disturbance d (when $P(s)$ and $P_n(s)$ are similar) and may be used for compensating the disturbance. When the mismatch between the plant $P(s)$ and the nominal model $P_n(s)$ can be lumped into the disturbance d, the compensation by the disturbance observer becomes also effective to the uncertainties in the plant.

This rough sketch of idea has been rigorously analyzed in the literature. In the beginning of this century, robustness and performance of DOB are discussed in the frequency domain [10], or an extension to nonlinear plants has been reported [11]. More rigorous analysis on stability has been conducted in [12] using the singular perturbation theory, where it has been emphasized that the conventional DOB configuration has the infinite gain in the feedback loop which explains the powerful performance of DOB, and that the closed-loop system may experience the peaking phenomenon [13] which might induce large overshoot during the transient period. Then, the authors have derived an almost necessary and sufficient condition for robust stability of the closed-loop system with DOB, and proposed a design guideline in [14]. Based on the analysis of DOB in the state space, the works [15] and [16] presented nonlinear versions of DOB-based inner-loop controller for nonlinear systems. Moreover, a design procedure is provided guaranteeing not only the robust stability but also the recovery of nominal transient performance.

Basically, two stable filters (called Q-filters) are present in the typical DOB structure as well as the inner-loop controller developed in [15] and [16]. Thus, the number of integrators required to implement these structures is $2\nu + n_0$ where ν is the dimension of Q-filter and n_0 the dimension of zero dynamics of the nominal plant (see [12] for more details). However, observing that the transfer functions of two Q-filters are actually the same $Q(s)$, it is natural to ask whether the order of final controller could be reduced. In fact, the reference [3] has already presented the reduced-order implementation of DOB when the plant is simply the

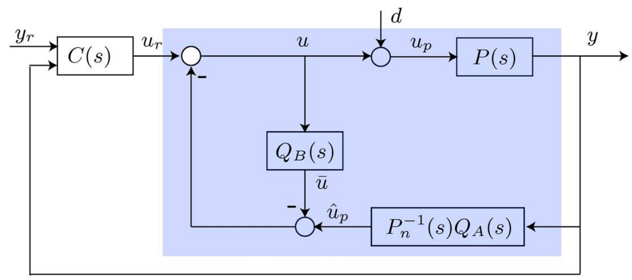

Fig. 1. Conventional disturbance observer configuration implemented with two Q-filters. Although they are marked as $Q_A(s)$ and $Q_B(s)$ for convenience, they are actually the same $Q(s)$.

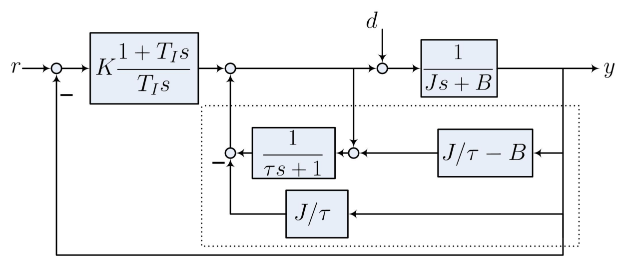

Fig. 2. Implementation used in [3] that has only one Q-filter with the transfer function $1/(\tau s + 1)$. The dotted area represents the DOB configuration.

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first order system, as can be seen in Fig. 2. In order to have the configuration of Fig. 2, one exploits the fact that $P_n^{-1}(s)Q(s) = \frac{Js+B}{\tau s+1} = \frac{J}{\tau} - \frac{J/\tau - B}{\tau s+1}$ and reorganizes the loop so that the controller can be implemented using just one integrator. We note that it is not very straightforward to generalize this idea for high order systems when the nominal system has zero dynamics. Moreover, it is not straightforward either whether the analysis and synthesis performed in [15] and [16] for the conventional DOB configuration still apply to the reduced-order implementation that will be proposed shortly.

This paper aims to present a reduced-order implementation of the DOB configuration, and to verify that the robust stability condition and the robust transient behavior (which have been dealt with for the conventional DOB) are still preserved with the reduced-order implementation. The idea of the reduced-order implementation is to view the inner-loop controller as a multi-input-single-output system whose inputs are the plant input and output, and realize it together with the inverse of the nominal plant in the state-space. On top of this realization, a suitable coordinate transformation (modified from [15]) is developed in order to convert the closed-loop system into the standard singular perturbation form. Then, the singular perturbation theory yields the robust stability condition of the closed-loop system, and guarantees the robust transient behavior with the introduction of a saturation function within the feedback-loop. As a result, it is seen that, despite the reduced-order implementation, all the properties of previous implementations in [15], [16] are preserved for uncertain linear systems.

The paper is organized as follows. At first we formulate the problem in Section II. In Section III, we present the reduced-order DOB with stability analysis, and a redesign method with saturation to have robust transient behavior. Some concluding remarks are given is Section IV.

Notation: 0_k stands for the zero vector in $\mathbb{R}^k$ and I_k the identity matrix in $\mathbb{R}^{k\times k}$. For two column vectors (or scalars) a and b, $[a;b] := [a^T, b^T]^T$.

II. PROBLEM FORMULATION

Consider a single-input-single-output system with a relative degree ν given, without loss of generality, by

$$\begin{aligned} \dot{z} &= Sz + Py \\ \dot{x} &= Ax + B(\psi^T z + \phi^T x + g(u+d)) \\ y &= Cx \end{aligned} \tag{1}$$

where $u \in \mathbb{R}$ is the control input, $d \in \mathbb{R}$ the unknown disturbance, $y \in \mathbb{R}$ the system output, $x = [x_1; \dots; x_\nu] \in \mathbb{R}^\nu$ and $z = [z_1; \dots; z_{n-\nu}] \in \mathbb{R}^{n-\nu}$ are the states of the system, and the matrices A, B, and C are given by

$$A := \begin{bmatrix} 0_{\nu-1} & I_{\nu-1} \\ 0 & 0_{\nu-1}^T \end{bmatrix}, B := \begin{bmatrix} 0_{\nu-1} \\ 1 \end{bmatrix}, C := \begin{bmatrix} 1 & 0_{\nu-1}^T \end{bmatrix}.$$

It is assumed that the parameter matrices S, P, ψ, ϕ, and g are unknown but belong to known compact sets Ω_S, Ω_P, Ω_ψ, Ω_ϕ, and Ω_g, respectively, in the matrix space. Moreover, we assume that the sign of g is known, and thus, suppose that $0 < g_- \le g \le g_+$ (by reversing the sign of the input u and the disturbance d if necessary), with known bounds g_- and g_+. It is also assumed that the unknown matrix S is Hurwitz (thus, we assume that the uncertain system is of minimum phase). Finally, the disturbance $d(t)$ is assumed to be C^2, and $|d(t)|$ and $|\dot{d}(t)|$ are bounded for all time t; for example, $|d(t)| \le d_+$, $\forall t \ge 0$, with a known constant d_+.

Suppose that an output feedback controller, represented by

$$\begin{aligned} \dot{c} &= \Gamma c + \Pi(y_r - \bar{y}), \quad c \in \mathbb{R}^l \\ u_r &= \Theta c + \Lambda(y_r - \bar{y}), \quad u_r \in \mathbb{R} \end{aligned} \tag{2}$$

where y_r is the reference command, has been designed for the disturbance-free nominal plant for (1) given by

$$\begin{aligned} \dot{\bar{z}} &= \bar{S}\bar{z} + \bar{P}\bar{y} \\ \dot{\bar{x}} &= A\bar{x} + B(\bar{\psi}^T\bar{z} + \bar{\phi}^T\bar{x} + \bar{g}u_r) \\ \bar{y} &= C\bar{x} \end{aligned} \tag{3}$$

where $\bar{S}$, $\bar{P}$, $\bar{\psi}$, $\bar{\phi}$, and $\bar{g}$ are nominal parameters of S, P, ψ, ϕ, and g, respectively, and we assume that they also belong to their corresponding parameter sets (e.g., $\bar{\psi} \in \Omega_\psi$). We now make an assumption for the output feedback controller (2).

Assumption 1: The reference input $y_r(t)$ and the closed-loop system (2)–(3) satisfy the following:

1) $|y_r(t)| \le y_{r+}, \forall t \ge 0$, with a known constant y_{r+} and $|\dot{y}_r(t)|$ is bounded.
2) The closed-loop system (2) and (3) is asymptotically stable.

□

We consider a dynamic inner-loop controller of the form

$$\begin{aligned} \dot{\chi} &= \begin{bmatrix} \dot{\chi}_1 \\ \dot{\chi}_2 \end{bmatrix} = \begin{bmatrix} \Upsilon_1(\chi, y, u_r) \\ \Upsilon_2(\chi, y, u_r) \end{bmatrix} = \Upsilon(\chi, y, u_r) \\ u &= \upsilon(\chi, y, u_r) \end{aligned} \tag{4}$$

where $\chi_1 \in \mathbb{R}^{n-\nu}$ and $\chi_2 \in \mathbb{R}^\nu$. With this controller the closed-loop system becomes

$$\begin{aligned} \dot{z} &= Sz + Py \\ \dot{x} &= Ax \\ &\quad + B\big[\psi^T z + \phi^T x + g(\upsilon(\chi, Cx, \Theta c + \Lambda(y_r - Cx)) + d)\big] \\ \dot{c} &= \Gamma c + \Pi(y_r - Cx) \\ \dot{\chi} &= \Upsilon(\chi, Cx, \Theta c + \Lambda(y_r - Cx)). \end{aligned} \tag{5}$$

In this paper, we are interested in designing the controller (4) which is an n dimensional system (same dimension as the plant) and guarantees **robust stability** and **robust transient performance**. By the robust stability, we mean that the closed-loop system is asymptotically stable under the variation of uncertain parameters within the compact parameter sets. We say that the closed-loop system has the robust transient performance if $[x(t); c(t)]$ of the closed-loop trajectory $[z(t); x(t); c(t); \chi(t)]$ of (5) can be made arbitrarily close to $[x_N(t); c_N(t)]$ of the nominal closed-loop trajectory $[z_N(t); x_N(t); c_N(t)]$ with $[z_N(0); x_N(0); c_N(0)] = [\chi_1(0); x(0); c(0)]$ (that is the solution of (2) and (3)) *in* *the whole time horizon*, under the variation of uncertain parameters.

III. Main Result

A. Reduced-order Implementation of DOB

As seen in Fig. 1, conventional implementations of DOB contain two Q-filters. However, the following argument suggests a reduced-order implementation using a single $Q(s)$. From the figure, it is observed that

$$\begin{aligned} w(s) &= Q(s)u(s) - Q(s)P_n^{-1}(s)y(s) \\ &= \begin{bmatrix} Q(s) & -Q(s)P_n^{-1}(s) \end{bmatrix} \begin{bmatrix} u(s) \\ y(s) \end{bmatrix} =: \mathcal{Q}(s) \begin{bmatrix} u(s) \\ y(s) \end{bmatrix} \end{aligned} \tag{6}$$

where, as in other literature including [15], [16], we take $Q(s)$ as

$$Q(s) = \frac{a_0}{(\tau s)^\nu + a_{\nu-1}(\tau s)^{\nu-1} + \cdots + a_0}.$$

Here, a_i's and τ are design parameters to be determined. Let us consider an n dimensional state-space representation of $\mathcal{Q}(s)$. Since $\bar{x}_i = \bar{y}^{(i-1)}$ $(i = 1, \cdots, \nu)$ and $\dot{\bar{x}}_\nu = \bar{y}^{(\nu)} = \bar{\psi}^T \bar{z} + \bar{\phi}^T \bar{x} + \bar{g} u_r$ from (3), the inverse dynamics of $P_n(s) = \bar{y}(s)/u_r(s)$ is obtained by

$$\begin{aligned} \dot{\bar{z}} &= \bar{S}\bar{z} + \bar{P}\bar{y} \\ u_r &= \frac{1}{\bar{g}}\left(\bar{y}^{(\nu)} - \bar{\phi}^T \bar{x} - \bar{\psi}^T \bar{z}\right) \\ &= \frac{1}{\bar{g}}\left(s^\nu \bar{y} - \bar{\phi}_\nu s^{\nu-1}\bar{y} - \cdots - \bar{\phi}_1 \bar{y} - \bar{\psi}^T \bar{z}\right) \end{aligned}$$

where s represents the differentiation operator. Since y is an input to $\mathcal{Q}(s)$, we obtain the dynamics for $P_n^{-1}(s) = y^\dagger(s)/y(s)$ as (the output is denoted by $y^\dagger$)

$$\begin{aligned} \dot{\bar{z}} &= \bar{S}\bar{z} + \bar{P}y \\ y^\dagger &= \frac{1}{\bar{g}}\left(s^\nu y - \bar{\phi}_\nu s^{\nu-1} y - \cdots - \bar{\phi}_1 y - \bar{\psi}^T \bar{z}\right). \end{aligned}$$

With this in mind, one can rewrite (6) as

$$\begin{aligned} &((\tau s)^\nu + a_{\nu-1}(\tau s)^{\nu-1} + \cdots + a_0) w(s) \\ &= a_0 \left(u(s) + \frac{1}{\bar{g}} \bar{\psi}^T \bar{z}(s)\right) \\ &\qquad - \frac{a_0}{\bar{g}}\left(s^\nu - \bar{\phi}_\nu s^{\nu-1} - \cdots - \bar{\phi}_1\right) y(s). \end{aligned}$$

Motivated by a state-space realization of the above relation, we propose the following form of the DOB implementation which corresponds to (4):

$$\dot{\bar{z}} = \bar{S}\bar{z} + \bar{P}y, \qquad \bar{z} \in \mathbb{R}^{n-\nu} \tag{7a}$$

$$\dot{q} = A_{q\tau} q + \frac{a_0}{\tau^\nu} B_{q\tau} \begin{bmatrix} u + \frac{1}{\bar{g}}\bar{\psi}^T \bar{z} \\ \frac{1}{\bar{g}} y \end{bmatrix}, \quad q \in \mathbb{R}^\nu \tag{7b}$$

$$w = Cq - \frac{1}{\bar{g}} \frac{a_0}{\tau^\nu} y \tag{7c}$$

$$u = u_r + w \tag{7d}$$

where

$$A_{q\tau} = \begin{bmatrix} -\frac{a_{\nu-1}}{\tau} & 1 & 0 & \cdots & 0 \\ -\frac{a_{\nu-2}}{\tau^2} & 0 & 1 & \dots & 0 \\ \vdots & \vdots & \vdots & \ddots & \vdots \\ -\frac{a_1}{\tau^{\nu-1}} & 0 & 0 & \cdots & 1 \\ -\frac{a_0}{\tau^\nu} & 0 & 0 & \cdots & 0 \end{bmatrix}, B_{q\tau} = \begin{bmatrix} 0 & \bar{\phi}_\nu + \frac{a_{\nu-1}}{\tau} \\ 0 & \bar{\phi}_{\nu-1} + \frac{a_{\nu-2}}{\tau^2} \\ \vdots & \vdots \\ 0 & \bar{\phi}_2 + \frac{a_1}{\tau^{\nu-1}} \\ 1 & \bar{\phi}_1 + \frac{a_0}{\tau^\nu} \end{bmatrix}.$$

Remark 1: The inner-loop controller developed in [15], [16] is of dimension $n + \nu = 2\nu + (n - \nu)$, in which 2ν is due to the two Q-filters of dimension ν and $(n-\nu)$ is due to the nominal zero dynamics. It was shown in [15] that one Q-filter acts like a high gain observer while the other provides the infinite gain property. However, this kind of interpretation is lost in the reduced-order implementation (7). □

B. Robust Stability Condition and Selection of Design Parameters

In this subsection, we investigate the stability of the closed-loop system (5) with (7) playing the role of (4) (i.e., $\chi = [\bar{z}; q]$). First of all, we show that the system (5) can be transformed into the standard singular perturbation form [13] with the time separation parameter τ. The proof is omitted due to page limit.

Lemma 1: The coordinate transformation defined by

$$\eta_i = \tau^{i-1}\left(q_1^{(i-1)} - \frac{a_0}{\tau^\nu}\frac{1}{\bar{g}} y^{(i-1)}\right), \quad i = 1, \dots, \nu, \tag{8}$$

transforms the closed-loop system (1), (2), and (7) into

$$\begin{aligned} \dot{z} &= Sz + Py \\ \dot{\bar{z}} &= \bar{S}\bar{z} + \bar{P}y \\ \dot{x} &= Ax + B(\psi^T z + \phi^T x + g(u_r + \eta_1 + d)) \\ \dot{c} &= \Gamma c + \Pi(y_r - Cx), \qquad u_r = \Theta c + \Lambda(y_r - Cx) \\ \tau\dot{\eta} &= A\eta - B\begin{bmatrix} a_0\frac{g}{\bar{g}} & a_1 & \cdots & a_{\nu-1} \end{bmatrix}\eta + a_0 B u_r \\ &\quad + \frac{a_0}{\bar{g}} B\left(\bar{\psi}^T \bar{z} - \psi^T z + (\bar{\phi} - \phi)^T x - g(u_r + d)\right). \end{aligned} \tag{9}$$

■

Remark 2: If we express (8) with the states q and x, it will be seen that

$$\eta_i = \sum_{j=1}^{i}\left(\tau^{j-1}\alpha_{ij} q_j + \frac{1}{\tau^\nu}\mu_j^i(\tau) x_j\right) \tag{10}$$

where α_{ij} is a constant coefficient and $\mu_j^i(\tau)$ is a *polynomial of order $i-1$ with respect to τ*. The implication of (10) is that, even though the initial conditions $q(0)$ and $x(0)$ are bounded, the initial condition $\eta(0)$ may become arbitrarily large as τ approaches zero. This behavior is actually due to the peaking phenomenon and becomes an obstacle for achieving the robust transient performance by reducing the design parameter τ (see [15]). Fortunately, since the term $\mu_j^i(\tau)/\tau^\nu$ is the polynomial of $(1/\tau)$, it can be seen that global exponential stability of the η-subsystem will yield arbitrarily small $|\eta(T) - \eta^*(T)|$ (η^* is defined in Section III-C) at arbitrarily small time T with sufficiently small τ, and that the mal-effect of the peaking on the plant state $x(t)$

can be blocked by introducing a saturation function in the feedback loop.

The equality (10) can be proved by induction. Indeed, it holds for $i = 1$ with $\alpha_{11} = 1$ and $\mu_1^1(\tau) = -a_0/\bar{g}$. Suppose it holds for $1 \le i \le \nu - 2$. Then we obtain

$$\begin{aligned} \eta_{i+1} =& \tau\dot{\eta}_i \\ =& \sum_{j=1}^{i} \Big[\tau^j \alpha_{ij}(-a_{\nu-j}/\tau^j \cdot q_1 + q_{j+1} \\ &\quad + a_0/\tau^\nu \cdot (\bar{\phi}_{\nu-j+1} + a_{\nu-j}/\tau^j)x_1/\bar{g}) \\ &\quad + \tau/\tau^\nu \cdot \mu_j^i(\tau)x_{j+1}\Big] \\ =& \sum_{j=1}^{i}(-\alpha_{ij}a_{\nu-j}q_1 + \tau^j\alpha_{ij}q_{j+1} + \tau/\tau^\nu \cdot \mu_j^i(\tau)x_{j+1}) \\ &+ \Big(\sum_{j=1}^{i} \tau^j \alpha_{ij} a_0 \bar{\phi}_{\nu-j+1} + \alpha_{ij}a_0 a_{\nu-j}\Big)x_1/(\tau^\nu \bar{g}). \end{aligned}$$

By taking $\alpha_{i+1,1} := -\sum_{j=1}^{i} \alpha_{ij}a_{\nu-j}$, $\alpha_{i+1,j} := \alpha_{i,j-1}$ for $2 \le j \le i+1$, $\mu_1^{i+1}(\tau) := \sum_{j=1}^{i}(\tau^j\alpha_{ij}a_0\bar{\phi}_{\nu-j+1} + \alpha_{ij}a_0a_{\nu-j})/\bar{g}$, and $\mu_j^{i+1}(\tau) := \tau\mu_{j-1}^i(\tau)$ for $2 \le j \le i+1$, the claim follows. □

Now we discuss the stability of the closed-loop system (9) applying the singular perturbation theory [13]. The theory says that if the dynamics of one subsystem (called fast dynamics) is sufficiently fast compared to the other part (called slow dynamics), and if the state of the fast dynamics converges to its equilibrium point which is parameterized by the states of the slow dynamics with the slow states frozen, then the trajectory of slow states of the original system is close to that of the so-called quasi-steady-state subsystem. The quasi-steady-state subsystem is defined as the dynamics of the slow system with the states corresponding to the fast dynamics replaced by their equilibrium points. In our case, the dynamics of η comprises the fast dynamics, while the dynamics of z, $\bar{z}$, x, and c corresponds to the slow dynamics. From now on, η is called the fast variable and z, $\bar{z}$, x, c, y_r, u_r, and d slow variables.

In order to apply the singular perturbation theory, it is firstly noted that with the slow variables frozen, the fast dynamics has one and only one equilibrium point given by

$$\begin{aligned} \eta_1^* &= \frac{\bar{g}-g}{g}u_r + \frac{1}{g}\left(\bar{\psi}^T\bar{z} - \psi^T z + (\bar{\phi}-\phi)^T x\right) - d, \\ \eta_2^* &= \cdots = \eta_\nu^* = 0. \end{aligned}$$

With this result at hand, the quasi-steady-state subsystem is obtained from (9) as follows:

$$\begin{aligned} \dot{z} &= Sz + Py \\ \dot{\bar{z}} &= \bar{S}\bar{z} + \bar{P}y \\ \dot{x} &= Ax + B(\bar{\psi}^T\bar{z} + \bar{\phi}^T x + \bar{g}u_r) \\ \dot{c} &= \Gamma c + \Pi(y_r - Cx), \qquad u_r = \Theta c + \Lambda(y_r - Cx). \end{aligned} \tag{11}$$

Then, it is seen that the input-output relation (from y_r to $y = Cx$) is exactly the same as that of the disturbance-free nominal closed-loop system (2) and (3). Since the state z is unobservable in the system (11), the stability of the quasi-steady-state subsystem is determined by the stability of the nominal closed-loop system (2) and (3), and by the stability of the zero dynamics $\dot{z} = Sz + Py$.

On the other hand, let us compute the boundary-layer subsystem [13] of (9), which can be obtained by rewriting the dynamics of $\tilde{\eta} := \eta - \eta^*$ in the new time scale $\sigma := t/\tau$ and setting $\tau = 0$; namely

$$\frac{d\tilde{\eta}}{d\sigma} = A\tilde{\eta} - B\begin{bmatrix} a_0\frac{g}{\bar{g}} & a_1 & \cdots & a_{\nu-1}\end{bmatrix}\tilde{\eta} =: \mathcal{A}_f\tilde{\eta}. \tag{12}$$

From the structure of A and B, we obtain the following.

Lemma 2: The boundary-layer subsystem (12) is exponentially stable if and only if the polynomial $s^\nu + a_{\nu-1}s^{\nu-1} + \cdots + a_1 s + (g/\bar{g})a_0$ is Hurwitz. ■

Remark 3: Since the bounds g_- and g_+ for the uncertain parameter g are known, one can always find a_i's such that $\mathcal{A}_f$ is Hurwitz over all possible parameters of g. For this, we recall the design proposed in [15]; that is, one can find $a_1, \ldots, a_{\nu-1}$ first such that $s^{\nu-1} + a_{\nu-1}s^{\nu-2} + \cdots + a_1$ is a Hurwitz polynomial, and then, pick $a_0 > 0$ sufficiently small. For details, see [15]. □

The robust stability condition, by which we mean that the system matrix of the closed-loop system (9) is asymptotically stable despite the uncertainties, is given as follows.

Theorem 1: Suppose that

(a) the quasi-steady-state subsystem (11) is exponentially stable, i.e., the matrix

$$\mathcal{A}_s = \begin{bmatrix} S & 0 & PC & 0 \\ 0 & \bar{S} & \bar{P}C & 0 \\ 0 & B\bar{\psi}^T & A + B(\bar{\phi}^T - \bar{g}\Lambda C) & B\bar{g}\Theta \\ 0 & 0 & -\Pi C & \Gamma \end{bmatrix} \tag{13}$$

is Hurwitz for all $S \in \Omega_S$ (which is equivalent to the fact that S is Hurwitz and the nominal closed-loop system (2) and (3) is exponentially stable),

(b) the boundary-layer subsystem (12) is exponentially stable, i.e., the matrix $\mathcal{A}_f$ is Hurwitz for all $g \in \Omega_g$.

Then, there exists $\tau^* > 0$ such that, for each $0 < \tau < \tau^*$, the linear closed-loop system (1), (2), and (7), or equivalently (9), is robustly stable. (The quantity τ^* depends on the sets Ω_S, Ω_P, Ω_ψ, Ω_ϕ, and Ω_g in general, and as the size of those sets gets larger, the value of τ^* gets smaller.) ■

Proof: Define $\tilde{\eta}(t) := \eta(t) - \eta^*(t)$. Then, we have

$$\tau\dot{\tilde{\eta}} = \mathcal{A}_f\tilde{\eta} - \tau\bar{B}\dot{\eta}_1^* \tag{14}$$

where $\bar{B} = [1; 0; \cdots; 0] \in \mathbb{R}^\nu$. By performing a tedious computation of $\dot{\eta}_1^*$ along (9), one can see that

$$\dot{\eta}_1^* = MC\tilde{\eta} + N_1[z; \bar{z}; x; c] + N_2[y_r; \dot{y}_r; d; \dot{d}] \tag{15}$$

where $M = -(\bar{g}-g)\Lambda CB + (\bar{\phi}-\phi)^T B$, and N_1 and N_2 are some constant matrices. With these results, the closed-loop system (9) is expressed as

$$\begin{aligned}\dot{z} &= Sz + Py\\ \dot{\bar{z}} &= \bar{S}\bar{z} + \bar{P}y\\ \dot{x} &= Ax + B(\bar{\psi}^T\bar{z} + \bar{\phi}^T x + \bar{g}(\Theta c + \Lambda(y_r - Cx)) + g\tilde{\eta}_1)\\ \dot{c} &= \Gamma c + \Pi(y_r - Cx)\\ \dot{\tilde{\eta}} &= \frac{1}{\tau}\mathcal{A}_f\tilde{\eta}\\ &\quad - \bar{B}(MC\tilde{\eta} + N_1[z;\bar{z};x;c] + N_2[y_r;\dot{y}_r;d;\dot{d}])\end{aligned}$$

whose system matrix is given by

$$\begin{bmatrix}\mathcal{A}_s & *\\ -\bar{B}N_1 & \frac{1}{\tau}\mathcal{A}_f - \bar{B}MC\end{bmatrix}$$

where $*$ represents some constant matrix. Since $\mathcal{A}_s$ and $\mathcal{A}_f$ are Hurwitz matrices by assumption, and all blocks of the above matrix except the $(2,2)$ block are independent of τ, it can be shown that there exists $\tau^* > 0$ (possibly depending on the size of uncertainties) such that, for all $0 < \tau < \tau^*$, the system matrix of (9) is Hurwitz. ■

The stability achieved in Theorem 1 is a global one, and therefore, the closed-loop system (1), (2), and (7) is stable for any initial conditions with sufficiently small τ. However, if $x(0) \neq 0$, then $\eta(0)$ in the equivalent closed-loop system (9) gets arbitrarily large with small τ (see Remark 2). This implies that, if we compare the solution of the closed-loop system with its nominal solution of (11) initiated at the same initial condition $\bar{z}(0)$, $x(0)$, and $c(0)$, it is seen that very large overshoot of $\eta(t)$ in its initial period may disrupt the closeness of two solutions. Therefore, if the robust transient response is of interest, we should block the mal-effect of the unwanted large overshoot in $\eta(t)$ propagating into the plant. This is achieved by saturating some signal in the feedback loop, sacrificing the global stability.

C. Redesign for Robust Transient Performance

We saturate the feedback signal generated by the DOB in order that the effect of large peaking does not propagate to the slow variables. It is important to note that the domain of interest in the state space (the set of initial conditions $q(0)$, $z(0)$, $\bar{z}(0)$, $x(0)$, and $c(0)$) covered by this method is bounded. In particular, we suppose that the trajectory $[\bar{z}(t);\bar{x}(t);c(t)]$ of the stable nominal closed-loop system (2) and (3) under a bounded reference command $y_r(t)$ remains in an open connected and bounded set $U \subset \mathbb{R}^{n+l}$ when the initial condition $[\bar{z}(0);\bar{x}(0);c(0)]$ is located in a known compact set $S_0 \subset U$. With the solution $\bar{x}(t)$, we also suppose that the solution $z(t)$ of $\dot{z} = Sz + Py$ with $y(t) = C\bar{x}(t)$ resides in a compact set Z even under the variations of S and P, when $z(0)$ is in the projection of S_0 into the z subspace.

The redesigned DOB for robust transient performance is given by

$$\begin{aligned}\dot{\bar{z}} &= \bar{S}\bar{z} + \bar{P}y, && \bar{z} \in \mathbb{R}^{n-\nu}\\ \dot{q} &= A_{q\tau}q + \frac{a_0}{\tau^\nu}B_{q\tau}\begin{bmatrix}u + \frac{1}{\bar{g}}\bar{\psi}^T\bar{z}\\ \frac{1}{\bar{g}}y\end{bmatrix}, && q \in \mathbb{R}^\nu\\ u &= u_r + \Phi(w), && w = Cq - \frac{1}{\bar{g}}\frac{a_0}{\tau^\nu}y\end{aligned} \tag{16}$$

where Φ is a globally bounded C^1 function satisfying

$$\Phi(w) = w, \forall w \in S_w, \text{ and } 0 \leq \frac{d\Phi}{dw}(w) \leq 1, \forall w \in \mathbb{R} \tag{17}$$

in which

$$\begin{aligned}S_w = \Big\{s &= \frac{\bar{g} - g}{g}(\Theta c + \Lambda(y_r - Cx))\\ &\quad + \frac{1}{g}\left(\bar{\psi}^T\bar{z} - \psi^T z + (\bar{\phi} - \phi)^T x\right) - d :\\ &\quad z \in Z_\delta, [\bar{z};x;c] \in U_\delta, |y_r| \leq y_{r+}, |d| \leq d_+,\\ &\quad g \in \Omega_g, \phi \in \Omega_\phi, \psi \in \Omega_\psi\Big\}\end{aligned} \tag{18}$$

with any small positive constant δ, where Z_δ implies $\{z + \bar{z} : z \in Z, |\bar{z}| \leq \delta\}$ and U_δ is similarly defined. In principle, the set S_w can be computed because all the bounds of unknown parameters are known, which however is quite a daunting task. Instead, by observing the fact that the set is designed so that the saturation function becomes inactive in the slow transient and in the steady-state, one can simply tune the saturation level sufficiently large by roughly overestimating such bounds or by repeating computer simulations.

In order to have robust transient performance, we also tune the parameter a_0 after choosing a_i's such that $s^{\nu-1} + a_{\nu-1}s^{\nu-2} + \cdots + a_1$ is a Hurwitz polynomial, which is explained below. Consider a disk $D(g_-/\bar{g}, g_+/\bar{g})$ which is defined as a closed disk in the complex plane whose diameter is the line segment $[-\bar{g}/g_-, -\bar{g}/g_+]$ on the real axis. Choose a sufficiently small a_0 such that the disk is disjoint from the Nyquist plot of

$$G(s) = \frac{1}{s}\frac{a_0}{s^{\nu-1} + a_{\nu-1}s^{\nu-2} + \cdots + a_1} \tag{19}$$

and the plot does not encircle the disk. Note that this is always possible since the Nyquist plot is bounded to the left (by assumption, all the poles of $G(s)$ except the pole at the origin are stable) and it shrinks towards the origin as a_0 becomes smaller.

By applying the transformation (8) one obtains the dynamics of new closed-loop system as follows:

$$\begin{aligned}\dot{z} &= Sz + Py\\ \dot{\bar{z}} &= \bar{S}\bar{z} + \bar{P}y\\ \dot{x} &= Ax + B(\psi^T z + \phi^T x + g(u_r + \Phi(\eta_1) + d))\\ \dot{c} &= \Gamma c + \Pi(y_r - Cx), \quad u_r = \Theta c + \Lambda(y_r - Cx)\end{aligned} \tag{20a}$$

$$\begin{aligned}\tau\dot{\eta} &= (A\eta - Ba^T)\eta + a_0\frac{\bar{g} - g}{\bar{g}}B(u_r + \Phi(\eta_1))\\ &\quad + \frac{a_0}{\bar{g}}B\left(\bar{\psi}^T\bar{z} - \psi^T z + (\bar{\phi} - \phi)^T x - gd\right)\end{aligned} \tag{20b}$$

where $a = [a_0;\cdots;a_{\nu-1}]$.

In spite of the saturation function Φ, the equilibrium point η^* for the boundary-layer subsystem of (20) is the same as before. Indeed, it immediately follows that $\eta_2^* = \cdots = \eta_\nu^* = 0$, and for η_1^*, we solve from (20b) that

$$\begin{aligned}&\bar{g}\eta_1 - (\bar{g} - g)\Phi(\eta_1)\\ &\quad = (\bar{g} - g)u_r + \bar{\psi}^T\bar{z} - \psi^T z + (\bar{\phi} - \phi)^T x - gd.\end{aligned}$$

Since the left-hand side of this equation is strictly increasing with respect to η_1, it has a unique solution $\eta_1^* = \frac{\bar{g}-g}{g} u_r + \frac{1}{g}\left(\bar{\psi}^T \bar{z} - \psi^T z + (\bar{\phi}-\phi)^T x\right) - d$ since $\Phi(\eta_1^*) = \eta_1^*$ by construction.

Now let us consider the dynamics of $\tilde{\eta} = \eta - \eta^*$. Then, we have that

$$\begin{aligned}\dot{\tilde{\eta}} &= \frac{1}{\tau}(A - Ba^T)\tilde{\eta} - \bar{B}MC\tilde{\eta} \\ &\quad + \frac{a_0}{\tau}(1 - g/\bar{g})B\left(\Phi(\tilde{\eta}_1 + \eta_1^*) - \Phi(\eta_1^*)\right) \\ &\quad - \bar{B}\theta^*(z, \bar{z}, x, c, y_r, \dot{y}_r, d, \dot{d})\end{aligned} \tag{21}$$

where

$$\theta^*(z, \bar{z}, x, c, y_r, \dot{y}_r, d, \dot{d}) = N_1[z; \bar{z}; x; c] + N_2[y_r; \dot{y}_r; d; \dot{d}]$$

and the matrices $\bar{B}$, M, N_1, and N_2 are defined in (14) and (15). For this system, we have the following result.

Lemma 3: Suppose that $z(t) \in Z$ and $[\bar{z}(t); x(t); c(t)] \in U$. Then, there exist positive constants τ_1, k, and λ such that, for each $0 < \tau \leq \tau_1$, it holds that

$$|\tilde{\eta}(t)| \leq k e^{-\lambda \frac{t}{\tau}} |\tilde{\eta}(0)| + \gamma(\tau), \quad \forall t \geq 0 \tag{22}$$

where γ is a class $\mathcal{K}$ function. ■

By virtue of this lemma, we obtain the robust transient behavior as follows.

Theorem 2: Suppose that the stability condition of the item (a) in Theorem 1 holds, and the coefficients a_i's are designed as discussed in this subsection. Let S_q be a compact set for the initial condition $q(0)$, and S_z be the projection of S_0 into the z subspace. Then, for a given $\epsilon > 0$, there exists a τ^* such that for any $0 < \tau < \tau^*$, the trajectory $[z(t); \bar{z}(t); x(t); c(t)]$ of the closed-loop system (1), (2), and (16), initiated at $[z(0); \bar{z}(0); x(0); c(0)] \in S_z \times S_0$, is bounded for all $t \geq 0$ and satisfies that

$$\left|[x(t); c(t)] - [\bar{x}_N(t); c_N(t)]\right| \leq \epsilon, \quad \forall t \geq 0 \tag{23}$$

where $[\bar{x}_N(t); c_N(t)]$ is from the solution $[\bar{z}_N(t); \bar{x}_N(t); c_N(t)]$ of the nominal closed-loop system (2) and (3), with $[\bar{z}_N(0); \bar{x}_N(0); c_N(0)] = [\bar{z}(0); x(0); c(0)]$. ■

This theorem shows that all the benefits of the inner-loop controller using the DOB in [15] and [16] are preserved with the reduced-order implementation proposed in this paper.

IV. Conclusion

We have proposed a reduced-order implementation of the conventional DOB structure and it is shown that all the benefits of previous implementations are preserved through rigorous stability analysis. Although the current paper is on linear systems, it seems that extensions to nonlinear systems are possible to some extent but we believe that it is not trivial for the general case. It is mainly because in our previous implementations, one Q-filter plays the role of high gain observer while it is not certain whether the plant state can be reconstructed with the proposed implementation. The authors are currently working on this problem and the result will be reported in the near future.

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