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# Noncommutative Formal Power Series and Noncommutative Functions

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*Abstract*— **In various applications of formal power series, their evaluations on linear operators (acting on an infinite-dimensional Hilbert space) or on square matrices (of any size or of size large enough) play an important role and allow one to develop a noncommutative analog of analytic function theory. On the other hand, functions defined on square matrices of any size which respect direct sums and similarities and satisfy a local boundedness condition behave in many ways as analytic functions and have power series expansions — a noncommutative analogue of Taylor series. We will discuss convergence of noncommutative power series and analyticity of noncommutative functions.**

## I. INTRODUCTION

Noncommutative polynomials or, more generally, noncommutative formal power series appear in many areas of mathematics: in enumerative combinatorics, probability, formal languages, theory of polynomial or rational identities in rings, theory of Lie algebras, just to mention a few. A noncommutative formal power series in $d$ indeterminates $z = (z_1, \ldots, z_d)$ with the coefficients in a vector space $\mathcal{V}$ over a field $\mathbb{K}$ has the form

$$f = \sum_{w \in \mathbf{F}_d} f_w z^w, \tag{1}$$

where $\mathbf{F}_d$ denotes the free semigroup on $d$ generators $g_1, \ldots, g_d$ (letters) with the identity element $\emptyset$ (the empty word), and we use the noncommutative power notation: $z^w = z_{i_1} \cdots z_{i_\ell}$ for a word $w = g_{i_1} \cdots g_{i_\ell}$. We will also assume that $\mathbf{F}_d$ is ordered, and the order respects the word length.

Rational noncommutative formal power series with scalar or matrix coefficients are those series of the form (1) which are obtained by applying a finite number of successive operations of summation, multiplication, and inversion to the indeterminates $z_1, \ldots, z_d$, where the inversion is defined for a series (1) with $f_\emptyset$ an invertible matrix. They first appeared in a system theoretical context in the theory of formal languages and finite automata; see Kleene [1], Schützenberger [2], [3], and Fliess [4], [5], [6]. (The work of Fliess was motivated also by applications to certain classes of nonlinear systems.) By the Kleene–Schützenberger–Fliess theorem, the series of the form (1) with coefficients in $\mathbb{K}^{p\times q}$ is rational if and only if the infinite Hankel matrix $H_f = [f_{vw}]_{v,w\in\mathbf{F}_d}$ has a finite rank $m$ if and only if the series is recognizable, i.e., there exist matrices $A_1, \ldots, A_d \in \mathbb{K}^{m\times m}$, $B \in \mathbb{K}^{m\times q}$, and $C \in \mathbb{K}^{p\times m}$ such that

$$f_w = CA^w B, \quad w \in \mathbf{F}_d, \tag{2}$$

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where $A = (A_1, \ldots, A_d)$. Alternatively, one can write (2) as

$$f = C(I - A_1 z_1 - \cdots - A_d z_d)^{-1} B, \tag{3}$$

which is a typical realization formula. A similar noncommutative realization formula was obtained in Fornasini–Marchesini [7] and then used to obtain a $d$-dimensional system realization of a given matrix-valued rational function in $d$ commuting variables. The noncommutative formal power series and the realization formulae in these works were formal algebraic objects and were not viewed as functions.

More recently, realizations of rational expressions in Hilbert space operators (modelling structured, possibly time varying, uncertainty) appeared in work on robust control of linear systems, see Beck [8], Beck–Doyle–Glover [9], Lu–Zhou–Doyle [10]. For a series of the form (1) and a $d$-tuple of bounded linear operators $X = (X_1, \ldots, X_d)$ on a common Hilbert space $\mathcal{H}$, the tensor substitutions were used as follows (of course, the operator norms should be small enough to guarantee the convergence of the series):

$$f(X) = \sum_{w\in\mathbf{F}_d} X^w \otimes f_w.$$

Similarly, matrix-valued noncommutative rational expressions can be evaluated on appropriate $d$-tuples of operators, e.g., for (3) we have

$$f(X) = (I_{\mathcal{H}} \otimes C)(I_{\mathcal{H}} \otimes I_m - X_1 \otimes A_1 - \cdots - X_d \otimes A_d)^{-1} \cdot (I_{\mathcal{H}} \otimes B).$$

Understanding noncommutative formal power series or rational expressions as functions on $d$-tuples of (not necessarily commuting) operators makes realization theory work better. In the works of Ball–Groenewald–Malakorn [11], [12], [13], a theory of structured noncommutative multidimensional systems with $\mathbf{F}_d$ as a "time domain" was developed, which provides various state-space models for systems with structured uncertainties (which include, in particular, noncommutative versions of Fornasini–Marchesini or Givone–Roesser systems); a noncommutative version of the Bounded Real Lemma has been obtained; a noncommutative analogue of the Schur–Agler class of contractive (in the sense of operator substitutions) analytic functions on the unit polydisk and its system realizations were described. This realization theory involves therefore not necessarily rational noncommutative formal power series and, possibly, infinite dimensional state-space realizations. We also mention that in Ball–Kaliuzhnyi-Verbovetskyi [14] an even more general form of noncommutative multidimensional systems was introduced and studied.

Noncommutative formal power series converging on certain sets of $d$-tuples of operators (matrices) as a noncommutative analog of analytic functions on a neighborhood of zero in $\mathbb{C}^d$ appear also in a number of works; see, e.g., Popescu [15], [16], [17], [18], Ball–Vinnikov [19], Helton et al. [20], [21], Kaliuzhnyi-Verbovetskyi [22], Alpay–Kaliuzhnyi-Verbovetskyi [23], [24], Kaliuzhnyi-Verbovetskyi–Vinnikov [25], [26], where various realization formulas are used as well, with or without transfer function interpretation.

Another set-up where matrix substitutions are used is noncommutative rational expressions. An important area where this set-up is natural is Linear Matrix Inequalities (LMIs); see, e.g., Nesterov–Nemirovski [27], Nemirovski [28], Skelton–Iwasaki–Grigoriadis [29]. As it turns out, most optimization problems appearing in systems and control have matrices as the natural variables, and the problem involves rational expressions in these matrix variables which have therefore the same form independent of matrix sizes; see Helton [30], Helton–McCullough–Putinar–Vinnikov [31]. Realizations of rational functions in noncommuting indeterminates are used in Helton–McCullough–Vinnikov [32] to convert (numerically unmanageable) rational matrix inequalities into (highly manageable) LMIs. Rational functions, scalar or matrix valued, are understood in [32] as classes of equivalence of matrix-valued noncommutative rational expressions (two such expressions are equivalent if they coincide on the intersection of their domains of regularity).

The concept of noncommutative rational functions (not necessarily regular at zero) was developed further in Kaliuzhnyi-Verbovetskyi–Vinnikov [26], [33]. In [26], the left and right shifts were defined for noncommutative rational functions regular at zero in terms of evaluations on matrices. These shift operators were used to show that the singularity set of a matrix-valued noncommutative rational function coincides with the singularity set of the resolvent in a minimal noncommutative Fornasini–Marchesini system realization. Moreover, the commutative analog of this result was obtained as a consequence of the noncommutative one via "lifting" to the noncommutative setting. In [33], the difference-differential calculus was developed for matrix-valued noncommutative rational functions. The shifts are a special case of difference-differential operators when the latter ones are applied at zero.

In fact, the difference-differential calculus can be developed in a more general setting of noncommutative functions. In our current joint project with Victor Vinnikov [34], the foundations of the theory of noncommutative functions, in particular analytic ones, are laid out. In what follows, we present a (necessarily) brief overview of this theory.

## II. Noncommutative functions, their difference-differential calculus, and Taylor–Taylor series

### A. The definition of noncommutative functions

For a vector space $\mathcal{V}$ over a field $\mathbb{K}$, we define the noncommutative (nc) space over $\mathcal{V}$,

$$\mathcal{V}_{\mathrm{nc}} = \coprod_{n=1}^{\infty} \mathcal{V}^{n\times n}.$$

For $X \in \mathcal{V}^{n\times n}$ and $Y \in \mathcal{V}^{m\times m}$ we define their direct sum,

$$X \oplus Y = \begin{bmatrix} X & 0 \\ 0 & Y \end{bmatrix} \in \mathcal{V}^{(n+m)\times(n+m)}.$$

Notice that matrices over $\mathbb{K}$ act from the right and from the left on matrices over $\mathcal{V}$ by the standard rules of matrix multiplication: if $X \in \mathcal{V}^{p\times q}$ and $T \in \mathbb{K}^{r\times p}$, $S \in \mathbb{K}^{q\times s}$, then

$$TX \in \mathcal{V}^{r\times q}, \quad XS \in \mathcal{V}^{p\times s}.$$

A subset $\Omega \subseteq \mathcal{V}_{\mathrm{nc}}$ is called a nc set if it is closed under direct sums; explicitly, denoting $\Omega_n = \Omega \cap \mathcal{V}^{n\times n}$, we have $X \oplus Y \in \Omega_{n+m}$ for all $X \in \Omega_n$, $Y \in \Omega_m$. In the case of $\mathcal{V} = \mathbb{K}^d$ we identify matrices over $\mathcal{V}$ with $d$-tuples of matrices over $\mathbb{K}$:

$$\left(\mathbb{K}^d\right)^{p\times q} \cong \left(\mathbb{K}^{p\times q}\right)^d.$$

Under this identification, for $d$-tuples $X = (X_1, \ldots, X_d) \in \left(\mathbb{K}^{n\times n}\right)^d$ and $Y = (Y_1, \ldots, Y_d) \in \left(\mathbb{K}^{m\times m}\right)^d$,

$$X \oplus Y = \left( \begin{bmatrix} X_1 & 0 \\ 0 & Y_1 \end{bmatrix}, \ldots, \begin{bmatrix} X_d & 0 \\ 0 & Y_d \end{bmatrix} \right) \in \left(\mathbb{K}^{(n+m)\times(n+m)}\right)^d;$$

and for a $d$-tuple $X = (X_1, \ldots, X_d) \in \left(\mathbb{K}^{p\times q}\right)^d$ and matrices $T \in \mathbb{K}^{r\times p}$, $S \in \mathbb{K}^{q\times s}$,

$$TX = (TX_1, \ldots, TX_d) \in \left(\mathbb{K}^{r\times q}\right)^d,$$

$$XS = (X_1S, \ldots, X_dS) \in \left(\mathbb{K}^{p\times s}\right)^d.$$

Let $\mathcal{V}$ and $\mathcal{W}$ be vector spaces over $\mathbb{K}$, and let $\Omega \subseteq \mathcal{V}_{\mathrm{nc}}$ be a nc set. A mapping $f\colon \Omega \to \mathcal{W}_{\mathrm{nc}}$ with $f(\Omega_n) \subseteq \mathcal{W}^{n\times n}$ is called a nc function if $f$ satisfies the following two conditions:

- $f$ respects direct sums:
$$f(X \oplus Y) = f(X) \oplus f(Y), \qquad X, Y \in \Omega. \tag{4}$$
- $f$ respects similarities: if $X \in \Omega_n$ and $S \in \mathbb{K}^{n\times n}$ is invertible with $SXS^{-1} \in \Omega_n$, then
$$f(SXS^{-1}) = Sf(X)S^{-1}. \tag{5}$$

*Proposition 1:* A mapping $f\colon \Omega \to \mathcal{W}_{\mathrm{nc}}$ with $f(\Omega_n) \subseteq \mathcal{W}^{n\times n}$ respects direct sums and similarities, i.e., (4) and (5) hold, if and only if $f$ respects intertwinings: for any $X \in \Omega_n$, $Y \in \Omega_m$, and $T \in \mathbb{K}^{n\times m}$ such that $XT = TY$,

$$f(X)T = Tf(Y). \tag{6}$$

We note that the condition (6) has appeared first in J.L. Taylor [35], [36] in the case where $\mathcal{V} = \mathbb{C}^d$ and an additional assumption of analyticity of $f(X)$ as a function of matrix entries $(X_i)_{jk}$, $i = 1, \ldots, d$, $j, k = 1, \ldots, n$, for every $n \in \mathbb{N}$ is used. The name of J.L. Taylor is well known mainly due to his most general definition of the joint spectrum of a $d$-tuple of commuting bounded linear operators on a Banach space, and the corresponding analytic functional calculus; see [37], [38]. His attempt in [35], [36] to extend these to $d$-tuples of noncommuting bounded linear operators are less known. Though the latter works did not achieve their ultimate goals, a machinery of nc functions has been developed. We use some of Taylor's ideas, however in a more general setting and with milder assumptions on nc functions. We also note that these ideas have found applications in free probability; see Voiculescu [39].

The following result, though looking technical at the first glance, provides us with the definition of the right nc difference-differential operator $\Delta_R$.

*Theorem 2:* Let $f\colon \Omega \to \mathcal{W}_{\rm nc}$ be a nc function on a nc set $\Omega$. Let $X \in \Omega_n$, $Y \in \Omega_m$, and $Z \in \mathcal{V}^{n\times m}$ be such that $\left[\begin{smallmatrix} X & Z \\ 0 & Y\end{smallmatrix}\right] \in \Omega_{n+m}$. Then

$$f\left(\begin{bmatrix} X & Z \\ 0 & Y\end{bmatrix}\right) = \begin{bmatrix} f(X) & \Delta_R f(X,Y)(Z) \\ 0 & f(Y)\end{bmatrix},$$

where the off-diagonal block entry $\Delta_R f(X,Y)(Z)$ is determined uniquely and is linear in $Z$.

In fact, $\Delta_R f(X,Y)(Z)$ can be extended by linearity to all $Z \in \mathcal{V}^{n\times m}$. As a consequence of Theorem 2, the following formula of finite differences holds:

$$f(X)-f(Y) = \Delta_R f(Y,X)(X-Y), \quad n\in\mathbb{N},\ X,Y\in\Omega_n.$$

The linear mapping $\Delta f(Y,Y)(\cdot)$ plays the role of a nc differential at $Y$. If $\mathbb{K}=\mathbb{R}$ or $\mathbb{C}$, setting $X = Y + tZ$ with $t\in\mathbb{R}$ ($t\in\mathbb{C}$), we obtain

$$f(Y+tZ)-f(Y) = t\Delta_R f(Y,Y+tZ)(Z).$$

Under appropriate continuity conditions, it follows that $\Delta_R f(Y,Y)(Z)$ is the directional derivative of $f$ at $Y$ in the direction $Z$. In the case where $\mathcal{V}=\mathbb{K}^d$, the finite difference formula turns into

$$f(X)-f(Y) = \sum_{i=1}^{N} \Delta_{R,i} f(Y,X)(X_i - Y_i), \quad X,Y\in\Omega_n,$$

with the right partial difference-differential operators $\Delta_{R,i}$:

$$\Delta_{R,i} f(Y,X)(C) := \Delta_R f(Y,X)(0,\ldots,0,\underbrace{C}_{i^{\rm th}\ \text{place}},0,\ldots,0).$$

The linear mapping $\Delta_{R,i} f(Y,Y)(\cdot)$ plays the role of a right nc $i$-th partial differential at the point $Y$.

*Example 3:* Let $\mathcal{V}=\mathcal{W}=\mathbb{C}$, $f(X)=X^2$. Then

$$\Delta_R f(X,Y)(Z) = \left(\begin{bmatrix} X & Z \\ 0 & Y\end{bmatrix}^2\right)_{12} = XZ+ZY.$$

In particular, if $n=1$ and $X=Y$, then $\Delta_R f(X,X)(Z) = 2XZ$, i.e., $\Delta_R f(X,X)$ coincides with the usual (commutative) differential.

The left nc full and partial difference-differential operators $\Delta_L$, $\Delta_{L,i}$, $i=1,\ldots,d$, are defined analogously, via evaluations of nc functions on lower block-triangular matrices.

The nc difference-differential operators obey natural calculus rules: linearity, the product rule, the chain rule, etc. On the other hand, as a function of $X$ and $Y$, $\Delta_R f(X,Y)(\cdot)$ respects direct sums and similarities, or equivalently, respects intertwinings in the following sense: if $X\in\Omega_n$, $Y\in\Omega_m$, $\widetilde{X}\in\Omega_{\tilde n}$, $\widetilde{Y}\in\Omega_{\tilde m}$, and $T\in\mathbb{K}^{\tilde n\times n}$, $S\in\mathbb{K}^{\tilde m\times m}$ are such that

$$TX = \widetilde{X}T, \quad SY=\widetilde{Y}S,$$

then

$$T\Delta_R f(X,Y)(ZS) = \Delta_R f(\widetilde{X},\widetilde{Y})(TZ)S.$$

Thus, $\Delta_R f$ can be viewed as a higher order nc function.

*Example 4:* For $f(X)=X^2$, we have $\Delta_R f(X,Y)(Z) = XZ+ZY$ (see Example 3) and

$$\begin{aligned} T\Delta_R f(X,Y)(ZS) &= T(XZS+ZSY) \\ &= (\widetilde{X}TZ + TZ\widetilde{Y})S = \Delta_R f(\widetilde{X},\widetilde{Y})(TZ)S. \end{aligned}$$

### *B. Higher order nc functions*

More generally, we define the class of nc functions of order $k$,

$$\mathcal{T}^k = \mathcal{T}^k(\Omega;\mathcal{W}_{0,\rm nc},\mathcal{W}_{1,\rm nc},\ldots,\mathcal{W}_{k,\rm nc})$$

as a class of functions on $\Omega^{k+1}$, where $\Omega\subseteq\mathcal{V}_{\rm nc}$ is a nc set, whose values on $\Omega_{n_0}\times\cdots\times\Omega_{n_k}$ are $k$-linear forms

$${\mathcal{W}_1}^{n_0\times n_1}\times\cdots\times{\mathcal{W}_k}^{n_{k-1}\times n_k}\to{\mathcal{W}_0}^{n_0\times n_k},$$

and which respect direct sums and similarities, or equivalently, respect intertwinings: if $T_jX^j = \tilde{X}^jT_j$, $j=0,\ldots,k$, then

$$\begin{aligned} &T_0 f\left(X^0,\ldots,X^k\right)\left(Z^1T_1,\ldots,Z^kT_k\right) \\ &\qquad = f\left(\tilde{X}^0,\ldots,\tilde{X}^k\right)\left(T_0Z^1,\ldots,T_{k-1}Z^k\right)T_k \end{aligned}$$

for $n_j,\tilde n_j\in\mathbb{N}$, $X^j\in\Omega_{n_j}$, $\tilde{X}^j\in\Omega_{\tilde n_j}$, $T_j\in\mathbb{K}^{\tilde n_j\times n_j}$, $j=0$, $\ldots$, $k$, and for $Z^j\in{\mathcal{W}_j}^{n_{j-1}\times\tilde n_j}$, $j=1,\ldots,k$. The class $\mathcal{T}^0=\mathcal{T}^0(\Omega;\mathcal{W}_{\rm nc})$ is the class of nc functions $f\colon\Omega\to\mathcal{W}_{\rm nc}$. If $f\in\mathcal{T}^0(\Omega;\mathcal{W}_{\rm nc})$, then $\Delta_R f\in\mathcal{T}^1(\Omega;\mathcal{W}_{\rm nc},\mathcal{V}_{\rm nc})$.

*Proposition 5:* Let $f\in\mathcal{T}^k(\Omega;\mathcal{W}_{0,\rm nc},\ldots,\mathcal{W}_{k,\rm nc})$. Let $X^0\in\Omega_{n_0}$, $\ldots$, $X^{k-1}\in\Omega_{n_{k-1}}$, $X^{k\prime}\in\Omega_{n_k'}$, $X^{k\prime\prime}\in\Omega_{n_k''}$, $Z^1\in{\mathcal{W}_1}^{n_0\times n_1}$, $\ldots$, $Z^{k-1}\in(\mathcal{W}_{k-1})^{n_{k-2}\times n_{k-1}}$, $Z^{k\prime}\in{\mathcal{W}_k}^{n_{k-1}\times n_k'}$, $Z^{k\prime\prime}\in{\mathcal{W}_k}^{n_{k-1}\times n_k''}$. Let $Z\in\mathcal{V}^{n_k'\times n_k''}$ be such that $\begin{bmatrix} X^{k\prime} & Z \\ 0 & X^{k\prime\prime}\end{bmatrix}\in\Omega_{n_k'+n_k''}$. Then

$$\begin{aligned} &f\left(X^0,\ldots,X^{k-1},\begin{bmatrix} X^{k\prime} & Z \\ 0 & X^{k\prime\prime}\end{bmatrix}\right) \\ &\qquad\left(Z^1,\ldots,Z^{k-1},\operatorname{row}\left[Z^{k\prime},Z^{k\prime\prime}\right]\right) \\ &= \operatorname{row}\left[f\left(X^0,\ldots,X^{k-1},X^{k\prime}\right)\left(Z^1,\ldots,Z^{k-1},Z^{k\prime}\right),\right. \\ &\quad \Delta_R f\left(X^0,\ldots,X^{k-1},X^{k\prime},X^{k\prime\prime}\right)\left(Z^1,\ldots,Z^{k-1},Z^{k\prime},Z\right) \\ &\qquad \left.+ f\left(X^0,\ldots,X^{k-1},X^{k\prime\prime}\right)\left(Z^1,\ldots,Z^{k-1},Z^{k\prime\prime}\right)\right]. \end{aligned}$$

Here

$$\Delta_R f\left(X^0,\ldots,X^{k-1},X^{k\prime},X^{k\prime\prime}\right)\left(Z^1,\ldots,Z^{k-1},Z^{k\prime},Z\right)$$

is determined uniquely, and is linear in $Z$. Moreover, this extended right nc difference-differential operator $\Delta_R$ is a mapping of the classes:

$$\Delta_R\colon \mathcal{T}^k(\Omega;\mathcal{W}_{0,\mathrm{nc}},\dots,\mathcal{W}_{k,\mathrm{nc}}) \to \mathcal{T}^{k+1}(\Omega;\mathcal{W}_{0,\mathrm{nc}},\dots,\mathcal{W}_{k,\mathrm{nc}},\mathcal{V}_{\mathrm{nc}}).$$

Iterating $\ell$ times the operator $\Delta_R$, we obtain the following result (which is clearly true for $\ell=1$ by Theorem 2).

*Theorem 6:* Let $f\in\mathcal{T}^0(\Omega;\mathcal{W}_{\mathrm{nc}})$. Then

$$\Delta_R^\ell f(X^0,\dots,X^\ell)(Z^1,\dots,Z^\ell) = f\left(\begin{bmatrix} X^0 & Z^1 & 0 & \cdots & 0\\ 0 & X^1 & \ddots & \ddots & \vdots\\ \vdots & \ddots & \ddots & \ddots & 0\\ \vdots & & \ddots & X^{\ell-1} & Z^\ell\\ 0 & \cdots & \cdots & 0 & X^\ell\end{bmatrix}\right)_{1,\ell+1} \in \mathcal{T}^\ell(\Omega;\underbrace{\mathcal{W}_{\mathrm{nc}},\dots,\mathcal{W}_{\mathrm{nc}}}_{\ell\text{ times}},\mathcal{V}_{\mathrm{nc}})$$

### C. The Taylor–Taylor formula

We use the calculus of higher order nc difference-differential operators to derive a nc analogue of the classical (Brook) Taylor formula, which we call the Taylor–Taylor (TT) expansion in honor of Brook Taylor [40] and of Joseph L. Taylor [36].

*Theorem 7:* Let $f\in\mathcal{T}^0(\Omega;\mathcal{W}_{\mathrm{nc}})$ with $\Omega\subseteq\mathcal{V}_{\mathrm{nc}}$ a nc set, $n\in\mathbb{N}$, and $Y\in\Omega_n$. Then for each $N\in\mathbb{N}$ and arbitrary $X\in\Omega_n$,

$$f(X)=\sum_{\ell=0}^{N}\Delta_R^\ell f(\underbrace{Y,\dots,Y}_{\ell+1\text{ times}})(\underbrace{X-Y,\dots,X-Y}_{\ell\text{ times}}) + \Delta_R^{N+1} f(\underbrace{Y,\dots,Y}_{N+1\text{ times}},X)(\underbrace{X-Y,\dots,X-Y}_{N+1\text{ times}}).$$

In the case where $\mathcal{V}=\mathbb{K}^d$, we obtain

$$f(X)=\sum_{\ell=0}^{N}\sum_{w=g_{i_1}\cdots g_{i_\ell}}\Delta_R^{w^\top} f(\underbrace{Y,\dots,Y}_{\ell+1\text{ times}})(X_{i_1}-Y_{i_1},\dots,X_{i_\ell}-Y_{i_\ell}) + \sum_{w=g_{i_1}\cdots g_{i_{N+1}}}\Delta_R^{w^\top} f(\underbrace{Y,\dots,Y}_{N+1\text{ times}},X)(X_{i_1}-Y_{i_1},\dots,X_{i_{N+1}}-Y_{i_{N+1}}),$$

where for a word $w=g_{i_1}\cdots g_{i_\ell}$,

$$\Delta_R^{w^\top}:=\Delta_{R,i_\ell}\cdots\Delta_{R,i_1}.$$

If $Y=(I_n\mu_1,\dots,I_n\mu_d)$, this is a genuine nc power expansion

$$f(X)=\sum_{\ell=0}^{N}\sum_{|w|=\ell}(X-I_n\mu)^w\,\Delta_R^{w^\top} f(\underbrace{\mu,\dots,\mu}_{\ell+1\text{ times}}) + \sum_{|w|=N+1}(X-I_n\mu)^w\,\Delta_R^{w^\top} f(\underbrace{\mu,\dots,\mu}_{N+1\text{ times}},X).$$

In the cases where $\mathbb{K}=\mathbb{R}$ or $\mathbb{K}=\mathbb{C}$, and the spaces $\mathcal{V}$ and $\mathcal{W}$ are Banach spaces, we study the convergence of TT series in the sense that the remainder, that is the second sum in these formulas, converges to 0. Here we consider the case where $\mathbb{K}=\mathbb{C}$, and $\mathcal{V}$ and $\mathcal{W}$ are operator spaces, which is most relevant in free probability and is a natural set-up for studying uniform convergence of nc power series.

Recall that a Banach space $\mathcal{W}$ over $\mathbb{C}$ is called an operator space (see, e.g., Paulsen [41]) if a sequence of norms $\|\cdot\|_n$ on $\mathcal{W}^{n\times n}$, $n=1,2,\dots$, is defined so that the following two conditions hold:

- For every $n,m\in\mathbb{N}$, $X\in\mathcal{W}^{n\times n}$ and $Y\in\mathcal{W}^{m\times m}$,
$$\|X\oplus Y\|_{n+m}=\max\{\|X\|_n,\|Y\|_m\}.$$
- For every $n\in\mathbb{N}$, $X\in\mathcal{W}^{n\times n}$ and $S,T\in\mathbb{C}^{n\times n}$,
$$\|SXT\|_n\le\|S\|\,\|X\|_n\|T\|,$$
where $\|\cdot\|$ denotes the operator norm of $\mathbb{C}^{n\times n}$ with respect to the standard Euclidean norm of $\mathbb{C}^n$.

Let $\mathcal{W}$ be an operator space. For $Y\in\mathcal{W}^{s\times s}$ and $r>0$, define a nc ball centered at $Y$ of radius $r$ as

$$B_{\mathrm{nc}}(Y,r) = \coprod_{m=1}^{\infty}\left\{X\in\mathcal{W}^{ms\times ms}\colon \left\|X-\bigoplus_{\alpha=1}^{m}Y\right\|_{ms}<r\right\}.$$

NC balls form a basis for a topology on $\mathcal{W}_{\mathrm{nc}}$. Open sets in this topology will be called uniformly open.

Let $\mathcal{V},\mathcal{W}$ be operator spaces, and let $\Omega\subseteq\mathcal{V}_{\mathrm{nc}}$ be a uniformly open nc set. A nc function $f\colon\Omega\to W_{\mathrm{nc}}$ is called uniformly locally bounded if for any $s\in\mathbb{N}$ and $Y\in\Omega_s$ there exists a nc ball $B_{\mathrm{nc}}(Y,r)\subseteq\Omega$ such that $f$ is bounded on $B_{\mathrm{nc}}(Y,r)$. A nc function $f\colon\Omega\to\mathcal{W}_{\mathrm{nc}}$ is called Gâteaux (G-) differentiable if for every $n\in\mathbb{N}$ the function $f|_{\Omega_n}$ is G-differentiable, i.e., for every $X\in\Omega_n$ and $Z\in\mathcal{V}^{n\times n}$ the G-derivative of $f$ at $X$ in direction $Z$,

$$\lim_{t\to0}\frac{f(X+tZ)-f(X)}{t}=\frac{d}{dt}f(X+tZ)\Big|_{t=0},$$

exists. A nc function is called uniformly analytic if $f$ is uniformly locally bounded and G-differentiable. We note that this is a stronger notion than just analyticity of $f|_{\Omega_n}$ for each $n\in\mathbb{N}$. We address the reader to the classical book of Hille and Phillips [42] and to a more recent source, Mujica [43], for the theory of analytic functions on Banach spaces.

*Theorem 8:* Let a nc function $f\colon\Omega\to\mathcal{W}_{\mathrm{nc}}$ be uniformly locally bounded. For $s\in\mathbb{N}$, $Y\in\Omega_s$, let $\delta:=\sup\{r>0\colon f\text{ is bounded on }B_{\mathrm{nc}}(Y,r)\}$. Then

$$f(X)=\sum_{\ell=0}^{\infty}\Delta_R^\ell f\Big(\underbrace{\bigoplus_{\alpha=1}^{m}Y,\dots,\bigoplus_{\alpha=1}^{m}Y}_{\ell+1\text{ times}}\Big)\underbrace{\Big(X-\bigoplus_{\alpha=1}^{m}Y,\dots,X-\bigoplus_{\alpha=1}^{m}Y\Big)}_{\ell\text{ times}} \qquad (7)$$

holds, with the TT series converging absolutely and uniformly on every open nc ball $B_{\rm nc}(Y,r)$ with $r<\delta$.

*Corollary 9:* Let $\Omega \subseteq \mathcal{V}_{\rm nc}$ be a uniformly open nc set. Then a nc function $f\colon \Omega \to \mathcal{W}_{\rm nc}$ is uniformly locally bounded if and only if $f$ is continuous with respect to the uniformly-open topologies on $\mathcal{V}_{\rm nc}$ and $\mathcal{W}_{\rm nc}$ if and only if $f$ is uniformly analytic.

In the case where $\mathcal{V}=\mathbb{C}^d$, we also study the convergence of the TT series along $\mathbf{F}_d$.

*Theorem 10:* Let a nc function $f\colon \Omega \to \mathcal{W}_{\rm nc}$ be uniformly locally bounded. For every $s\in\mathbb{N}$, $Y\in\Omega_s$, let $\delta := \sup\{r>0\colon f \text{ is bounded on } B_{\rm nc}(Y,r)\}$. Then

$$f(X)=\sum_{\ell=0}^{\infty}\sum_{w=g_{i_1}\cdots g_{i_\ell}\in\mathbf{F}_d}\Delta_R^{w^\top} f\Big(\underbrace{\bigoplus_{\alpha=1}^{m}Y,\ldots,\bigoplus_{\alpha=1}^{m}Y}_{\ell+1 \text{ times}}\Big) \Big(X_{i_1}-\bigoplus_{\alpha=1}^{m}Y_{i_1},\ldots,X_{i_\ell}-\bigoplus_{\alpha=1}^{m}Y_{i_\ell}\Big) \tag{8}$$

holds, with the series converging absolutely and uniformly on every open nc diamond about $Y$,

$$\diamondsuit_{\rm nc}(Y,r):=\coprod_{m=1}^{\infty}\Big\{X\in\Omega_{sm}\colon \sum_{j=1}^{d}\|e_j\|_1\Big\|X_j-\bigoplus_{\alpha=1}^{m}Y_j\Big\|<r\Big\}$$

with $r<\delta$. Here $e_j\in\mathbb{C}^d$ are the standard basis vectors.

## III. Convergence of nc Power Series

Let $\mathcal{V}$ be a $\mathbb{K}$-vector space and let $s,\ \ell,\ m_0,\ \ldots,\ m_\ell\in\mathbb{N}$. For $Z^1\in\mathcal{V}^{sm_0\times sm_1},\ \ldots,\ Z^\ell\in\mathcal{V}^{sm_{\ell-1}\times sm_\ell}$, we define

$$Z^1\odot_s\cdots\odot_s Z^\ell\in\Big(\big(\mathcal{V}^{s\times s}\big)^{\otimes\ell}\Big)^{m_0\times m_\ell}$$

as follows:

$$\begin{aligned}(Z^1\odot_s\cdots\odot_s Z^\ell)^{\alpha,\beta} \\ =\sum_{\alpha_0=\alpha,\alpha_j=1,\ldots,m_j,\alpha_\ell=\beta} Z^{1\alpha_0,\alpha_1}\otimes\cdots\otimes Z^{\ell\alpha_{\ell-1},\alpha_\ell},\end{aligned}$$

where $\alpha=1,\ \ldots,\ m_0$, $\beta=1,\ \ldots,\ m_\ell$, and $Z^j=[Z^{j\gamma,\delta}]_{\gamma=1,\ldots,m_{j-1},\delta=1,\ldots,m_j}$. This is nothing but the product of matrices over the tensor algebra $\mathbf{T}(\mathcal{V}^{s\times s})$. We write

$$Z^{\odot_s\ell}=\underbrace{Z\odot_s\cdots\odot_s Z}_{\ell \text{ times}}$$

in the case where $m_0=\ldots=m_\ell=m$ and $Z^1=\ldots=Z^\ell=Z$. We study the convergence of power series

$$\sum_{\ell=0}^{\infty}Z^{\odot_s\ell}f_\ell, \tag{9}$$

where $Z\in\mathcal{V}^{sm\times sm}$, $m=1,2,\ldots$, $\mathcal{V}$ and $\mathcal{W}$ are operators spaces, $f_\ell\colon(\mathcal{V}^{s\times s})^\ell\to\mathcal{W}^{s\times s}$, $\ell=0,1,\ldots$, is a given sequence of $\ell$-linear mappings, i.e., a given linear mapping $\mathbf{T}(\mathcal{V}^{s\times s})\to\mathcal{W}^{s\times s}$. The mappings $f_\ell$ are naturally extended to matrices over $\mathcal{V}^{s\times s}$ by their action on matrix entries and assumed to be completely bounded (in the sense of Christensen and Sinclair; see, e.g., [41, Chapter 17]), i.e.,

$$\|f_\ell\|_{\mathcal{L}^\ell_{\rm cb}}:=\sup_{\|Z^1\|=\cdots=\|Z^\ell\|=1}\|(Z^1\odot_s\cdots\odot_s Z^\ell)f_\ell\|_{m_0,m_\ell}$$

is finite, where $Z^1\in\mathcal{V}^{sm_0\times sm_1},\ \ldots,\ Z^\ell\in\mathcal{V}^{sm_{\ell-1}\times sm_\ell}$, $m_j=1,\ \ldots,\ \infty$, $j=0,\ldots,\ell$. Here $\|\cdot\|_{p,q}$ is the uniquely determined norm for $p\times q$ matrices over $\mathcal{W}^{s\times s}$ (which is itself an operator space) such that the "bullet" conditions in Section II-C hold for rectangular matrices in the place of square matrices; see [41, p. 185, Exercise 13.2]. We consider here, without loss of generality, the power series centered at $0_{sm\times sm}$, for every $m=1,2,\ldots$. Clearly, the results can be extended to power series centered at $\bigoplus_{\alpha=1}^m Y$, for every $m=1,2,\ldots$, with $Y\in\mathcal{V}^{s\times s}$, i.e., the series with powers of $(Z-\bigoplus_{\alpha=1}^m Y)$ in the place of $Z$. Notice that interpreting the values of $\Delta_R^\ell f(X^0,\ldots,X^\ell)$ as $\ell$-linear mappings, we can write the right-hand side of (7) as a power series:

$$f(X)=\sum_{\ell=0}^{\infty}\Big(X-\bigoplus_{\alpha=1}^{m}Y\Big)^{\odot_s\ell}\Delta_R^\ell f\Big(\underbrace{\bigoplus_{\alpha=1}^{m}Y,\ldots,\bigoplus_{\alpha=1}^{m}Y}_{\ell+1 \text{ times}}\Big).$$

Thus, the results in this section can be considered as the converse of the results in section II-C.

*Theorem 11:* The series (9) converges uniformly on every nc ball $B_{\rm nc}(0_{s\times s},\delta)$ with

$$\delta<\rho_{\rm cb}:=\Big(\limsup_{\ell\to\infty}\sqrt[\ell]{\|f_\ell\|_{\mathcal{L}^\ell_{\rm cb}}}\Big)^{-1},$$

moreover

$$\sum_{\ell=0}^{\infty}\sup_{W\in B_{\rm nc}(0_{s\times s},\delta)}\|W^{\odot_s\ell}f_\ell\|<\infty.$$

The series (9) fails to converge uniformly on every nc ball $B_{\rm nc}(0_{s\times s},\delta)$ with $\delta>\rho_{\rm cb}$.

In the case where $\mathcal{V}=\mathbb{C}^d$, we also consider noncommutative power series

$$\sum_{w\in\mathbf{F}_d}Z^{\odot_s w}f_w, \tag{10}$$

where $Z=(Z_1,\ldots,Z_d)\in(\mathbb{C}^{sm\times sm})^d$, $m=1,2,\ldots$,

$$Z^{\odot_s w}=Z_{i_1}\odot_s\cdots\odot_s Z_{i_\ell}$$

for $w=g_{i_1}\cdots g_{i_\ell}$, and for a given sequence of $\ell$-linear mappings $f_w\colon(\mathbb{C}^{s\times s})^\ell\to\mathcal{W}^{s\times s}$, $w\in\mathbf{F}_d$, $\ell=|w|$. Notice that interpreting the values of $\Delta_R^{w^\top}f(X^0,\ldots,X^\ell)$ as $\ell$-linear mappings, we can write the right-hand side of (8) as a power series:

$$f(X)=\sum_{w\in\mathbf{F}_d}\Big(X-\bigoplus_{\alpha=1}^{m}Y\Big)^{\odot_s w}\Delta_R^{w^\top}f\Big(\underbrace{\bigoplus_{\alpha=1}^{m}Y,\ldots,\bigoplus_{\alpha=1}^{m}Y}_{\ell+1 \text{ times}}\Big).$$

There is also an analog of Theorem 11 for the convergence of the series (10).

## IV. CONCLUSIONS

We have demonstrated that the evaluations of noncommutative formal power series on square matrices of all sizes (so that the series become "not so formal") are important in various applications and allow one to think of them as of functions on the corresponding domains of convergence. We have obtained certain results on the uniform convergence of such series. On the other hand, functions defined on square matrices of any size which respect direct sums and similarities (we call them noncommutative functions) and satisfy a local boundedness condition behave in many ways as analytic functions and have power series expansions — a noncommutative analogue of Taylor series. We have shown this via developing the difference-differential calculus of noncommutative functions. The main conclusion is that analytic noncommutative functions and convergent noncommutative power series are essentially the same object, as it is in the classical (commutative) case.

## REFERENCES


[1] S.C. Kleene, "Representation of events in nerve nets and finite automata", in *Automata studies*, Annals of mathematics studies, vol. 34, Princeton University Press, Princeton, NJ, 1956, pp. 3-41.

[2] M.P. Schützenberger, On the definition of a family of automata, *Information and Control*, vol. 4, 1961, pp. 245-270.

[3] M.P. Schützenberger, "Certain elementary families of automata", *in Proc. Sympos. Math. Theory of Automata (New York, 1962)*, Polytechnic Press of Polytechnic Inst. of Brooklyn, Brooklyn, New York, 1963, pp. 139-153.

[4] M. Fliess, Sur le plongement de l'algèbre des séries rationnelles non commutatives dans un corps gauche, *C. R. Acad. Sci. Paris, Ser. A*, vol. 271, 1970, pp. 926-927.

[5] M. Fliess, Matrices de Hankel, *J. Math. Pures Appl.*, vol. 53(9), 1974, pp. 197-222.

[6] M. Fliess, Sur divers produits de séries formelles, *Bull. Soc. Math. France*, vol. 102, 1974, 181-191.

[7] E. Fornasini and G. Marchesini, Doubly-indexed dynamical systems: state-space models and structural properties, *Math. Systems Theory*, vol. 12(1), 1978/79, pp. 59-72.

[8] C. Beck, On formal power series representations for uncertain systems, *IEEE Trans. Automat. Control*, vol. 46(2), 2001, pp. 314-319.

[9] C.L. Beck, J. Doyle, and K. Glover, Model reduction of multidimensional and uncertain systems, *IEEE Trans. Automat. Control*, vol. 41(10), 1996, pp. 1466-1477.

[10] W.-M. Lu, K. Zhou, and J.C. Doyle, Stabilization of uncertain linear systems: an LFT approach, *IEEE Trans. Automat. Control*, vol. 41(1), 1996, pp. 50-65.

[11] J.A. Ball, G. Groenewald, and T. Malakorn, Structured noncommutative multidimensional linear systems, *SIAM J. Control Optim.*, vol. 44(4), 2005, pp. 1474-1528 (electronic).

[12] J.A. Ball, G. Groenewald, and T. Malakorn, Bounded real lemma for structured noncommutative multidimensional linear systems and robust control, *Multidimens. Syst. Signal Process.*, vol. 17(2-3), 2006, pp. 119-150.

[13] J.A. Ball, G. Groenewald, and T. Malakorn, "Conservative structured noncommutative multidimensional linear systems", *in The state space method generalizations and applications*, vol. 161 of *Oper. Theory Adv. Appl.*, Birkhäuser, Basel, 2006, pp. 179-223.

[14] J.A. Ball and D.S. Kaliuzhnyi-Verbovetskyi, Conservative dilations of dissipative multidimensional systems: The commutative and non-commutative settings, *Multidim. Syst. Signal Processing*, vol. 19, 2008, pp. 79-122.

[15] G. Popescu, Multi-analytic operators on Fock spaces, *Math. Ann.*, vol. 303(1), 1995, pp. 31-46.

[16] G. Popescu, Free holomorphic functions on the unit ball of $B(\mathcal{H})^n$, *J. Funct. Anal.*, vol. 241(1), 2006, pp. 268-333.

[17] G. Popescu, Free holomorphic functions on the unit ball of $B(\mathcal{H})^n$. II, *J. Funct. Anal.*, vol. 258(5), 2010, pp. 1513-1578.

[18] G. Popescu, Free holomorphic automorphisms of the unit ball of $B(\mathcal{H})^n$, *J. Reine Angew. Math.*, vol. 638, 2010, pp. 119-168.

[19] J.A. Ball and V. Vinnikov, Lax-Phillips scattering and conservative linear systems: a Cuntz-algebra multidimensional setting, *Mem. Amer. Math. Soc.*, vol. 178(837), 2005, iv+101 pp.

[20] J.W. Helton, I. Klep, S. McCullough, N. Slinglend, Noncommutative ball maps, *J. Funct. Anal.*, vol. 257(1), 2009, pp. 47-87.

[21] J.W. Helton, I. Klep, and S. McCullough, Analytic mappings between noncommutative pencil balls, *Journal of Mathematical Analysis and Applications*, Vol. 376(2), 2011, pp. 407-428.

[22] D.S. Kalyuzhnyĭ-Verbovetzkiĭ, Carathéodory interpolation on the non-commutative polydisk, *J. Funct. Anal.*, vol. 229, 2005, pp. 241-276.

[23] D. Alpay and D.S. Kalyuzhnyi-Verbovetzkii, On the intersection of null spaces for matrix substitutions in a non-commutative rational formal power series, *C. R. Math. Acad. Sci. Paris*, vol. 339(8), 2004, pp. 533-538.

[24] D. Alpay and D.S. Kalyuzhnyĭ-Verbovetzkiĭ, "Matrix-$J$-unitary non-commutative rational formal power series", *in Linear operators and linear systems*, Operator Theory: Adv. Appl., vol. 161, Birkhäuser Verlag, Basel, 2006, pp. 49-113.

[25] D.S. Kalyuzhnyĭ-Verbovetzkiĭ and V. Vinnikov, Non-commutative positive kernels and their matrix evaluations, *Proc. Amer. Math. Soc.*, vol. 134(3), 2005, pp. 805-816.

[26] D.S. Kaliuzhnyi-Verbovetskyi and V. Vinnikov, Singularities of non-commutative rational functions and minimal factorizations, *Lin. Alg. Appl.*, vol. 430, 2009, pp. 869-889.

[27] Yu. Nesterov and A. Nemirovskii, Interior-point polynomial algorithms in convex programming, vol. 13 of *SIAM Studies in Applied Mathematics*, Society for Industrial and Applied Mathematics (SIAM), Philadelphia, PA, 1994.

[28] A. Nemirovskii, Advances in convex optimization: conic programming, *Plenary Lecture, International Congress of Mathematicians (ICM)*, Madrid, Spain, 2006.

[29] R.E. Skelton, T. Iwasaki, and K.M. Grigoriadis, *A Unified Algebraic Approach to Linear Control Design*, Taylor & Francis; 1997.

[30] J.W. Helton, "Manipulating matrix inequalities automatically", *in Mathematical systems theory in biology, communications, computation, and finance (Notre Dame, IN, 2002)*, vol. 134 of *IMA Vol. Math. Appl.*, Springer, New York, 2003, pp. 237-256.

[31] J. W. Helton, S. A. McCullough, M. Putinar, and V. Vinnikov, Convex matrix inequalities versus linear matrix inequalities, *IEEE Trans. Aut. Control*, vol. 54(5), 2009, pp. 952-964.

[32] J.W. Helton, S.A. McCullough, and V. Vinnikov, Noncommutative convexity arises from linear matrix inequalities, *J. Funct. Anal.*, vol. 240(1), 2006, pp. 105-191.

[33] D.S. Kaliuzhnyi-Verbovetskyi and V. Vinnikov, Noncommutative rational functions, their difference-differential calculus and realizations, *Multidimens. Syst. Signal Process.*, to appear, arXiv:1003.0695.

[34] D.S. Kaliuzhnyi-Verbovetskyi and V. Vinnikov, Foundations of non-commutative function theory, in preparation.

[35] J.L. Taylor, A general framework for a multi-operator functional calculus, *Advances in Math.*, vol. 9, 1972, pp. 183-252.

[36] J.L. Taylor, Functions of several noncommuting variables, *Bull. Amer. Math. Soc.*, vol. 79, 1973, pp. 1-34.

[37] J.L. Taylor, A joint spectrum for several commuting operators, *J. Functional Analysis*, vol. 6, 1970, pp. 172191.

[38] J.L. Taylor, The analytic-functional calculus for several commuting operators, *Acta Math.*, vol. 125, 1970, pp. 138.

[39] D.-V. Voiculescu, Free analysis questions II: the Grassmannian completion and the series expansions at the origin, *J. Reine Angew. Math.*, vol. 645, 2010, pp. 155-236.

[40] B. Taylor, *Methodus Incrementorum Directa et Inversa*, London: Gul. Innys; 1715.

[41] V. Paulsen, *Completely bounded maps and operator algebras.* Cambridge Studies in Advanced Mathematics, vol. 78, Cambridge University Press, Cambridge; 2002.

[42] E. Hille and R.S. Phillips, *Functional analysis and semi-groups*, Third printing of the revised edition of 1957, American Mathematical Society Colloquium Publications, vol. XXXI, Providence, RI; 1974.

[43] J. Mujica, *Complex analysis in Banach spaces. Holomorphic functions and domains of holomorphy in finite and infinite dimensions*, North-Holland Mathematics Studies, vol. 120, North-Holland Publishing Co., Amsterdam; 1986.