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String Instability in Coupled Harmonic Oscillator Systems

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Abstract— **In this paper we consider the problem of disturbance response for a simple system of coupled harmonic oscillators. We suppose that the oscillators are connected in a string in which each oscillator tries to track its predecessor. Motivated by terminology from the problem of vehicle platooning, we say that the system is string unstable if the effect of a disturbance to the lead oscillator is amplified as it propagates along the string. By using a new Bode-like integral relation that must be satisfied by the complementary sensitivity function, we provide sufficient conditions for string instability.**

I. Introduction

Many researchers have studied the problem of synchronization in systems of coupled oscillators. As noted in [1], [2], this problem may be viewed as a special case of consensus control in multi-agent systems, in which each oscillator communicates with a subset of its neighbors for the purpose of achieving synchronization. Depending on the communication topology, the oscillators may or may not be able to do so. The ability to achieve synchronization also depends on the presence of communication time delays and changes in the communication topology. In the present paper, we study the effect of a disturbance on a system of coupled oscillators. Specifically, we wish to know whether the effect of a disturbance to one oscillator will be amplified or diminished as it propagates through the system.

Our approach to the problem of disturbance propagation for a system of oscillators is inspired by the literature on the problem of string instability that may arise in vehicle platooning (e.g. [3]–[7]). Specifically, we consider a string of oscillators, in which one is the leader, and with which the remainder try to synchronize their oscillations by tracking only their immediate predecessor in the string. It is known that this *predecessor following* strategy may exhibit string instability for vehicle platoons. More complex communication schemes, on the other hand, may allow the design of control laws that are string stable.

For example, each vehicle may communicate with both its immediate predecessor and successor. Early studies of string instability were done in the context of specific control laws, such as PID [3]. This made comparison between different communication schemes problematic, in that the observed string instability may have been due to a poor choice of controller gains rather than the communication scheme adopted. The authors of [5], on the other hand, show that, under appropriate hypotheses, certain communication topologies will lead to string instability for any linear controller. To show this, they applied the theory of fundamental design limitations [8], which enables such general statements to be made assuming only that the controller is stabilizing. In [5], it is assumed that all the vehicles have the same model and use the same control law, and it is shown that the predecessor following control law will necessarily lead to problems of string instability for constant spacing between vehicles. The authors of [7] greatly extend the results in [5] by considering heterogenous platoons, and more general spacing policies and communication topologies.

Many papers on oscillator synchronization use the first order, nonlinear Kuramoto model, or an appropriate extension thereof [1], [2]. In order to apply the theory of fundamental design limitations, we instead use the second order, linear oscillator model described in [9]. This will enable us to use the fact that such oscillators have poles on the imaginary axis, and to generalize the results from the theory of fundamental limitations that were used in [5]. We also use a very simple communication topology, corresponding to the predecessor-following strategy used in vehicle platooning studies. (We shall consider more general topologies in subsequent work.)

The remainder of this paper is outlined as follows. In Section II, we provide an explicit problem statement and review the integral constraint on the complementary sensitivity function that was used in [5]. This integral constraint is not applicable to our problem, and thus in Section III we propose a more general integral relation that may be applied to oscillator systems. We use this result to derive two sufficient conditions for string instability. In Section IV we derive a third sufficient condition for string instability. Specifically, we assume that a controller has been designed that satisfies certain time and frequency domain design specifications, and show that this assumption implies a lower bound on the peak in the complementary sensitivity function; if this lower bound exceeds one, then string instability is present. The results of the paper are illustrated with numerical examples in Section V. Conclusions and future research directions are given in Section VI.

Notation: Denote by OLHP, CLHP, ORHP, and CRHP respectively the open-left, closed-left, open-right and closed-right halves of the complex plane. We use Re and Im to represent the real and imaginary parts of a complex number, respectively. We use log to denote the natural logarithm and

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arg to denote the principal branch of the argument of a complex number. The relative degree r of a rational transfer function is the degree of its denominator minus the degree of its numerator polynomial.

II. Problem Formulation

Consider the series connection, or *string*, of n single-loop feedback systems depicted in Figure 1. We assume that these systems are all identical, with each plant described by a proper rational transfer function of the form

$$P(s) = P_0(s)\frac{1}{s^2+\alpha^2}, \tag{1}$$

where $P_0(s)$ has no zeros at $s = \pm j\alpha$, and with rational and stabilizing controller $C(s)$. Each plant thus contains the dynamics of a harmonic oscillator with natural frequency α radians/second.

Suppose that we desire each oscillator in the string to track the position of its immediate predecessor. Following the terminology used in vehicle platooning, we refer to the system in Figure 1 as a *predecessor-following* control architecture. Denote the commanded position to the lead oscillator by $r_1(t)$, and the positions and tracking errors of the ith oscillator as $y_i(t)$ and $e_i(t)$, respectively. Let $d_{out}(t)$ denote a disturbance entering at the output of the first oscillator. Each error signal can thus be expressed as

$$e_i(t) = \begin{cases} r_1(t) - y_1(t), & i = 1, \\ y_{i-1}(t) - y_i(t), & i \geq 2. \end{cases} \tag{2}$$

Define the *open loop* transfer function $L(s) = P(s)C(s)$, and the *sensitivity* and *complementary sensitivity* functions by

$$S(s) = \frac{1}{1+L(s)}, \qquad T(s) = \frac{L(s)}{1+L(s)}, \tag{3}$$

respectively. Then the Laplace transforms of the tracking error signals satisfy

$$E_1(s) = S(s)R_1(s) - S(s)D_{out}(s), \tag{4}$$

$$E_k(s) = T(s)E_{k-1}(s), \quad k \geq 2, \tag{5}$$

and thus

$$E_k(s) = T^{k-1}(s)E_1(s). \tag{6}$$

The presence of the plant poles at $\pm j\alpha$ implies that $T(\pm j\alpha) = 1$ and $S(\pm j\alpha) = 0$. Hence the steady state error in response to an input of the form $r_1(t) = A\sin(\alpha t + \phi)$ will be equal to zero for each oscillator in the string. In this way the motion of all the oscillators in the string will synchronize to that of the lead oscillator. We see from (4) that the command $r_1(t)$ and output disturbance $d_{out}(t)$ affect the system symmetrically, and thus conclusions drawn about the command response also apply to the disturbance response.

Suppose there exists a frequency for which $|T(j\omega)| > 1$. Then (6) implies that any disturbance to the lead oscillator at this frequency will be amplified as it propagates to successive oscillators. As the number of oscillators increases, the error will be amplified without bound, and the string in Figure 1 will be *string unstable*. In studies of string instability in vehicle platooning, one may derive *sufficient* conditions for string instability using the following integral relation, dual to the Bode sensitivity integral, that must be satisfied by the complementary sensitivity function [8, Theorem 3.1.5].

Theorem 1: (a) Consider a feedback system with plant $P(s)$ and stabilizing controller $C(s)$. Assume that $L(s)$ is rational and proper, with N_z zeros in the ORHP, $\{z_i : i = 1, \ldots, N_z\}$. Assume further that $L(s)$ may be factored as $L(s) = L_0(s)/s^k$, where $k \geq 1$ and $L_0(s)$ has neither poles nor zeros at $s = 0$. Then

$$\int_0^\infty \log|T(j\omega)|\frac{d\omega}{\omega^2} = \frac{\pi}{2}T'(0) + \pi\sum_{i=1}^{N_z}\frac{1}{z_i}, \tag{7}$$

where $T'(0) = \lim_{s\to 0} dT(s)/ds$.

(b) Suppose, in addition, that $k \geq 2$. Then $T'(0) = 0$, and

$$\int_0^\infty \log|T(j\omega)|\frac{d\omega}{\omega^2} = \pi\sum_{i=1}^{N_z}\frac{1}{z_i}. \tag{8}$$

■

Since complex zeros must occur in conjugate pairs, it follows that the right hand side of (8) is real and nonnegative. It follows immediately from (8) that if $L(s)$ has a double integrator, then necessarily there must exist a frequency for which $|T(j\omega)| > 1$. This fact was used in [5] to show that a platoon of identical vehicles in the predecessor-following control architecture must be string unstable. Recently, the results of [5] were generalized in [7] to provide sufficient conditions for string instability with heterogenous platoons and more general control architectures. The assumption of a double integrator is reasonable for study of vehicle platoons. If only a single integrator is present, then an integral constraint still holds, but need not imply that $|T(j\omega)| > 1$ due to the term $T'(0)$, which may be negative. As discussed in [8], this term is inversely proportional to the *velocity constant* of a Type 1 feedback system.

III. A Generalized Complementary Sensitivity Integral

The results of Theorem 1 are not applicable to our study of oscillators. Instead, we propose a new integral relation that the complementary sensitivity function must satisfy whenever $L(s)$ contains a pair of poles on the imaginary axis.

Theorem 2: (a) Consider a feedback system with plant $P(s)$ given by (1), and stabilizing controller $C(s)$. Assume that $L(s)$ is rational and proper, with N_z ORHP zeros $\{z_i : i = 1, \ldots, N_z\}$. Assume further that $L(s)$ may be factored as

$$L(s) = L_0(s)\frac{1}{(s^2+\alpha^2)^k}, \tag{9}$$

where $k \geq 1$, and $L_0(s)$ has no zeros at $s = \pm j\alpha$. Then

$$\int_0^\infty \log|T(j\omega)|\, W(\omega,\alpha)d\omega = \frac{\pi}{2}\mathrm{Re}\,(K_\alpha) + \pi\sum_{i=1}^{N_z}\left(\frac{z_i}{z_i^2+\alpha^2}\right), \tag{10}$$

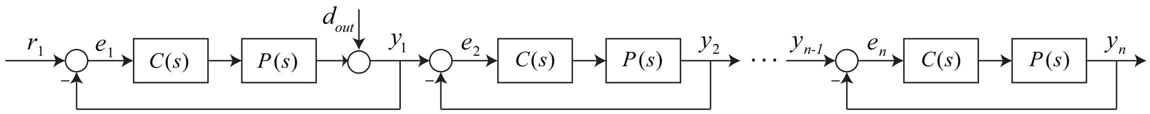

Fig. 1. Block diagram depicting a string of stabilized oscillators with length n.

where

$$K_\alpha \triangleq \lim_{s\to j\alpha} \frac{dT(s)}{ds}, \tag{11}$$

and the weighting function $W(\omega, \alpha)$ is defined as

$$W(\omega, \alpha) = \frac{\omega^2 + \alpha^2}{\left(\omega^2 - \alpha^2\right)^2}. \tag{12}$$

(b) Suppose, in addition, that $k \geq 2$. Then $K_\alpha = 0$, and

$$\int_0^\infty \log|T(j\omega)|\, W(\omega,\alpha) d\omega = \pi \sum_{i=1}^{N_z} \left(\frac{z_i}{z_i^2 + \alpha^2} \right). \tag{13}$$

Proof: See the longer version of this paper [10]. ■

Note that the right hand side of (13) is nonnegative, and that $W(\omega, \alpha) > 0$ for all frequencies except $\omega = \alpha$. It follows that if $L(s)$ has at least *two* pairs of poles at $\pm j\alpha$, then there must exist a frequency for which $|T(j\omega)| > 1$, and thus that the string of oscillators in Figure 1 is string unstable.

Suppose that $L(s)$ contains only a single pair of poles at $\pm j\alpha$, namely, those due to the plant (1). Then K_α defined in (11) may be negative and, as a consequence, $|T(j\omega)|$ may be less than one at all frequencies and string instability may not be present. (On the other hand, unless the controller also has poles at $\pm j\alpha$, the system will not be able to reject the effect of a disturbance $d_{in}(t) = \sin \alpha t$ entering at the plant input [11].)

Recall that the term corresponding to K_α in Theorem 1 is inversely proportional to the velocity constant that describes the steady state error of a Type 1 feedback system in response to a ramp input. The following result provides a corresponding interpretation for K_α, and shows that it describes the steady state error in response to an input of the form $r_1(t) = t \sin \alpha t$.

Theorem 3 (Interpretation of K_α): (a) Consider the series connection of feedback systems in Figure 1, with plant (1) and stabilizing compensator $C(s)$. Assume that $r_1(t) = t \sin \alpha t$, and define the steady state error for the first system as the response that persists after the transient response decays, denoted by $e_1^{ss}(t)$. Then

$$e_1^{ss}(t) = |K_\alpha| \sin\left(\alpha t + \arg\left(-K_\alpha\right)\right). \tag{14}$$

(b) Suppose in addition that $\arg\left(-K_\alpha\right) = 0$. Then in steady state $y_1(t)$ is in phase with $r_1(t)$, and the steady state response $y_1^{ss}(t)$ is given by

$$y_1^{ss}(t) = (t - |K_\alpha|) \sin \alpha t. \tag{15}$$

Proof: See the longer version of this paper [10]. ■

Our next result uses Theorem 3, together with the fact that all the subsystems in Figure 1 are identical, to show that the steady state tracking errors for each subsystem are identical.

Corollary 1: (a) Let $e_k^{ss}(t)$ denote the steady state tracking error of the k'th subsystem in Figure 1 in response to the input $r_1(t) = t \sin \alpha t$. Then

$$e_k^{ss}(t) = e_1^{ss}(t), \quad k = 1, \ldots, n. \tag{16}$$

(b) Suppose in addition that $\arg(-K_\alpha) = 0$. Then in steady state $y_k(t)$ is in phase with $r_1(t)$:

$$y_k^{ss}(t) = (t - k|K_\alpha|) \sin \alpha t, \qquad k = 1, \ldots, n. \tag{17}$$

Proof: See the longer version of this paper [10]. ■

Motivated by (17), we say that if $\arg(-K_\alpha) = 0$, then the *steady state phase error* for each oscillator is equal to zero. We now show that if the steady state phase error is nonzero, then the string of oscillators will be string unstable.

Theorem 4: Suppose that $\arg(-K_\alpha) \neq 0$. Then there exists a frequency ω such that $|T(j\omega)| > 1$.

Proof: First consider the case $\arg(-K_\alpha) = \pi$. Then K_α is real and positive and the result follows immediately from (10). Suppose next that $\arg(-K_\alpha) \neq 0, \pi$. Then K_α has a nonzero imaginary component. Using the fact that $T(j\alpha) = 1$, we have by definition (11) of K_α that

$$K_\alpha = \lim_{s\to j\alpha} \frac{d \log|T(s)|}{ds} + j \lim_{s \to j\alpha} \frac{d \arg T(s)}{ds}.$$

Letting $s = \sigma + j\omega$, it follows from the Cauchy-Riemann equations [12, Section 21], [13, p. 41] that

$$K_\alpha = \lim_{\omega\to\alpha} \frac{\partial \arg T(j\omega)}{\partial \omega} - j \lim_{\omega\to\alpha} \frac{\partial \log|T(j\omega)|}{\partial\omega}.$$

Together, the facts that $|T(j\alpha)| = 1$ and that $\lim_{\omega\to\alpha} \partial \log|T(j\omega)|/\partial\omega \neq 0$ imply that there exists a frequency ω near α such that $|T(j\omega)| > 1$. ■

We have now provided two sufficient conditions for string instability. One is that the open loop transfer function contains at least two pairs of complex poles at $\pm j\alpha$. The other is that the phase error in response to an input $t \sin \alpha t$ is nonzero: $\arg(-K_\alpha) \neq 0$. The following example illustrates that string stability is possible when neither of these conditions is satisfied.

Example 1: Suppose that $P(s) = 1/(s^2 + \alpha^2)$ and $C(s) = ks$, $k > 0$. Then $T(s)$ has stable poles, and $K_\alpha = -2/k$, so that $\arg(-K_\alpha) = 0$. It is easy to verify that $|T(j\omega)| \leq 1$, $\forall\omega$. ■

In the next section we suppose that the system must satisfy certain performance specifications, and show that these may provide another sufficient condition for string instability.

IV. A Lower Bound on the Peak in Complementary Sensitivity

Our goal in the present section is to derive a lower bound on $\sup_\omega |T(j\omega)|$ that holds whenever the system is assumed to satisfy appropriate performance specifications. If this lower bound exceeds unity, then we may conclude that the system in Figure 1 is string unstable. We will be interested in the case for which neither sufficient condition for string instability derived in Section III is satisfied; however, our methods will also yield a lower bound for the case in which $L(s)$ has at least two pairs of poles at $\pm j\alpha$.

We first assume that a specification on the steady state error (SSE) in response to an input $t\sin\alpha t$ must be satisfied.

Assumption 1 (Magnitude Bound on SSE): Assume that the steady state error (14) is uniformly bounded by $q > 0$:

$$|e_1^{ss}(t)| \leq q, \quad \forall t \geq 0. \tag{18}$$

■

Recall from Corollary 1 that the steady state error for each of the oscillators is identical. The transient error, defined by

$$e_i^{tr}(t) \triangleq e_i(t) - e_i^{ss}(t), \tag{19}$$

will in general be different for different oscillators. We assume an *IATE* performance specification on the sum of the integrals of the absolute values of the transient errors.

Assumption 2 (IATE Specification): Let $e_i^{tr}(t)$ in (19) denote the transient error response of the ith oscillator in response to the command $r_1(t) = t\sin(\alpha t)$. We assume that the sum of the integrals of the absolute values of the transient errors must satisfy the specification

$$\sum_{i=1}^{n} \int_0^\infty \left|e_i^{tr}(t)\right| dt \leq u(n), \tag{20}$$

for some positive function $u(n)$. ■

We now show that Assumptions 1 and 2, combined with one additional hypothesis, imply an upper bound on the gain of $T(j\omega)$.

Lemma 1: Suppose that Assumptions 1 and 2 are satisfied.

(a) Assume in addition that $C(s)P(s)$ possesses one pair of poles at $\pm j\alpha$, and that the phase error is zero: $\arg(-K_\alpha) = 0$. Then

$$|T(j\omega)| \leq \left(1 + \eta(u(n), q, \alpha, \omega)\left(\omega^2 - \alpha^2\right)^2\right)^{\frac{1}{2n}}, \tag{21}$$

where

$$\eta(u(n), q, \alpha, \omega) = \frac{u(n)}{\alpha\omega} + \frac{n^2 q^2}{4\omega^2} + \left|\omega^2 - \alpha^2\right| \frac{n u(n) q}{2\alpha\omega^2} + (\omega^2 - \alpha^2)^2 \frac{u(n)^2}{2\alpha^2\omega^2}. \tag{22}$$

(b) Assume instead that $C(s)P(s)$ possesses at least two pairs of poles at $\pm j\alpha$. Then

$$|T(j\omega)| \leq \left(1 + \frac{\left(\omega^2 - \alpha^2\right)^2}{2\alpha\omega} u(n)\right)^{\frac{1}{n}}. \tag{23}$$

Proof: See the longer version of this paper [10]. ■

In either case, $T(j\alpha) = 1$ due to the presence of the oscillator poles. The bounds (21)-(23) constrain the rate at which $|T(j\omega)|$ converges to one as ω approaches α, and are a consequence of the requirement (20) that the transient response converges rapidly to zero.

The following assumption implies that the system in Figure 1 has the ability to track low frequency commands with a specified error. For example, we may wish to apply a command that "steers" the entire string of coupled oscillators to a new position while oscillating.

Assumption 3 (Steering Performance): We make the following assumption for the steering performance. Let $0 < \omega_L < \alpha$. For $\omega \in (0,\ \omega_L)$, the following inequality holds

$$|T(j\omega)^n - 1| < \epsilon, \tag{24}$$

where $0 \leq \epsilon < 1$. ■

Finally, we assume that the system satisfies a bandwidth limitation.

Assumption 4 (Bandwidth Limitation): The transfer function $T(s)$ obeys the high frequency roll-off

$$|T(j\omega)| \leq \left(\frac{\omega_H}{\omega}\right)^r, \quad \text{for all } \omega > \omega_H \tag{25}$$

for some $\omega_H > \alpha$ and relative degree $r \geq 1$. ■

The following theorem shows that Assumptions 1-4, together with one additional hypothesis, imply the existence of a lower bound on the peak magnitude response of the complementary sensitivity function (3).

Theorem 5: Suppose that Assumptions 1-4 are satisfied.

(a) Assume in addition that $C(s)P(s)$ possesses one pair of poles at $\pm j\alpha$, and that the phase error is zero: $\arg(-K_\alpha) = 0$. Then for any $\omega_M \in (\alpha, \omega_H)$, we have the following inequality:

$$\max_{\omega\in[\omega_M,\omega_H]} \log|T(j\omega)| \geq \frac{\Omega_H - \Omega_\alpha - \Omega_L - \frac{\pi}{2}q + \pi\sum_{i=1}^{N_z}\left(\frac{z_i}{z_i^2+\alpha^2}\right)}{\int_{\omega_M}^{\omega_H} W(\omega,\alpha)d\omega}, \tag{26}$$

where Ω_L, Ω_α, and Ω_H are bounds on the integral of $\log|T(j\omega)|$ over different frequency ranges:

$$\Omega_L \triangleq \frac{1}{n}\int_0^{\omega_L} \log(1+\epsilon) W(\omega,\alpha) d\omega, \tag{27}$$

$$\Omega_\alpha \triangleq \frac{1}{2n}\int_{\omega_L}^{\omega_M} \log\left(1 + \eta(u(n), q, \alpha, \omega)\left(\omega^2 - \alpha^2\right)^2\right) \times W(\omega,\alpha) d\omega, \tag{28}$$

$$\Omega_H \triangleq r\int_{\omega_H}^{\infty} \log\frac{\omega}{\omega_H} W(\omega,\alpha) d\omega. \tag{29}$$

(b) Assume instead that $C(s)P(s)$ possesses at least two pairs of poles at $\pm j\alpha$. Then $T(s)$ must satisfy the lower bound (26), where Ω_H and Ω_L are as defined in (27) and (29), and

$$\Omega_\alpha \triangleq \frac{1}{n}\int_{\omega_L}^{\omega_M} \log\left(1+\frac{\left(\omega^2-\alpha^2\right)^2}{2\alpha\omega}u(n)\right) W(\omega,\alpha)d\omega. \tag{30}$$

Proof: We establish this result by splitting the integration interval in (10). In particular,

$$\begin{aligned}\int_{\omega_M}^{\omega_H} \log|T(j\omega)|W(\omega,\alpha)d\omega = \\ &- \int_0^{\omega_L} \log|T(j\omega)|\,W(\omega,\alpha)d\omega \\ &- \int_{\omega_L}^{\omega_M} \log|T(j\omega)|\,W(\omega,\alpha)d\omega \\ &- \int_{\omega_H}^{\infty} \log|T(j\omega)|\,W(\omega,\alpha)d\omega \\ &+ \frac{\pi}{2}\mathrm{Re}\,(K_\alpha) + \pi\sum_{i=1}^{N_z}\left(\frac{z_i}{z_i^2+\alpha^2}\right). \end{aligned} \tag{31}$$

It follows from Assumption 3 and the triangle inequality that

$$-\int_0^{\omega_L} \log|T(j\omega)|\,W(\omega,\alpha)d\omega \geq -\Omega_L. \tag{32}$$

Similarly, Lemma 1 implies that

$$-\int_{\omega_L}^{\omega_M} \log|T(j\omega)|W(\omega,\alpha)d\omega \geq -\Omega_\alpha, \tag{33}$$

where Ω_α is defined either by (28) or (30). Together, Assumption 1 and (14) imply that Re$\,(K_\alpha) \geq -q$. Also note

$$\begin{aligned}&\int_{\omega_M}^{\omega_H} \log|T(j\omega)|W(\omega,\alpha)d\omega \\ &\leq \max_{\omega\in[\omega_M,\omega_H]}\{\log|T(j\omega)|\}\int_{\omega_M}^{\omega_H} W(\omega,\alpha)d\omega.\end{aligned}$$

The result follows by combining the preceding inequalities and applying the high frequency bound (25). ■

It follows from Theorem 5 that time and frequency domain specifications, such as those in Assumptions 1-4, impose a *lower bound* on the peak value of $|T(j\omega)|$. For case (a), should this lower bound prove to be greater than unity, then it provides another sufficient condition for string instability. For case (b), already known to be string unstable, the lower bound provides an estimate of the severity of the instability.

In fact, the lower bound (26) is conservative for the purpose of predicting string instability in case (a). To see this, note that the first two terms on the right hand side of (31) will be nonnegative if $|T(j\omega)| \leq 1$ in the frequency range $(0, \omega_M)$. (If $|T(j\omega)| > 1$ in this frequency range, then the system is known to be string unstable without considering behavior at other frequencies.) Hence we have the following corollary to the proof of Theorem 5. For purposes of simplicity, we also assume that $L(s)$ has no ORHP zeros.

Corollary 2: In addition to the hypotheses of Theorem 5, assume that $|T(j\omega)| \leq 1,\ \forall\omega \in (0, \omega_M)$, and that $N_z = 0$. Then, for any $\omega_M \in (\alpha, \omega_H)$, we have that

$$\max_{\omega\in[\omega_M,\omega_H]} \log|T(j\omega)| \geq \frac{\Omega_H - \frac{\pi}{2}q}{\int_{\omega_M}^{\omega_H} W(\omega,\alpha)d\omega}. \tag{34}$$

■

It follows immediately from (34) that a necessary condition for string stability is that

$$q > \frac{2}{\pi}\Omega_H, \tag{35}$$

where Ω_H is defined by (29). If (35) is not satisfied, then the limit as $\omega_M \to \omega_H$ of the right hand side of (34) is equal to infinity, and thus the specifications are infeasible. Hence, the desirability of string stability imposes a tradeoff between bandwidth limitations of the form imposed in Assumption 4, and steady state tracking error requirements as imposed in Assumption 1.

V. Numerical Example

Consider a string of n identical oscillators with frequency $\alpha = 1$ and plant transfer function $P(s) = (s+0.5)/(s^2+1)$. A controller that achieves zero steady state phase error, $\arg(-K_\alpha) = 0$, is given by

$$C(s) = \frac{40(s+10)(s+2)}{s^2+0.05s+1.5}. \tag{36}$$

A plot of the lower bound (26) as a function of ω_H, the frequency at which the bandwidth limitation becomes effective, is given in Figure 2 for various values of the parameter q that governs the size of the tracking error via (18). As expected, smaller values of ω_H increase the size of the lower bound, and for a given value of ω_H, the bound increases as the constraint on the tracking error decreases.

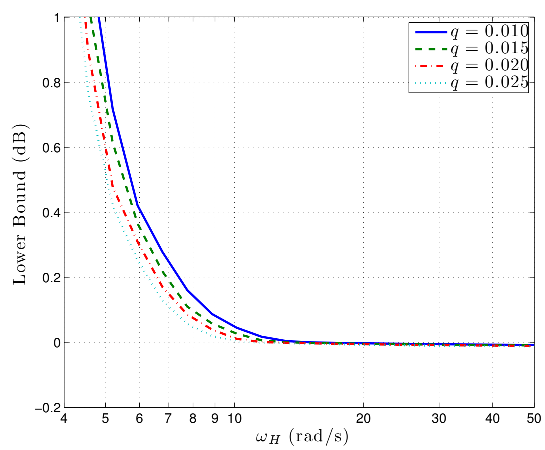

Fig. 2. The lower bound (26) vs. ω_H, for parameters $n = 10$, $r = 1$, $\epsilon = 0.1$, $u(10) = 1$, and $\omega_L = 0.6$.

The corresponding complementary sensitivity function is

$$T(s) = \frac{40s^3+500s^2+1040s+400}{s^4+40.05s^3+502.5s^2+1040s+401.5}. \tag{37}$$

As it happens, the DC gain of $|T(j\omega)|$ is nearly unity, and it is straightforward to verify that $T(j1) = 1$ and $K_\alpha = -0.001$. The Bode magnitude plot for (37), depicted in Figure 3, shows a peak value of 1.70 dB, or 1.22 in absolute terms. As a consequence, the string of oscillators is string unstable. The tracking errors (2) in response to

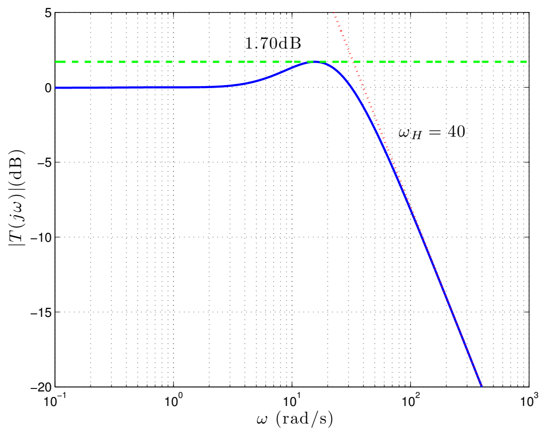

Fig. 3. Magnitude response of $T(s)$.

an input $r_1(t) = t \sin t$ are plotted in Figure 4, and show transient peaks that, as expected, increase in magnitude along the string. In all cases, the steady state value of the tracking error is given by $e_k^{ss}(t) = 0.001 \sin t$, as predicted from Theorem 3 and Corollary 1.

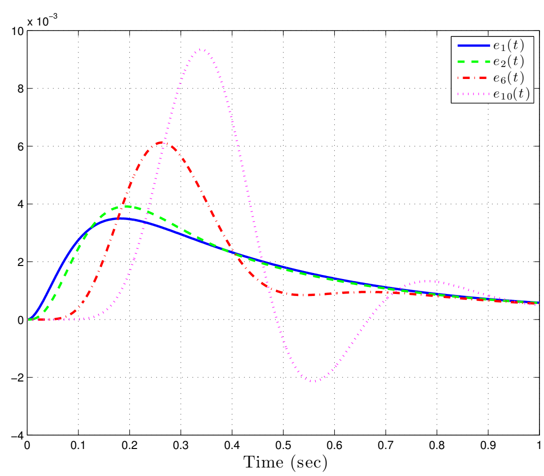

Fig. 4. Tracking errors $e_i(t)$ defined in (2).

To illustrate the bound (26), we find that the various parameters used to construct the bound have the values depicted in Table I. With these parameter values, we predict that $|T(j\omega)|$ must have a peak greater than 1.0146 (0.126 dB), which is less than the observed peak value of 1.70 dB. The difference is due in part to conservativeness in the lower bound (26), and in part due to controller design. A different controller might yield a smaller peak, but no smaller than the guaranteed lower bound provided that the rest of the design satisfied the parameter values from Table I.

TABLE I

PARAMETERS TO CALCULATE THE LOWER BOUND

n	ϵ	$u(n)$	q	r	ω_H	ω_L	ω_M
10	0.0367	0.072	0.001	1	40	0.536	1.95

VI. Conclusion and Future Directions

In this paper we have introduced the problem of string instability in a simple system of identical coupled harmonic oscillators. By using a new integral relation that must be satisfied by the complementary sensitivity function, we provided three sufficient conditions for string instability. In future work, we shall consider more complex control and communication strategies, and suppose that each oscillator may communicate with more than one other oscillator in the string. We will also consider heterogeneous strings, in which the controllers for different oscillators may be tuned differently. In doing so, we plan to develop results that parallel those for vehicle platoons in the recent comprehensive study [7]. We also plan to study the effects of time delays in communication, which will be important in practice.

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