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Robust non-fragile $\mathcal{H}_\infty$ control with regional pole location of discrete-time systems with multiple delays in the state

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***Abstract*— The problem of robust and non-fragile control by static state feedback gains assuring both the $\mathcal{H}_\infty$ guaranteed cost and regional pole location of the closed loop eigenvalues is proposed in this paper, for uncertain discrete-time system with multiple delays in the states. The regional pole location, or the $\mathcal{D}$-stabilization, concerns with the problem of locating the closed-loop system eigenvalues inside a circular region of the complex plane, called $\mathcal{D}(\alpha, r)$-region, with center in $(-\alpha, 0)$ and radius r. Besides this performance specification, the robust control gains are designed assuring an $\mathcal{H}_\infty$ guaranteed cost between an exogenous input and the output signals. An iterative algorithm is proposed to solve the conditions achieving better results than previous results in the literature. The robust gains that feedback the delayed states are designed in a non-fragile way. Contrary to the most of the approaches presented in the literature, it is possible to prescribe an explicit percentage of perturbation for elements of these gains. A numerical design example is given to show the effectiveness of the proposed conditions.**

I. Introduction

The objective of this paper is to study the design of robust and non-fragile feedback gains, assuring simultaneously regional pole location — $\mathcal{D}(\alpha, r)$-stabilization — and $\mathcal{H}_\infty$ guaranteed cost for discrete-time systems with multiple delays in the state. The non-fragility property is related to the fact that controllers can have implementation errors which can yield loss of robustness [7], [6].

In [7] it is shown that robust controllers assuring some performance index, such as $\mathcal{H}_\infty$, $\mathcal{H}_2$ or ℓ^1, can be *fragile*, that is, in some cases small perturbations in their parameters might result in closed-loop instability, despite the robust controller design. Moreover, the so-called *non-fragile controllers* admit some uncertainties in their parameters, keeping both the closed-loop stability and some performance index.

As presented in [6], the fragility problem usually defines a trade-off between the accuracy of controller implementation and the performance deterioration.

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Most of the results available in the literature employs the norm-bounded approach to model uncertainties in the parameters of the controller. This means that the uncertainties associated to the implementation of the controllers can be time-varying, which is not the practical case in general. In such context, a technique for design a non-fragile state feedback gains for discrete-time descriptor systems is presented in [17]. Another technique is proposed in [19] for the design of non-fragile dynamic output feedback controllers that are affected by additive uncertainties. Non-fragility is investigated for neutral systems in [1], where delay-dependent non-fragile $\mathcal{H}_\infty$ observer-based control is proposed. Both controllers and observers have additive gain variations. In [20], non-fragile guaranteed cost controllers for uncertain stochastic nonlinear systems with time-varying delays are designed by linear matrix inequalities (LMIs) conditions.

Despite the large number of results based on norm-bounded assumptions, the polytopic domain leads to an interesting framework for representing errors in controllers. Besides this, polytopic representation can handle time-invariant errors on the controller parameters in a less conservative fashion. Also, for decentralized control gains the implementation hardware may differs among each control gain element. The polytopic approach has been exploited in [16] where the non-fragility of a robust state feedback control gain has been addressed by an iterative algorithm based on LMIs. However, the requirements over the controller parameters are also given in an iterative way. Thus, the percent error for the achieved gains must be checked *a posteriori*.

In this paper the problem of non-fragility of state feedback gains is addressed in the context of discrete time systems with multiple state delays. Two performance specifications are included in the design: *i)* the $\mathcal{D}(\alpha, r)$-stabilization and *ii)* the guaranteed $\mathcal{H}_\infty$ cost between an exogenous input and the output of the system. The $\mathcal{D}(\alpha, r)$-stabilization concerns the design of a controller for a discrete-time systems with multiple delays, such that the closed-loop eigenvalues rely inside a circle centered at $(\alpha, 0)$ with radius r in the complex plane. This kind of regional pole location have been widely used in the literature, specially for delay free systems [9], [14], [3], and only a few deal with discrete-time systems with multiple delays in the state [12], [15], [2], [18]. Similarly, there is a wide set of works for controller design assuring an $\mathcal{H}_\infty$ guaranteed performance for delay free systems [5] as well as a number of results for discrete time

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systems with delay in the states [10]. But none of those approaches deal with the problem of controller fragility.

The objective here is to obtain conditions for $\mathcal{D}(\alpha, r)$-stabilization with $\mathcal{H}_\infty$ guaranteed performance robust controllers for systems with multiple delays in the states. Besides this, the gains that feedback the delayed states are required to be non-fragile. This non-fragility is in the sense that for a given bound on implementation errors in these gains does not affect neither the stability nor the assured performance level ($\mathcal{H}_\infty$ guaranteed cost and $\mathcal{D}(\alpha, r)$ stability) of the closed loop system. A side result is an algorithm that improves the search for solution of the optimization problems proposed in [15].

Notation: the notation used in this work is quite standard: $\mathbb{N}^*$, $\mathbb{R}$, $\mathbb{R}^+$ denote, respectively, the sets of natural numbers excluded the zero, real numbers and positive real numbers. M^{-1}, M' and M^H denote the inverse, the transpose and the hermitian of matrix M, respectively. ℓ_2 denote the space of sequence of the real vectors with finite energy and $\|x_k\|^2$ is the energy of $x_k \in \ell_2$. $\overline{\mathbf{z}}$ is the conjugate of $\mathbf{z}$. I and $\mathbf{0}$ denote the identity matrices and null, respectively, with appropriate dimensions. In the text, if necessary the dimensions of these matrices are identified by subindex. $M < \mathbf{0}$ ($M > \mathbf{0}$) means that the matrix M is negative defined (positive). The symbol $\star$ represents the symmetric blocks regarding diagonal.

II. Problem Statement

Consider the uncertain discrete-time system with multiple delays in the state given by

$$\mathcal{S}(\beta) : \begin{cases} x_{k+1} = & A_0(\beta)x_k + \sum_{\ell=1}^{L} A_\ell(\beta)x_{k-d_\ell} \\ & +B_u(\beta)u_k + B_w(\beta)w_k \\ z_k = & C_0(\beta)x_k + \sum_{\ell=1}^{L} C_\ell(\beta)x_{k-d_\ell} \\ & +D_u(\beta)u_k + D_w(\beta)w_k \end{cases} \tag{1}$$

where $x_k \in \mathbb{R}^n$, $u_k \in \mathbb{R}^q$, $\omega_k \in \mathbb{R}^{\tilde{q}}$ and $z_k \in \mathbb{R}^p$ denote the state vector, control input, exogenous input and measure output, respectively. The time-invariant delays are denoted by $d_\ell \in \mathbb{N}^*$, $\ell = 1, \ldots, L$ with $0 \le d_\ell \le \overline{d}_\ell$, with $\overline{d}_\ell = \max d_\ell$. The stabilization conditions proposed here can be established considering only the current state x_k, or both, the delayed states x_{k-d_ℓ} and x_k. The latter can be specially useful in batch processes, where d_ℓ may assume a different value between 0 and $\overline{d}_\ell$ at each run. All matrices in (1) belong to a polytopic domain parametrized on $\beta \in \Omega_\beta$

$$\Omega_\beta = \left\{ \beta \in \mathbb{R}^N : \beta_j \ge 0, \sum_{j=1}^{N} \beta_j = 1 \right\} \tag{2}$$

Thus, system $\mathcal{S}(\beta)$ is also described by a polytopic domain in terms of β. In this case each matrix in (1) is given by

$$R(\beta) = \sum_{i=1}^{N} \beta_i R_i, \quad \beta \in \Omega_\beta \tag{3}$$

where R can be replaced by any matrix of $\mathcal{S}(\beta)$. Matrices R_i, $i = 1, \ldots, N$, are vertex matrices with known values.

The following control law is considered

$$u_k = K_0 x_k + \sum_{\ell=1}^{L} K_\ell x_{k-d_\ell} \tag{4}$$

with $K_0 \in \mathbb{R}^{q\times n}$ and $K_\ell \in \mathbb{R}^{q\times n}$, $\ell = 1, \ldots, L$. Gains K_ℓ are subject to time-invariant uncertainties being modeled as

$$[K_\ell]_{ij} = (1 + \Delta_{\ell ij})\, [\tilde{K}_\ell]_{ij} \quad \begin{cases} i = 1, \ldots, q, \\ j = 1, \ldots, n, \\ \ell = 1, \ldots, L \end{cases} \tag{5}$$

where $[K_\ell]_{ij}$ and $[\tilde{K}_\ell]_{ij}$ correspond to the (i, j)-th element of the matrix K_ℓ and to the nominal matrix $\tilde{K}_\ell$, respectively. In (5), the uncertainty is casted by $\Delta_{\ell ij}$:

$$\Delta_{\ell ij} = \frac{\rho_{\ell ij}}{100\%}\delta_{\ell ij}, \quad |\delta_{\ell ij}| \le 1 \tag{6}$$

Note that, the time-invariant variables $\delta_{\ell ij}$ in (6) are not precisely known and are associated to each element $[K_\ell]_{ij}$, $i = 1, \ldots, q$, $j = 1, \ldots, n$, $\ell = 1, \ldots, L$. If $\delta_{\ell ij} = 0$, then $[K_\ell]_{ij} = [\tilde{K}_\ell]_{ij}$, i.e., it assumes its nominal value. Constants $\rho_{\ell ij}$ are given in percentage and are related to the maximum perturbation value affecting the nominal value of each element $[\tilde{K}_\ell]_{ij}$. Gains K_ℓ can be casted in a polytopic representation obtained by the combination of the extreme values of $\delta_{\ell ij}$ as

$$\mathcal{K}(\eta) = [K_1 \ \cdots \ K_L](\eta), \text{ with } \eta \in \Omega_\eta \tag{7}$$

where the number of vertices in $\mathcal{K}(\eta)$ is χ. Each vertex contains L representations of each of the controllers which are indicated by $[\mathcal{K}]_{\ell\kappa}$: the ℓ-th controller of the κ-th vertex of $\mathcal{K}$, $\kappa = 1, \ldots, \chi$, $\ell = 1, \ldots, L$. Thus, the representation introduced in (3) also applies to $K_\ell(\eta)$. For example, consider (4) with $L = 2$ and $K_\ell \in \mathbb{R}^{1\times 2}$, i.e., $K_1 = [\hat{k}_a \ \ \hat{k}_b]$ and $K_2 = [\hat{k}_c \ \ \hat{k}_d]$. In this case the polytope vertices of Ω_η are given by $\mathcal{K}_1 = [\hat{k}_a(1-\Delta_{111}) \ \ \hat{k}_b(1-\Delta_{112}) \ \ \hat{k}_c(1-\Delta_{211}) \ \ \hat{k}_d(1-\Delta_{212})]$, $\ldots$, $\mathcal{K}_4 = [\hat{k}_a(1-\Delta_{111}) \ \ \hat{k}_b(1-\Delta_{112}) \ \ \hat{k}_c(1+\Delta_{211}) \ \ \hat{k}_d(1+\Delta_{212})]$, $\mathcal{K}_5 = [\hat{k}_a(1-\Delta_{111}) \ \ \hat{k}_b(1+\Delta_{112}) \ \ \hat{k}_c(1-\Delta_{211}) \ \ \hat{k}_d(1-\Delta_{212})]$, $\ldots$, $\mathcal{K}_{16} = [\hat{k}_a(1+\Delta_{111}) \ \ \hat{k}_b(1+\Delta_{112}) \ \ \hat{k}_c(1+\Delta_{211}) \ \ \hat{k}_d(1+\Delta_{212})]$. Note that, only 4 variables are necessary to be determined despite the 16 vertices.

Uncertainties on K_0 are not addressed in this paper, but they can be handle using other approaches such as those in [16], [19], [17] but with the implementation errors given as absolute deviation values.

Using control law (4)-(6) into (1), it is possible to get the closed loop system

$$\hat{\mathcal{S}}(\beta,\eta):\begin{cases} x_{k+1} = & \hat{A}_0(\beta)x_k + \sum_{\ell=1}^{L}\hat{A}_\ell(\beta,\eta)x_{k-d_\ell} \\ & +B_\omega(\beta)\omega_k \\ z_k = & \hat{C}_0(\beta)x_k + \sum_{\ell=1}^{L}\hat{C}_\ell(\beta,\eta)x_{k-d_\ell} \\ & +D_\omega(\beta)\omega_k \end{cases} \tag{8}$$

with $\hat{A}_0(\beta) = A_0(\beta) + B_u(\beta)K_0$, $\hat{C}_0(\beta) = C_0(\beta) + D_u(\beta)K_0$, $\hat{A}_\ell(\beta,\eta) = A_\ell(\beta)+B_u(\beta)K_\ell(\eta)$ and $\hat{C}_\ell(\beta,\eta) = C_\ell(\beta) + D_u(\beta)K_\ell(\eta)$, $\ell = 1,\ldots,L$, and where each of these matrix belongs to polytopes established in terms of β or η.

The following definition is used to establish the main problem addressed in this paper.

Definition 1: System $\hat{\mathcal{S}}(\beta,\eta)$ is said $\mathcal{D}(\alpha,r)$-stable if all of its eigenvalues belong to the circle with center at $(-\alpha,0)$ and radius r of the complex plane, $\forall\beta\in\Omega_\beta$ and $\forall\eta\in\Omega_\eta$.

Problem 1: Determine, if possible, state feedback gains K and $\tilde{K}_\ell$ subject to uncertainties (5)–(6) such that the control law (4) robustly $\mathcal{D}(\alpha,r)$-stabilizes (1)–(6) and verifies

$$\|z_k\|_2 < \gamma\|\omega_k\|_2 \tag{9}$$

for all $\omega_k \in \ell_2$, $z_k \in \ell_2$, $\forall\ \beta \in \Omega_\beta$ and all admissible values of $\delta_{\ell ij}$ and $\rho_{\ell ij}$. In this case, γ is called an $\mathcal{H}_\infty$-guaranteed cost for the resulting uncertain closed-loop system and gains $\tilde{K}_\ell$, $\ell = 1,\ldots,L$ are said non-fragile.

To solve Problem 1, $\mathcal{S}(\beta)$ is considered with null initial conditions, i.e., $x_k = \mathbf{0}$, for $k = -\bar{h},\ldots,0$, and $\bar{h} = \max_{\ell=1,\ldots,L}\bar{d}_\ell$.

The following lemmas [12] are used in this paper.

Lemma 1: Consider real matrices X, Y and a complex matrix Q with $Q^H Q \leq I$. Then, for any scalar real $\lambda > 0$,

$$XQY + Y'Q^H X' \leq \lambda XX' + \frac{1}{\lambda}Y'Q^H QY \tag{10}$$

Lemma 2: System (8) with $B_\omega = \mathbf{0}$ is $\mathcal{D}(\alpha,r)$-stable if

$$U^H P_j U - r^2 P_j < 0, \text{ for } |v| \geq 1 \tag{11}$$

where $\mathbf{0} < P_j' = P_j \in \mathbb{R}^{n\times n}, j = 1,2,\ldots,N$ and $U = \hat{A}_{0j} - \alpha I + \sum_{i=1}^{L}\hat{A}_{ij}(rv+\alpha)^{-h_i}$.

III. Robust and non-fragile controller design

In this section it is provided a matrix inequality condition that solves Problem 1. This proposal encompasses the synthesis method proposed in [15] in two aspects. Firstly, it includes the non-fragility issue for the controller gain synthesis. Secondly, if the fragility issue is not taken into account, the present proposal encompasses the formulation in [15], providing, in general, less conservative results, thanks to a relaxation algorithm proposed in this section.

Theorem 1: Consider the uncertain discrete-time system with multiple delays in the states (1)-(3) with null initial conditions. This system is $\mathcal{D}(\alpha,r)$-stabilizable with $\mathcal{H}_\infty$-guaranteed cost γ by means of the robust state feedback gain K_0 and the robust and non-fragile gains $\tilde{K}_\ell$, $\ell = 1,\ldots,L$ subject to (5)-(7), if there exist a scalar $\tau = \min_{|v|\geq 1}|rv+\alpha|$ and optimization variables $0 < \theta \leq 1$, $\lambda_{ij} \in \mathbb{R}^+$; $i = 1,\ldots,L$, $j = 1,\ldots,N$, $H, W \in \mathbb{R}^{n\times n}$, $Z \in \mathbb{R}^{q\times n}$, $K_\ell \in \mathbb{R}^{n\times q}$, $\ell = 1,\ldots,L$, $\mathbf{0} < \tilde{P}_j^T = \tilde{P}_j \in \mathbb{R}^{n\times n}$, $j = 1,\ldots,N$, and $\sqrt{\mu} = \gamma \in \mathbb{R}_+$ such that (12) is verified for $j = 1,\ldots,N$, $\kappa = 1,\ldots,\chi$. In this case, the robust state feedback gain is given by

$$K_0 = ZW^{-1} \tag{13}$$

and the non-fragile gains K_ℓ, $\ell = 1,\ldots,L$, are obtained directly from the solution of (12). ■

The synthesis condition (12) can be casted in an optimization problem to minimize μ as follows

$$\Pi_{\mathcal{H}_\infty}:\begin{cases} \min\limits_{\substack{\mu>0,\lambda_{ij}>0,W,Z,H,\\ \tilde{P}_j=\tilde{P}_j^T>\mathbf{0},0<\theta<1,\\ K_\ell,\ell=1,\ldots,L}} & \mu \\ \text{such that} & \Pi_{j\kappa} < \mathbf{0} \end{cases} \tag{14}$$

Matrix inequalities in (12) are nonlinear due to the product between optimization variables H and W. This problem is overcame in [15] by replacing H by $\phi\mathbf{I}$ and performing a linear search on ϕ using a standard functions, such as `fminsearch(·)` found in MatLab. However, for some initial values of ϕ, the achieved result can be very conservative, as it is shown in section IV. The algorithm presented below is proposed to overcome this issue and it is used in this paper to solve optimization problem $\Pi_{\mathcal{H}_\infty}$.

Algorithm 1

$\epsilon \leftarrow$ *accuracy*, $\overline{count} \leftarrow$ *max iterations*, $i \leftarrow 0$, *count* $\leftarrow 0$, $\mu \leftarrow 10^3$.
For a given $0 < \phi \in \mathbb{R}_-^*$, solve (12), with $H_i \leftarrow \phi\mathbf{I}$. Get W and μ.
$W_i \leftarrow W$, $\mu_W \leftarrow \mu$.
repeat
  $i \leftarrow i + 1$
  Solve $\Pi_{\mathcal{H}_\infty}$,(14), with W_{i-1}. Get H and μ.
  $H_i \leftarrow H$, $\mu_H \leftarrow \mu$.
  Solve $\Pi_{\mathcal{H}_\infty}$,(14), with H_i. Get W and μ.
  $W_i \leftarrow W$, $\mu_W \leftarrow \mu$.
until $|\mu_H - \mu_W| < \epsilon$ **or** *count* $> \overline{count}$
$\gamma \leftarrow \sqrt{\mu_W}$.

It is worth of mention that the initial condition of Algorithm 1 involves the determination of $\phi < 0$, such that (12) is feasible with a high value of μ. In the experience of the authors, such value is very easy to find. Note that, differently from [15], this initial guess for ϕ does not need to be close to value of ϕ that leads to a minimal value of μ.

Another relevant characteristic is that a linear matrix inequality (LMI) set is solved at each step of Algorithm 1. In these LMIs, only one variable remains constant, H or W, allowing very good conditions for convergence and minimization of γ.

$$
\Pi_{j\kappa} = \begin{bmatrix} \sum_{i=1}^{L}\lambda_{ij}I + H + H^T & -HW & \mathbf{0} & \mathbf{0} & \mathbf{0} & \mathbf{0} \\ \star & -r^2\tilde{P}_j & W^T(A_{0j}-\alpha I)^T + Z^T B_{uj}^T & W^T C_{0j}^T + Z^T D_{uj}^T & \mathbf{0} & \mathbf{0} \\ \star & \star & \tilde{P}_j - W^T - W & \mathbf{0} & B_{\omega j} & F_{j\kappa} \\ \star & \star & \star & -\theta I & D_{\omega j} & G_{j\kappa} \\ \star & \star & \star & \star & -\mu I & \mathbf{0} \\ \star & \star & \star & \star & \star & -S_j \end{bmatrix} < \mathbf{0},
$$

$$
j = 1, 2, \cdots, N, \quad \kappa = 1, \ldots, \chi \quad (12)
$$

$$
F_{j\kappa} = \begin{bmatrix} A_{1j} + B_{uj}[\mathcal{K}]_{1\kappa} & \cdots & A_{Lj} + B_{uj}[\mathcal{K}]_{L\kappa} \end{bmatrix}, \quad G_{j\kappa} = \begin{bmatrix} C_{1j} + D_{uj}[\mathcal{K}]_{1\kappa} & \cdots & C_{Lj} + D_{uj}[\mathcal{K}]_{L\kappa} \end{bmatrix}
$$

$$
S_j = \mathrm{diag}\{\lambda_{1j}\tau^{2\overline{h}_1}I, \lambda_{2j}\tau^{2\overline{h}_2}I, \cdots, \lambda_{Lj}\tau^{2\overline{h}_L}I\},
$$

As usual in the LMI framework, conditions based on quadratic stability approach can be recovered from (12) by imposing $P_j = \tilde{P}_j = P$, $j = 1, \ldots, N$. In general, the obtained conditions lead to more conservative results than those obtained from the use of parameter dependent Lyapunov functions. Decentralized controller design can be easily incorporate by following steps similar to those presented in [10]. Finally, note that this algorithm, although takes LMIs in each step, is a non-convex formulation and thus cannot assure that the optimal value is achieved. In this case the final value of controllers yielded by the algorithm depends on the initial value of ϕ.

IV. Numerical example

Example 1: Consider system (1)-(2) given by

$$
A_0(\epsilon) = \begin{bmatrix} 0.1+\epsilon_1 & 1.0 \\ 0.05 & 0.8-\epsilon_1 \end{bmatrix}, A_1(\epsilon) = \begin{bmatrix} 0 & 0 \\ 0 & 0.1+\epsilon_2 \end{bmatrix},
$$

$$
A_2(\epsilon) = \begin{bmatrix} 0.1-\epsilon_2 & 0 \\ 0 & 0 \end{bmatrix}, \ B_u(\epsilon) = B_\omega(\epsilon) = \begin{bmatrix} 1.0+\epsilon_3 \\ 0.6-\epsilon_3 \end{bmatrix},
$$

$$
C_0(\epsilon) = \begin{bmatrix} 0 & 1 \end{bmatrix}, C_1(\epsilon) = C_2(\epsilon) = \begin{bmatrix} 0 & 0 \end{bmatrix}, D_u = D_\omega = 0
$$

with $|\epsilon_1| \leq 0.1$, $|\epsilon_2| \leq 0.05$ and $|\epsilon_3| \leq 0.1$, $L = 2$. In this case, the uncertain parameters define a polytopic representation with 8 vertex made up with all the combinations of the extreme values of ϵ_1, ϵ_2 and ϵ_3.

In this example Algorithm 1 is used to design robust gains K_ℓ, $\ell = 0, 1, 2$. Unlike the algoritm proposed in [15], where the final value of the $\mathcal{H}_\infty$-guaranteed cost, γ, depends on the initial value of ϕ, Algorithm 1 can achieved the same minimal value of γ, independently of the initial feasible value of ϕ. For instance, for $\mathcal{D}(0.1, 0.65)$ with $\phi = -100$, Algorithm 1 yields $\gamma = 4.8$ while in [15], this same initial value yields $\gamma = 23.55$. Thus, this confirm the advantage of using Algorithm 1 w.r.t. the one in [15].

Further, Algorithm 1 is used to design robust state feedback gains K_ℓ, $\ell = 0, 1, 2$ where gains $\hat{K}_1$ and $\hat{K}_2$ are also non-fragile. Consider the region given by $\mathcal{D}(0.1, 0.65)$ and the parameters of K_ℓ ($\ell = 1, 2$) having an error bound $\rho = 0.5\%$ (see (6)). In this case, Algorithm 1 achieves $K = [-0.0019 \ \ -1.1153]$, $\tilde{K}_1 = [0.0019 \ \ 0.0507]$ and $\tilde{K}_2 = [-0.0608 \ \ -0.0171]$ and $\gamma = 4.82$. Considering the same percent error but with a larger region $\mathcal{D}(0.05, 0.9)$, it was obtained $\gamma = 1.4079$ with gains $K_0 = [-0.1328 \ \ -0.9800]$, $\tilde{K}_1 = [-0.0638 \ \ -0.0149]$ and $\tilde{K}_2 = [-0.0215 \ \ -0.2028]$, which represents a reduction of 70.7% w.r.t. the $\mathcal{H}_\infty$ guaranteed cost obtained for region $\mathcal{D}(0.1, 0.65)$.

Note that, these two nominal non-fragile gains, $\tilde{K}_\ell$ define a polytope with 16 vertices that encompass variations of 0.5% in each element of $\tilde{K}_i$, considering all possible combinations.

To study how the $\mathcal{D}(\alpha, r)$ regions and admissible errors on $\tilde{K}_\ell$ affect the $\mathcal{H}_\infty$-guaranteed cost, it was implemented simulations for different regions (α, r) and the percent error in each element of $\tilde{K}_\ell$, $\rho_{\ell ij}$. In Figure 1 it is presented the relation between the radius r, percent error $\rho_{\ell ji}$ and γ for $\alpha = 0$, 0.1 and 0.2. As expected, the $\mathcal{H}_\infty$-guaranteed cost decreases as the value of r increases and the admissible error, $\rho_{\ell ij}$ decreases. In this figure, it is presented the solutions for the percent error varying between 0% and 1.5%. Also note the dependency of γ on values of α. For $\alpha = 0$, i.e, the $\mathcal{D}$-region with center in $(0, 0)$, the non-fragile controllers result in a $\mathcal{H}_\infty$-guaranteed cost smaller than that for $\alpha = 0.1$ or 0.2. For instance, for $\mathcal{D}(\alpha, 0.80)$, the values for γ, for $\alpha = 0$, 0.1 and 0.2 are, respectively, $\gamma = 1.57, 1.94$ and 2.6270. As mentioned in [7], [11], [8] one of the drawbacks of the fragility, when designing for a given $\mathcal{D}(\alpha, r)$-stability, is the fact that, in general, robust controllers tend to put "the closed-loop poles on the boundary of the specified $\mathcal{D}$-stability region". As shown in Figure 1, for this example, there is a straightforward relation between $\mathcal{H}_\infty$-guaranteed cost, $\mathcal{D}(\alpha, r)$ region and uncertainties in $[\tilde{K}_\ell]_{ij}$, which illustrates the trade-off between $\mathcal{D}(\alpha, r)$, γ and $\rho_{\ell ij}$.

V. Conclusion

In this paper, some design conditions for state feedback gains are addressed in the context of uncertain discrete-time systems with multiple delays in the state. It was considered the $\mathcal{D}(\alpha, r)$-stabilization and, simultaneously, the minimization of the $\mathcal{H}_\infty$-guaranteed cost. Besides, the gains used to feedback delayed states can have uncertainties in their parameters and, thus, are designed

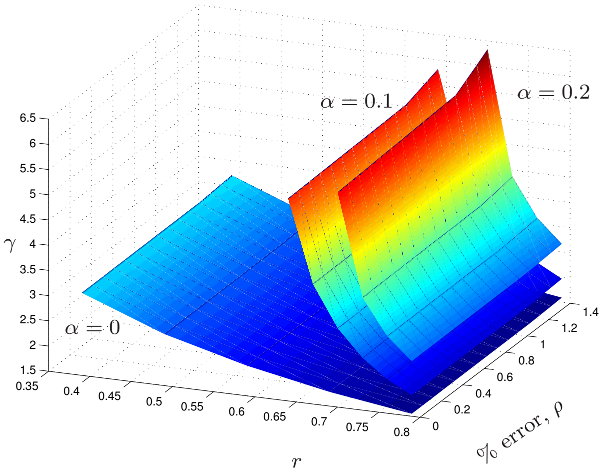

Fig. 1. Values of γ for different values of $\rho_{\ell ij} = \rho$ (the percent error), radius, r, and center, $(-\alpha, 0)$ of the $\mathcal{D}$-region.

in a non-fragile way. The proposed technique is based on a iterative algorithm where an LMI is solved at each step. The trade off between the minimal value of $\mathcal{H}_\infty$-guaranteed cost of the closed-loop system and the considered region $\mathcal{D}(\alpha, r)$ is evidenced by a numerical example. One advantage of the proposed algorithm is that the achieved value of the $\mathcal{H}_\infty$-guaranteed cost is not sensitive to the initial feasible values used to start the algorithm. The present proposal can deal with non-fragile gains design where the uncertainties in the controllers are considered time-invariant. Besides, the specification of these uncertainties can be done by a percentage value of the nominal controller. This is new w.r.t. the results found in the literature and can motivate new studies on non-fragile control design.

Appendix: proof of Theorem 1

Proof of Theorem 1

Only the main steps of the proof are presented. It is necessary to show that (12) is sufficient for the robust $\mathcal{D}(\alpha, r)$-stability of the closed-loop system (1)-(3) with K_0 given by (13) and $\tilde{K}_\ell$, $\ell = 1, \ldots, L$ obtained with the solution of (12).

Consider $V(\beta, x_k)$ a candidate to Lyapunov's function given by

$$V(\beta, x_k) = x_k^T \hat{P}(\beta) x_k > 0 \tag{15}$$

where $\hat{P}(\beta)$ is defined as in (3) and $\mathbf{0} < \hat{P}_j^T = \hat{P}_j \in \mathbb{R}^n$, ensuring the positiveness of (15). For the robust stability it is also required that

$$\Delta V(\beta, x_k) = V(\beta, x_{k+1}) - V(\beta, x_k) < 0 \tag{16}$$

Allying (16) with restriction (9) is possible to find [13]:

$$\Delta V(\beta, x_k) + z_k^T z_k - \gamma^2 \omega_k^T \omega_k < 0 \tag{17}$$

which, with Lemma 2, yields

$$x_{k+1}^T \hat{P}(\beta) x_{k+1} - r^2 x_k^T \hat{P}(\beta) x_k + z_k^T z_k - \gamma^2 \omega_k^T \omega_k < 0 \tag{18}$$

Thus, it is necessary to show that (12) is sufficient for (18).

Firstly, multiply (12) by η_κ, $\eta \in \Omega_\eta$ and sum it up from $\kappa = 1$ to χ. This yields a inequality where the gains K_ℓ, $\ell = 1, \ldots, L$, depend on η. Then, replace Z by $K_0 W$, multiply the obtained condition by β_j, $\beta \in \Omega_\beta$, and sum it up from $j = 1$ to N. Replace the closed-loop terms as given below equation (8). Here, the obtained condition is called $\Pi(\beta, \eta)$ which is negative definite. In the sequel, pre- and post-multiply the $\Pi(\beta, \eta) < \mathbf{0}$ by $\mathcal{T}_1$ and $\mathcal{T}_1^T$, with

$$\mathcal{T}_1 = \left[\begin{array}{cc|c} W & \mathbf{I}_n & \mathbf{0} \\ \hline \mathbf{0} & \mathbf{0} & \mathbf{I}_{(1+L)n+p+q} \end{array}\right] \tag{19}$$

and replace $W P(\beta) W^T$ by $\hat{P}(\beta)$. Note that block $(4,4)$ of $\mathcal{T}_1 \Pi(\beta, \eta) \mathcal{T}_1^T < \mathbf{0}$, $0 < \theta < 1$, can be rewritten as $-\varphi(\varphi - 2)$, $0 < \varphi < 2$. Then, apply the congruence transformation $\mathcal{T}_2 \mathcal{T}_1 \Pi(\beta, \eta) \mathcal{T}_1^T \mathcal{T}_2^T < \mathbf{0}$, where

$$\mathcal{T}_2 = \mathrm{diag}\{\mathbf{I}_2 \otimes W^{-1}, G, \mathbf{I}_{p+Ln}\} \tag{20}$$

with $G = -\dfrac{1}{\varphi}\mathbf{I}$, allowing to rewrite block $(4,4)$ as $\mathbf{I} + G + G^T$ (see [10, Theorem 3.3] for details). The resulting matrix inequality can be rewritten by using Schur's complement and Lemma 1 yielding

$$\Lambda(\beta, \eta) =$$

$$\left[\begin{array}{cccc} -r^2 \hat{P}(\beta) & \begin{pmatrix} \hat{A}_0(\beta) - \alpha\mathbf{I} \\ + \sum_{\ell=1}^{L} \hat{A}_\ell(\beta, \eta)(rv - \alpha)^{\overline{d}_\ell} \end{pmatrix}^T W^{-T} & \begin{array}{c} (\hat{C}_0(\beta) \\ + \sum_{\ell=1}^{L} \hat{C}_\ell(\beta, \eta)(rv - \alpha)^{\overline{d}_\ell})^T G^T \end{array} & \mathbf{0} \\ \star & \hat{P}(\beta) - W^{-T} - W^{-1} & \mathbf{0} & W^{-1} B_w(\beta) \\ \star & \star & \mathbf{I} + G + G^T & G D_w(\beta) \\ \star & \star & \star & -\mu\mathbf{I} \end{array}\right] < \mathbf{0} \tag{21}$$

This inequality can be rewritten as $\mathcal{Q}(\beta) + \mathcal{X}\mathcal{B}(\beta, \eta) + \mathcal{B}(\beta, \eta)^T \mathcal{X}^T < \mathbf{0}$, with

$$\mathcal{Q}(\beta) = \mathrm{diag}\{-r^2 \hat{P}(\beta),\ \hat{P}(\beta),\ I_p,\ \mu I_q\}, \tag{22}$$

and

$$\mathcal{X} = \begin{bmatrix} W^{-T} & \mathbf{0} & \mathbf{0} & \mathbf{0} & \mathbf{0} \\ \mathbf{0} & \mathbf{0} & G & \mathbf{0} & \mathbf{0} \end{bmatrix}^T$$

From the well known Finsler's Lemma, see [4], this is a sufficient condition for $\zeta_k^T \mathcal{Q}(\beta) \zeta_k < \mathbf{0}$ subject to $\mathcal{B}(\beta, \eta)\zeta_k = \mathbf{0}$ where

$$\zeta_k = \begin{bmatrix} x_k^T & x_{k+1}^T & z_k^T & \omega_k^T \end{bmatrix}^T \tag{23}$$

$$\mathcal{B}(\beta,\eta) = \begin{bmatrix} \hat{A}_0(\beta) - \alpha I + \sum_{\ell=1}^{L} \hat{A}_\ell(\beta,\eta)(rv+\alpha)^{-\overline{d}_\ell} \\ \hat{C}_0(\beta) + \sum_{\ell=1}^{L} \hat{C}_i(\beta,\eta)(rv+\alpha)^{-\overline{d}_\ell} \end{bmatrix.$$

$$\left. \begin{matrix} -\mathbf{I} & \mathbf{0} & B_\omega(\beta) \\ \mathbf{0} & -\mathbf{I} & D_\omega(\beta) \end{matrix} \right] \qquad (24)$$

which recovers (18), completing the proof.

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