2011 50th IEEE Conference on Decision and Control and
European Control Conference (CDC-ECC)
Orlando, FL, USA, December 12-15, 2011

Necessary and Sufficient Conditions for Stabilizability subject to Quadratic Invariance

Şerban Sabău and Nuno C. Martins

Abstract— **In this paper we deal with the problem of stabilizing linear, time–invariant plants using feedback control configurations that are subject to sparsity constraints. Recent results show that given a strongly stabilizable plant, the class of all stabilizing controllers that satisfy certain given sparsity constraints admits a convex representation via Zames's Q–parametrization. More precisely, if the pre–specified sparsity constraints imposed on the controller are *quadratically invariant* with respect to the plant, then such a convex representation is guaranteed to exist. The most useful feature of the aforementioned results is that the sparsity constraints on the controller can be recast as convex constraints on the Q–parameter, which makes this approach suitable for optimal controller design (in the $\mathcal{H}^2$ sense) using numerical tools readily available from the classical, centralized optimal $\mathcal{H}^2$ synthesis. All these procedures rely crucially on the fact that some stabilizing controller that verifies the imposed sparsity constraints is *a priori* known, while design procedures for such a controller to initialize the aforementioned optimization schemes are not yet available. This paper provides necessary and sufficient conditions for such a plant to be *stabilizable* with a controller having the given sparsity pattern. These conditions are formulated in terms of the existence of a doubly coprime factorization of the plant with additional sparsity constraints on certain factors. We show that the computation of such a factorization is equivalent to solving an exact model–matching problem. We also give the parametrization of the set of all decentralized stabilizing controllers by imposing additional constraints on the Youla parameter. These constraints are for the Youla parameter to lie in the set of all stable transfer function matrices belonging to a certain linear subspace.**

I. INTRODUCTION

In this paper we deal with the problem of stabilization and optimal synthesis for linear, time–invariant plants using feedback control configurations that are subject to sparsity constraints. In general, the sparsity constraints arise from the layout of the pre–specified information flows, such as when the overall controller consists of interconnected sub–controllers that can only act on certain entries of the controls vector while they only have access to a restricted set of entries of the measurements vector. For a general linear time–invariant plant G and a given graph of interconnections among sub–controllers, there is no theoretical framework that can provide a systematic analysis of the stabilization problem, let alone cope with additional optimality criteria. In fact, the solutions to particular instances of this problem have been known for a long time to be from notoriously difficult [1] to downright intractable [3], [2].

Ş. Sabău is with the Electrical and Computer Engineering Dept., Univ. of Maryland College Park, MD 20742-3285, USA email: sserban@isr.umd.edu.

N. C. Martins is with the Faculty of the Electrical and Computer Engineering Dept. and the Institute for Systems Research, University of Maryland, College Park, MD 20742-3285, USA. email: nmartins@isr.umd.edu

Work reported on this paper has been funded by NSF CNS grant 1037586, AFOSR MURI Grant "Distributed Learning and Information Dynamics in Networked Autonomous Systems" and the ONR AppEl Center, Task 12.

A. Previous Results

The authors of [12], [13], [14], [15], [16] identify certain specific distributed configurations (such as nested, chained, hierarchical, symmetric configurations, "funnel" causal systems, systems with delayed interaction and communication) that have the distinctive property that the set of all stabilizing controllers can be characterized via convex constraints on the Youla parameter Q. For example, in [15] for the case of plants having a triangular sparsity pattern, the problem of constraining the stabilizing controller to have itself a triangular sparsity pattern is recast as the condition on the Youla parameter Q to be triangular (via a doubly–coprime factorization in which all the factors are themselves triangular). Using the key attribute of convexity on the constraints of Q, a tractable, sequential algorithm is presented that computes the solution to an optimal control problem featuring a fairly general formulation of the performance cost. At each step, the algorithm yields a controller that complies with the sparsity constraints, together with a measure of its performance with respect to the optimal. In the subsequent work [17], the discussion is further elaborated taking into account possible uncertainties in the way sub–controllers are able to communicate information.

While the pioneering work in [12], [13], [14], [15], [16] succeeded to achieve convex parametrizations under the condition that the "structures" of the plant and controller must be invariant to cascade, parallel connections and inversion, the authors of [10] have showed that a more general class of configurations are those whose structure is "invariant under the feedback" transformation of the controller with the plant. This big leap forward was done in [10] by succeeding to provide a unifying treatment that encompasses many of the previously studied distributed structures and outlining the largest known class of convex problems in decentralized control. The main result in [10] revolves around a convex parametrization, whose existence is assured by a necessary and sufficient, algebraic test (named quadratic invariance) involving only the sparsity pattern of the plant and the sparsity constraints to be imposed on the controller. More precisely, it is shown via Zames's Q–parametrization, that assuming the hypothesis of quadratic invariance, given a strongly stabilizable plant, the class of all stabilizing controllers that satisfy certain given sparsity constraints admits a convex representation. The most useful feature of the aforementioned results is that the sparsity constraints on the controller can also be recast as convex constraints on the Q–parameter, which makes this approach suitable for optimal controller design (in the $\mathcal{H}^2$ sense) using numerical tools readily available from the classical, centralized optimal $\mathcal{H}^2$ synthesis. Later, this method has been extended to the general (not necessarily strongly stabilizable) case in [11], [24], via the coordinate–free parametrization of all stabilizing controllers.

B. Motivation and Scope of Paper

All the available algorithms for optimal synthesis ([10], [11]) rely crucially on the fact that some stabilizing controller that verifies the imposed sparsity constraints is *a priori* known, while synthesis methods for such a controller, (needed to initialize the aforementioned optimization schemes) are not yet available. This provided the motivation to the work presented here, as in this paper we develop necessary and sufficient conditions for such a plant to be *stabilizable* with a controller having the pre–selected sparsity pattern. These conditions are formulated in terms of the existence of a of doubly coprime factorization of the plant, with additional sparsity constraints on certain factors. We show that the computation of such a factorization is equivalent to solving an exact model–matching problem. We also give the parametrization of the set of all decentralized stabilizing controllers by imposing additional constraints on the Youla parameter. These constraints are for the Youla parameter to lie in the set of all stable transfer function matrices belonging to a certain linear subspace.

These results, were the missing link that prevented the full exploitation of the powerful tools from [10]. They allow us to initialize the tractable formulations of the optimal disturbance attenuation problem ([10]) and the optimal mixed sensitivity problem with sparsity constrained controllers ([11], [24]).

C. Outline of the Document

This paper is organized as follows: after the introductive section we follow with a preliminaries section, introducing the feedback control stabilization problem and a short primer on coprime factorizations of LTI systems. The third section contains mostly notation and introduces the notion of sparsity constraints for linear systems along with a summary of the main results on quadratic invariance from [10]. The fourth section contains the main results of this paper. We provide a necessary and sufficient condition for a plant to be *stabilizable* with a controller satisfying a pre–selected sparsity pattern that is quadratically invariant with respect to the plant. These conditions are formulated in terms of the existence of a doubly coprime factorization of the plant with additional sparsity constraints on certain factors. We prove that the computation of this particular doubly coprime factorization (when it does exist) is equivalent to solving an exact model–mathing problem. Along the way we obtain the set of all decentralized stabilizing controllers, characterized via the Youla parametrization. The sparsity constraints on the controller are recast as a linear subspace type of constraint on the Youla parameter.

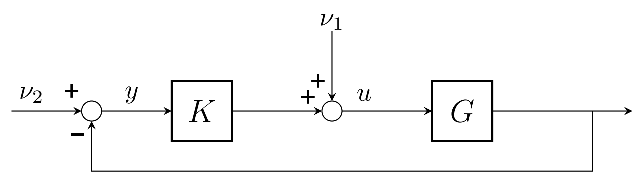

Fig. 1. Standard unity feedback interconnection

II. Preliminaries

Throughout this paper we make the leading assumptions that all systems are linear and time invariant (LTI), finite dimensional, proper, with either continuous or discrete–time. We deal with the frequency domain input/output operators of LTI systems. These operators are transfer function matrices (TFM), meaning matrices with all entries real–rational functions. By $\mathbb{R}(\lambda)$ we denote the set of all real–rational functions and by $\mathbb{R}(\lambda)^{n_y \times n_u}$ the set of $n_y \times n_u$ matrices having all entries in $\mathbb{R}(\lambda)$. The undeterminate λ is either s for continuous–time systems or z for discrete–time systems, respectively. Almost everywhere in the sequel, the λ argument following a TFM is omitted if it is clear from the context.

This paper gives a unified treatment for both the continuous and discrete-time cases. Henceforth, we will denote by Ω the open left half complex plane or the open unit disk, according to the type of system: continuous or discrete–time, respectively. The standard interpretation of Ω in systems theory is related to the *stability domain* of linear systems. We qualify a TFM $G(\lambda)$ as *stable* if all its poles are in Ω.

A. The Control Problem

In Fig.1 we depict the standard feedback interconnection between a plant and a controller, with the plant G belonging to $\mathbb{R}(\lambda)^{n_y \times n_u}$ and the controller K in the set $\mathbb{R}(\lambda)^{n_u \times n_y}$. Here, ν_1 and ν_2 are the disturbances and sensor noise, respectively. In addition, u is the control and y are the measurements. The integers n_u and n_y denote the dimensions of u and y respectively. Denote by $H(G, K) \in \mathbb{R}(\lambda)^{(n_u+n_y)\times(n_u+n_y)}$ the TFM from $[\nu_1^T \quad \nu_2^T \quad]^T$ to $[y^T \;\; u^T \;\;]^T$ (provided that the feedback loop is *well–posed*, *i.e.* $(I + KG)$ is invertible as a TFM). For the complete expressions of $H(G, K)$ in terms of G and K, we refer the reader to [4, Ch. 5.1, (7)]. If the transfer matrix $H(G, K)$ is *stable* we say that K is a *stabilizing controller* of G or equivalently that K *stabilizes* G. If a stabilizing controller of G exists, we say that G is *stabilizable*.

B. Coprime and Doubly Coprime Factorization for LTI Systems

Let $G(\lambda)$ be an arbitrary $n_y \times n_u$ TFM and Ω the stability domain in the complex plane. A *right coprime factorization* (RCF) of G over Ω is a fractional representation of the form $G = NM^{-1}$, with N and M having poles only in Ω, and for which $YM + XN = I$ holds for certain TFMs X and

Y with poles in Ω ([4, Ch. 4, Corollary 17]). Analogously, a *left coprime factorization* (LCF) of G (over Ω) is defined by $G = \widetilde{M}^{-1}\widetilde{N}$, where $\widetilde{N}$ and $\widetilde{M}$ are TFMs having poles only in Ω and satisfying $\widetilde{M}\widetilde{Y} + \widetilde{N}\widetilde{X} = I$ for certain TFMs $\widetilde{X}$ and $\widetilde{Y}$ with all poles in Ω. Due to the natural interpretation of the coprime factorizations as fractional representations, the invertible $\widetilde{M}$ and M factors are sometimes called the "denominator" TFMs of the coprime factorization.

Definition 2.1: [4, Ch.4, Remark pp. 79] A collection of eight TFMs $\big(M(\lambda), N(\lambda),\ \widetilde{M}(\lambda), \widetilde{N}(\lambda),\ X(\lambda), Y(\lambda),\ \widetilde{X}(\lambda), \widetilde{Y}(\lambda)\big)$ having all poles in Ω is called a *doubly coprime factorization* (DCF) of $G(\lambda)$ over Ω if the "denominator" TFMs $\widetilde{M}(\lambda)$ and $M(\lambda)$ are invertible and satisfy $G(\lambda) = \widetilde{M}(\lambda)^{-1}\widetilde{N}(\lambda) = N(\lambda)M(\lambda)^{-1}$ and

$$\begin{bmatrix} Y(\lambda) & X(\lambda) \\ -\widetilde{N}(\lambda) & \widetilde{M}(\lambda) \end{bmatrix} \begin{bmatrix} M(\lambda) & -\widetilde{X}(\lambda) \\ N(\lambda) & \widetilde{Y}(\lambda) \end{bmatrix} = I_{n_y+n_u}. \quad (1)$$

To avoid excessive terminology throughout this paper, we will simply refer to doubly coprime factorizations over Ω simply as doubly coprime factorizations (DCFs).

C. The Youla Parametrization of All Stabilizing Controllers

The following theorem is a central result in linear systems theory. We state it next, as it stands at the core of our main result.

Theorem 2.2: **(Youla)** [4, Ch.5, Theorem 1] Given a plant with the TFM $G \in \mathbb{R}(\lambda)^{n_y \times n_u}$, and *any* of its DCF (1), the set of *all* controllers K stabilizing G (in the standard feedback configuration from Figure 1) is given by

$$\begin{aligned} K &= \big(\widetilde{X} + MQ\big)\big(\widetilde{Y} - NQ\big)^{-1} \\ &= \big(Y - Q\widetilde{N}\big)^{-1}\big(X + Q\widetilde{M}\big) \end{aligned} \quad (2)$$

with Q any stable TFM in the set $\mathbb{R}^{n_u \times n_y}(\lambda)$.

Definition 2.3: Given the plant G and a certain DCF (1) of G, when taking the Youla–parameter Q equal to zero in (2) we get $K = \widetilde{X}\widetilde{Y}^{-1} = Y^{-1}X$, which is called the *central controller* (associated with the corresponding DCF (1)).

III. Feedback Control Configurations with Sparsity Constraints

Throughout this paper, the information constraints that are to be imposed on the controller are modeled via *sparsity constraints* ([10, pp. 283]). The precise formulation of the sparsity constrained stabilization problem is achieved by imposing a certain pre–selected sparsity pattern on the set of admissible stabilizing controllers. The notation we introduce next is entirely concordant with the one used in [9], [10].

A. Conformal Block Partitioning

For $p \geq 1$, we denote the set of integers from 1 to p as $\overline{1,p}$. Throughout the sequel we consider that the transfer function matrix $G(\lambda) \in \mathbb{R}(\lambda)^{n_y \times n_u}$ is partitioned in p block–rows and m block–columns. The i-th block–row has n_y^i rows, while the j-th block–column has n_u^j columns. Obviously, $\sum_{i=1}^{p} n_y^i = n_y$ and $\sum_{j=1}^{m} n_u^j = n_u$. For every pair (i,j) in the set $\overline{1,p} \times \overline{1,m}$, we denote by $[G]_{ij} \in \mathbb{R}^{n_y^i \times n_u^j}(\lambda)$ the $n_y^i \times n_u^j$ TFM at the intersection of the i-th block–row and j-th block–column of $G(\lambda)$. Accordingly,

$$G(\lambda) = \begin{bmatrix} [G]_{11} & \dots & [G]_{1m} \\ \vdots & & \vdots \\ [G]_{p1} & \dots & [G]_{pm} \end{bmatrix}, \quad [G]_{ij} \in \mathbb{R}^{n_y^i \times n_u^j}(\lambda). \quad (3)$$

Henceforth, we shall use this square bracketed notation for block indexing of transfer function matrices.

Analogously, the controller's transfer function matrix $K(\lambda) \in \mathbb{R}^{n_u \times n_y}(\lambda)$ is partitioned in m block–rows and p block–columns, where the j-th block–row has n_u^j rows and the i-th block–column has n_y^i columns. Correspondingly, $[K]_{ji}$ is the notation for the $n_u^j \times n_y^i$ TFM at the intersection of the j-th block–row and i-th block–column of $K(\lambda)$.

B. Sparsity Constraints

For the boolean algebra, the operations $(+,\cdot)$ are defined as usual: $0+0 = 0\cdot 1 = 1\cdot 0 = 0\cdot 0 = 0$ and $1+0 = 0+1 = 1+1 = 1\cdot 1 = 1$. By a binary matrix we mean a matrix whose entries belong to the set $\{0,1\}$. With the usual extension of notation, $\{0,1\}^{m\times p}$ stands for the set of all binary matrices with m rows and p columns. The addition and multiplication of binary matrices is carried out in the usual way, keeping in mind that the binary operations $(+,\cdot)$ follow the boolean algebra.

Binary matrices are denoted by capital letters with the "bin" superscript, in order to be distinguished from transfer function matrices over $\mathbb{R}(\lambda)$, which are represented in the sequel by plain capital letters. Henceforth, we adopt the convention that the transfer function matrices are indexed by blocks while binary matrices are indexed by each individual entry.

Furthermore, for binary matrices only, having the same dimensions, the notation $A^{\mathrm{bin}} \leq B^{\mathrm{bin}}$ means that $a_{ij} \leq b_{ij}$ for all i and j.

With the conformable block partitioning for K introduced in Subsection III-A, for any $K \in \mathbb{R}(\lambda)^{n_u \times n_y}$, define $\mathrm{Pattern}(K) \in \{0,1\}^{m\times p}$ to be the binary matrix

$$\mathrm{Pattern}(K)_{ij} \overset{def}{=} \begin{cases} 0 & \text{if the block } [K]_{ij} = 0; \\ 1 & \text{otherwise}. \end{cases} \quad (4)$$

Conversely, for any binary matrix with m rows and p columns, $K^{\mathrm{bin}} \in \{0,1\}^{m\times p}$, we can define the following linear subspace of $\mathbb{R}(\lambda)^{n_u \times n_y}$:

$$\mathrm{Sparse}(K^{\mathrm{bin}}) \overset{def}{=} \Big\{K \in \mathbb{R}(\lambda)^{n_u \times n_y} \,\Big|\, \mathrm{Pattern}(K) = K^{\mathrm{bin}}\Big\} \quad (5)$$

Hence $\mathrm{Sparse}(K^{\mathrm{bin}})$ is the set of all controllers K in the set $\mathbb{R}(\lambda)^{n_u \times n_y}$ for which $[K]_{ij} = 0$ whenever $K^{\mathrm{bin}}_{ij} = 0$. Accordingly, the binary value of $\mathrm{Pattern}(K)_{kl}$ determines whether controller k may read the block-row l of the output of the plant G.

Let $K^{\mathrm{bin}} \in \{0,1\}^{m\times p}$ be the pre-specified sparsity pattern to be imposed on the controller. Define the linear subspace $\mathcal{S}$ of $\mathbb{R}(\lambda)^{n_u \times n_y}$ as:

$$\mathcal{S} \stackrel{def}{=} \left\{ K \in \mathbb{R}(\lambda)^{n_u \times n_y} \,\middle|\, \text{Pattern}(K) \leq K^{\text{bin}} \right\}, \quad (6)$$

that is, the set of controllers whose transfer function matrices satisfy the imposed sparsity structure. With the terminology from [10], the linear space $\mathcal{S}$ (of admissible, sparsity constrained, causal controllers) will be called the *information constraint*.

The following matrix G^{bin} in the set $\{0,1\}^{p\times m}$ is the sparsity pattern of the plant which is defined as:

$$G^{\text{bin}} \stackrel{def}{=} \text{Pattern}(G) \quad (7)$$

Finally, from the matrix multiplication of matrices over $\mathbb{R}(\lambda)$ we note that for any $K \in \mathbb{R}(\lambda)^{n_u \times n_y}$ and any $G \in \mathbb{R}(\lambda)^{n_y \times n_u}$ with arbitrary sparsity patterns

$$\text{Pattern}(K\,G) \leq \text{Pattern}(K)\,\text{Pattern}(G). \quad (8)$$

C. Quadratic Invariance

Assumption 1. From this point on we make the assumption on the plant G to be strictly proper, *i.e.* for any of the entries of the transfer function matrix G (which is a rational function) the degree of the denominator is strictly greater than the degree of the numerator.

Definition 3.1: [10, Definition 13] Given the plant $G \in \mathbb{R}(\lambda)^{n_y \times n_u}$ and the subset $\mathcal{S}$ of $\mathbb{R}(\lambda)^{n_u \times n_y}$, we call $\mathcal{S}$ *inert* with respect to G if it satisfies the definition in [10, Definition 13].

Remark 3.2: [10] Throughout this section, both for continuous–time and discrete–time systems, the constraint set $\mathcal{S}$ is always inert with respect to the plant G, since G is assumed strictly proper and $\mathcal{S}$ is a subset of the set of proper LTI systems. Note also, that $\mathcal{S}$ is a closed set since it is a linear subspace (6).

Definition 3.3: [10, Definition 2] Given the plant $G \in \mathbb{R}(\lambda)^{n_y \times n_u}$ and the set $\mathcal{S} \subset \mathbb{R}(\lambda)^{n_u \times n_y}$, the set $\mathcal{S}$ is called *quadratically invariant* under the plant G if

$$KGK \in \mathcal{S} \qquad \text{for all } \; K \in \mathcal{S}. \quad (9)$$

Definition 3.4: Define the feedback transformation of G with K, as the following function from $\mathbb{R}(\lambda)^{n_u \times n_y}$ to $\mathbb{R}(\lambda)^{n_u \times n_y}$

$$h_G(K) \stackrel{def}{=} K\big(I + GK\big)^{-1}. \quad (10)$$

Proposition 3.5: The feedback transformation $h_G(\cdot)$ from (10) is an invertible function from $\mathbb{R}(\lambda)^{n_u \times n_y}$ to $\mathbb{R}(\lambda)^{n_u \times n_y}$ and its inverse is given by

$$h_G^{-1}(K) \stackrel{def}{=} K\big(I - GK\big)^{-1}. \quad (11)$$

Proof: First note that $h_G(\cdot)$ from (10) is indeed a well–posed function from $\mathbb{R}(\lambda)^{n_u \times n_y}$ to $\mathbb{R}(\lambda)^{n_u \times n_y}$, due to fact that the inverse of $\big(I+GK\big)$ exists for any K in $\mathbb{R}(\lambda)^{n_u \times n_y}$. This is guaranteed by the fact that K is proper and G is strictly proper (Assumption 1). The rest of the proof follows by direct algebraic computations and is omitted for brevity. ■

We restate next, for ease of reference, the main result from [9], [10], frequently invoked throughout the next section.

Theorem 3.6: [10, Theorem 14] Given the plant $G \in \mathbb{R}(\lambda)^{n_y \times n_u}$, the set $\mathcal{S} \subseteq \mathbb{R}(\lambda)^{n_u \times n_y}$ closed, inert with respect to G and quadratically invariant under G, then

$$\mathcal{S} \text{ is quadratically invariant under } G \Longleftrightarrow h_K(\mathcal{S}) = \mathcal{S}. \quad (12)$$

Assumption 2. Throughout this entire paper, we assume that the set $\mathcal{S}$ that defines the sparsity constraints to be imposed on the controller is quadratically invariant under the plant G.

IV. Main Result

In this section we develop a necessary and sufficient condition for a plant to be *stabilizable* with a controller satisfying a pre–selected sparsity pattern that is quadratically invariant with respect to the plant. These conditions are formulated in terms of the existence of a doubly coprime factorization of the plant featuring additional sparsity constraints on certain factors. This result has an especially important computational value, as it turns out that such a factorization (when it exists) is equivalent to solving for the Youla parameter a TFM linear equation (an exact model matching problem) .

The following preparatory result will be needed.

Proposition 4.1: Given any DCF (1) of the plant G denote by $K = \widetilde{X}\,\widetilde{Y}^{-1} = Y^{-1}X$ the "central" controller (from Definition 2.3). Then the following identities hold

$$\begin{aligned} MX &= \big(I + KG\big)^{-1}K, \\ \widetilde{X}\widetilde{M} &= K\big(I + GK\big)^{-1}. \end{aligned} \quad (13)$$

Proof: For the proof we refer to [24]. ■

The next Theorem makes out for the main result of this paper.

Theorem 4.2: Given a plant G in the set $\mathbb{R}(\lambda)^{n_y \times n_u}$ then G is stabilizable with a sparsity constrained controller K belonging to the set $\mathcal{S}$ if and only if there exists a DCF (1) of G such that

$$\text{Pattern}(\widetilde{X}\widetilde{M}) \leq K^{\text{bin}} \quad \text{or} \quad \text{Pattern}(MX) \leq K^{\text{bin}}. \quad (14)$$

Proof: Throughout the proofs, we shall make use of the following identities (that hold true in any ring, provided the inverses involved exist).

$$(I + AB)^{-1}A = A(I + BA)^{-1}, \quad (15)$$

$$(I + AB)^{-1} = I - A(I + BA)^{-1}B. \quad (16)$$

"Necessity". Suppose that there exists a stabilizing controller K in the set $\mathcal{S}$. Then as a consequence of Youla's Theorem 2.2, there exists a DCF (1) of the plant G for which K is the central controller. According to Proposition 4.1 we get from (13) that

$$\widetilde{X}\widetilde{M} = K\big(I + GK\big)^{-1}. \quad (17)$$

We apply the Pattern operator (4) on both sides of equation (17) and using Definition 3.4 get that $\text{Pattern}(\widetilde{X}\widetilde{M}) = \text{Pattern}(h_G(K))$. But $h_G(K)$ belongs to $\mathcal{S}$ because of

Assumption 2 and Theorem 3.6 and so $\mathrm{Pattern}(h_G(K)) \leq K^{\mathrm{bin}}$.

For $\mathrm{Pattern}(MX)$ we employ (13) and identity (15) to get that $\mathrm{Pattern}(MX) = \mathrm{Pattern}(h_G(K))$. Then by the same arguments as before we also get that $\mathrm{Pattern}(MX) \leq K^{\mathrm{bin}}$.

"Sufficiency". Suppose that $\mathrm{Pattern}(\widetilde{X}\widetilde{M}) \leq K^{\mathrm{bin}}$ holds, hence $\widetilde{X}\widetilde{M}$ belongs to the set $\mathcal{S}$. Take each side of (17) as an argument for $h_G^{-1}(\cdot)$ in order to get via Definition 3.4 that $h_G^{-1}(\widetilde{X}\widetilde{M}) = h_G^{-1}(h_G(K))$ and equivalently that $K = h_G^{-1}(\widetilde{X}\widetilde{M})$. Furthermore, via Proposition 3.5, Assumption 2 and Theorem 3.6 we get that $h_G^{-1}(\mathcal{S}) = \mathcal{S}$ which in turn implies that $h_G^{-1}(\widetilde{X}\widetilde{M})$ belongs to the set $\mathcal{S}$. This means that $K = h_G^{-1}(\widetilde{X}\widetilde{M})$ is also in $\mathcal{S}$.

The sufficiency of the second condition ($\mathrm{Pattern}(MX) \leq K^{\mathrm{bin}}$) follows by a similar line of reasoning and so is omitted for brevity.

■

Kronecker Products and Linear Matrix Equations([7, Chapter 13]) Given two matrices $P \in \mathbb{R}(\lambda)^{a\times b}$ and $S \in \mathbb{R}(\lambda)^{c\times d}$ let the Kronecker product of P and S be denoted by $P \otimes S$ and belonging to the set $\mathbb{R}(\lambda)^{ac\times bd}$. Given the matrix P, we write P in terms of its columns as

$$P = \left[\begin{array}{cccc} p_1 & p_2 & \dots & p_a \end{array}\right]$$

and then associate a column vector $\mathrm{vec}(P) \in \mathbb{R}(\lambda)^{ab}$ defined as

$$\mathrm{vec}(P) \stackrel{def}{=} \left[\begin{array}{c} p_1 \\ \vdots \\ p_a \end{array}\right]. \tag{18}$$

All the presented results related to matrix vectorization and Kronecker products do not depend in any way on the ring of matrices involved, therefore they are valid for the ring of TFMs (matrices over the field of real–rational functions).

Proposition 4.3: [7, Theorem 13.26] Let $P \in \mathbb{R}(\lambda)^{a\times b}$, $R \in \mathbb{R}(\lambda)^{b\times c}$ and $S \in \mathbb{R}(\lambda)^{c\times d}$. Then

$$\mathrm{vec}(PRS) = (S^T \otimes P)\mathrm{vec}(R) \tag{19}$$

A. Outline of the Sparse Controller Synthesis Algorithm

In this subsection, given the plant G we provide a numerically tractable algorithm (based on Theorem 4.2 above) for the computation of a sparse, stabilizing controller, belonging to the set $\mathcal{S}$ (when such a controller exists). We start with *any* DCF (1) of the plant, which can be computed using the standard state–space techniques from [6]. If this DCF satisfies relations (14) then according to Theorem 4.2 its associated central controller will be in the set $\mathcal{S}$.

Suppose now that this DCF we start with does not satisfy (14), which is generically speaking the case. An immediate consequence of Youla's Theorem 2.2 states that for any Youla parameter Q, the following identity represents another DCF of the plant G

$$\left[\begin{array}{cc} (Y - Q\widetilde{N}) & (X + Q\widetilde{M}) \\ -\widetilde{N} & \widetilde{M} \end{array}\right] \left[\begin{array}{cc} M & -(\widetilde{X} + MQ) \\ N & (\widetilde{Y} - NQ) \end{array}\right] = I. \tag{20}$$

We want to find that particular Youla parameter Q, for which the factors of the newly obtained DCF (20) satisfy the relations (14), namely that

$$\mathrm{Pattern}\Big((\widetilde{X} + MQ)\widetilde{M}\Big) \leq K^{\mathrm{bin}} \quad \text{or}$$
$$\mathrm{Pattern}\Big(M(X + Q\widetilde{M})\Big) \leq K^{\mathrm{bin}}$$

or equivalently

$$\mathrm{Pattern}\big(MQ\widetilde{M} + \widetilde{X}\widetilde{M}\big) \leq K^{\mathrm{bin}} \quad \text{or}$$
$$\mathrm{Pattern}\big(MQ\widetilde{M} + MX\big) \leq K^{\mathrm{bin}}. \tag{21}$$

Corollary 4.4: Given a plant G in the set $\mathbb{R}(\lambda)^{n_y\times n_u}$ then G is stabilizable with a sparsity constrained controller K belonging to the set $\mathcal{S}$ if and only if, starting from *any* DCF (1) of G, there exists a Youla parameter Q (stable TFM, belonging to the set $\mathbb{R}(\lambda)^{n_u\times n_y}$) such that (21) holds.

Proof: "Sufficiency" If there exists a Youla parameter Q, such that (21) holds, then exactly as in the "Sufficiency" part of the proof of Theorem 4.2, the controller (depending on Q) $K = h_G^{-1}\Big((\widetilde{X} + MQ)\widetilde{M}\Big)$ will belong to the set $\mathcal{S}$.

"Necessity" Suppose that a stabilizing controller K of G, belonging to the set $\mathcal{S}$ does exist and we consider K fixed. Then, a direct consequence of Youla's Theorem 2.2 states that for any DCF (1), there exist a (unique) Youla parameter Q (depending on the DCF), such that $K = \big(\widetilde{X} + MQ\big)\big(\widetilde{Y} - NQ\big)^{-1}$ (is the central controller associated with the DCF (20) of G). Then exactly as in the "Necessity" part of the proof of Theorem 4.2, it follows that (20) must satisfy (21).

■

Remark 4.5: We will provide our further argumentation only for the first relation from (21), since all the needed results for the second relation from (21) follow by a similar line of reasoning.

B. Sparse Controller Synthesis as An Exact Model–Matching Problem

For the remaining part of this section only, we briefly revisit the assumptions made in Subsection III-A. Specifically, we make the assumption that all the blocks in the conformal partition (3) of the plant G have the size 1×1, meaning that $\forall i \in \overline{1,p}$ and $\forall j \in \overline{1,m}$ it holds that $n_y^i = n_u^j = 1$. This hypothesis does not imply any loss of generality whatsoever, since all the vectorization and matrix Kronecker product results can be naturally adapted when the factors involved are conformally block–partitioned. However, this hypothesis does considerably simplify the notation while outlining all the essential ideas needed for the proof of the general case (for any conformal block–partition (3) of G).

As a consequence of the assumption made at the beginning of the current subsection we get (see Subsection III-A) that $G \in \mathbb{R}(\lambda)^{p\times m}$, $K \in \mathbb{R}(\lambda)^{m\times p}$ and consequently $K^{\mathrm{bin}} \in \{0,1\}^{m\times p}$. Define n_G as the number of the zero entries in the K^{bin} binary matrix (and also in the $\mathrm{vec}(K^{\mathrm{bin}}) \in \{0,1\}^{mp\times 1}$ binary vector). (It follows that the number of one entries in K^{bin} is equal to $(mp - n_G)$.)

We define the $(mp \times mp)$ binary matrix $\mathrm{diag}(K^{\mathrm{bin}})$ to be the diagonal matrix, which has the K^{bin}_{ij} entry on its $(i+j)$ diagonal entry. Note that $\mathrm{diag}(K^{\mathrm{bin}})$ has exactly $(mp - n_G)$ non–zero entries. Define next

$$\Phi \overset{def}{=} I_{mp} - \mathrm{diag}(K^{\mathrm{bin}}) \tag{22}$$

meaning that Φ has exactly n_G non–zero diagonal entries and since $\Phi\,\mathrm{vec}(K^{\mathrm{bin}}) = 0^{mp\times 1}$ we observe that Φ "selects" only the n_G zero rows of $\mathrm{vec}(K^{\mathrm{bin}})$.

Theorem 4.6: Given a plant G in the set $\mathbb{R}(\lambda)^{p\times m}$, we assume without any loss of generality that all the blocks in the conformal partition (3) of the plant G have the size 1×1, meaning that $\forall i \in \overline{1,p}$ and $\forall j \in \overline{1,m}$ it holds that $n^i_y = n^j_u = 1$. Th plant G is stabilizable with a sparsity constrained controller K belonging to the set $\mathcal{S}$ if and only if, starting from *any* DCF (1) of G, there exists a Youla parameter Q (stable TFM, belonging to the set $\mathbb{R}(\lambda)^{m\times p}$) such that $\mathrm{vec}(Q)$ is a stable solution to the linear system of TFM equations

$$\Phi\big(M^T \otimes \widetilde{M}\big)\mathrm{vec}(Q) = -\Phi\,\mathrm{vec}\big(\widetilde{X}\widetilde{M}\big), \tag{23}$$

where Φ is the matrix defined in (22).

Proof:

First, note that out of the set of (mp) linear equations in $\mathrm{vec}(Q)$ from (22), only n_G are nontrivial, while the rest are identities of the zero row vector.

We also remind here that the $\mathrm{vec}(\cdot)$ operator (18) is linear. Also note that the $\mathrm{vec}(\cdot)$ operator and the $\mathrm{Pattern}(\cdot)$ operator (4) are commutative.

We prove next that the existence of a Youla parameter (*stable* TFM, belonging to the set $\mathbb{R}(\lambda)^{m\times p}$) to satisfy the first relation in (21) (see also Remark 4.5) is equivalent with $\mathrm{vec}(Q)$ being a *stable* solution to (23). The rest of the proof will follow via Corollary 4.4.

$$\begin{aligned}
&\mathrm{Pattern}\big(\widetilde{X}\widetilde{M} + MQ\widetilde{M}\big) \leq K^{\mathrm{bin}} \iff \\
&\mathrm{vec}\Big(\mathrm{Pattern}\big(\widetilde{X}\widetilde{M} + MQ\widetilde{M}\big)\Big) \leq \mathrm{vec}(K^{\mathrm{bin}}) \iff \\
&\mathrm{Pattern}\Big(\mathrm{vec}\big(\widetilde{X}\widetilde{M} + MQ\widetilde{M}\big)\Big) \leq \mathrm{vec}(K^{\mathrm{bin}}) \quad \overset{(18)}{\iff} \\
&\mathrm{Pattern}\Big(\mathrm{vec}\big(\widetilde{X}\widetilde{M}\big) + \mathrm{vec}\big(MQ\widetilde{M}\big)\Big) \\
&\qquad\qquad \leq \mathrm{vec}(K^{\mathrm{bin}}) \quad \overset{Prop.\ 4.3}{\iff} \\
&\mathrm{Pattern}\Big(\mathrm{vec}\big(\widetilde{X}\widetilde{M}\big) + \big(M^T \otimes \widetilde{M}\big)\mathrm{vec}(Q)\Big) \\
&\qquad\qquad \leq \mathrm{vec}(K^{\mathrm{bin}}) \iff \\
&\mathrm{Pattern}\Big(\Phi\Big(\mathrm{vec}\big(\widetilde{X}\widetilde{M}\big) + \big(M^T \otimes \widetilde{M}\big)\mathrm{vec}(Q)\Big)\Big) \\
&\qquad\qquad \leq \Phi\,\mathrm{vec}(K^{\mathrm{bin}}) \iff \\
&\mathrm{Pattern}\Big(\Phi\,\mathrm{vec}\big(\widetilde{X}\widetilde{M}\big) + \Phi\big(M^T \otimes \widetilde{M}\big)\mathrm{vec}(Q)\Big)\Big) \\
&\qquad\qquad \leq 0^{mp\times 1} \iff \\
&\Phi\big(M^T \otimes \widetilde{M}\big)\mathrm{vec}(Q) = -\Phi\,\mathrm{vec}\big(\widetilde{X}\widetilde{M}\big)
\end{aligned} \tag{24}$$

■

Remark 4.7: Problems of type (23) were formulated and proposed for the first time by Wolovich ([19]) who also coined the terminology *exact model–matching problem* in the early 70's. Exact model–matching has a particular significance and importance in the control of LTI systems. When the additional constraint of stability on the solution $\mathrm{vec}(Q)$ is added, the problem becomes an exact model–matching problem with stability (see [20]). Reliable and efficient, state–space algorithms for solving (23) are available in the literature, see for example the most recent reference [21].

Remark 4.8: We remark here, that it can happen that the exact model–matching problem with stability from (23) does not have a solution. A reliable, computational method to detect this situation is also given in [21]. If this is the case, then the plant G is not stabilizable with a sparse controller belonging to the set $\mathcal{S}$.

Remark 4.9: Once a sparse, stabilizing controller (if one exists) is computed by solving the exact model–matching problem (23), one can use the results from [11], [24] to obtain the parametrization of *all* sparse, stabilizing controllers. The attractive feature of the main results from [11], [24] is that it recasts the sparsity constraints on the controller as sparsity constraints on the Q parameter. Also, since the aforementioned parametrization is affine in Q, it brings along the tractable computation (via [10]) of the optimal, sparse controller for both the disturbance attenuation and the mixed–sensitivity $\mathcal{H}_2$ problems.

C. A Numerical Example

Suppose we are given as input data the plant G and K^{bin}, as

$$G(\lambda) = \begin{bmatrix} \dfrac{1}{\lambda-1} & \dfrac{1}{\lambda+2} & 0 \end{bmatrix}, \quad K^{\mathrm{bin}} = \begin{bmatrix} 1 & 0 & 1 \end{bmatrix}^T \tag{25}$$

where all blocks in the partition (3) of G are 1×1 and both Assumptions 1 and 2 are met. We can start up our synthesis algorithm with *any* DCF (1) of the plant which can be computed for instance via the classical state–space formulas from [6]:

$$M(\lambda) = \begin{bmatrix} \frac{\lambda-1}{\lambda+5} & \frac{\lambda-1}{\lambda+6} & 0 \\ 0 & \frac{\lambda+2}{\lambda+6} & 1 \\ 0 & 0 & 1 \end{bmatrix}, \quad -\widetilde{X}(\lambda) = \begin{bmatrix} \frac{40}{\lambda+5} \\ -\frac{8}{3}\frac{1}{\lambda+6} \\ 0 \end{bmatrix},$$

$$N(\lambda) = \begin{bmatrix} \frac{1}{\lambda+5} & \frac{2}{\lambda+6} & \frac{1}{\lambda+2} \end{bmatrix},$$

$$\widetilde{Y}(\lambda) = \frac{\lambda^2 + 17\lambda + 66 + 2/3}{(\lambda+5)(\lambda+6)} \tag{26}$$

and also

$$-\widetilde{N}(\lambda) = \begin{bmatrix} \frac{\lambda+2}{(\lambda+3)(\lambda+4)} & \frac{\lambda-1}{(\lambda+3)(\lambda+4)} & 0 \end{bmatrix},$$

$$\widetilde{M}(\lambda) = \frac{(\lambda-1)(\lambda+2)}{(\lambda+3)(\lambda+4)}. \tag{27}$$

The remaining factors X and Y that complete the DCF (1) of G are not needed in view of Remark 4.5. By looking at (26) we can see that $\text{Pattern}(\widetilde{X}\widetilde{M}) = \begin{bmatrix} 1 & 1 & 0 \end{bmatrix}^T$. We need to find a Youla parameter Q, such that $\big(MQ\widetilde{M}+\widetilde{X}\widetilde{M}\big)$ has a zero in the second row entry. This is necessary and sufficient such as to make $\text{Pattern}\big(MQ\widetilde{M}+\widetilde{X}\widetilde{M}\big) \leq K^{\text{bin}}$.

We discuss next the exact model–matching problem $MQ\widetilde{M} = -\widetilde{X}\widetilde{M}$. Linear matrix equations of this type (also named Sylvester matrix equations) can be solved for Q via Proposition (4.3), by solving for $\text{vec}(Q)$ the following equivalent linear system of TFM equations.:

$$(\widetilde{M}^T \otimes M)\text{vec}(Q) = \text{vec}(-\widetilde{X}\widetilde{M}) \tag{28}$$

(For this particular example, it happens that $\text{vec}(Q) = Q$ and also $\text{vec}(K^{\text{bin}}) = K^{\text{bin}}$, but this does not change the mechanic of the algorithm for the general case.) One can compute the Φ matrix from (22) in order to get that $\Phi = \text{diag}(\{0,1,0\})$. So we must solve only the second linear equation from (28), composed precisely from the rows identified by the zero entries in the $\text{vec}(K^{\text{bin}})$ binary matrix. (The only zero in $\text{vec}(K^{\text{bin}})$ is in the second row, hence we must solve only the equation in the second row of (28)):

$$\widetilde{M}^T(\lambda)\begin{bmatrix} 0 & \frac{\lambda+2}{\lambda+6} & 1 \end{bmatrix} Q(\lambda) = -\frac{8}{3}\frac{1}{\lambda+6}\widetilde{M}(\lambda) \tag{29}$$

We choose a solution Q for (29)

$$Q = \frac{(\lambda+6+8/3)}{(\lambda+2)(\lambda+6)}\begin{bmatrix} (\lambda+5) & (\lambda+6) & \frac{(\lambda+2)(\lambda+6)}{(\lambda+6+8/3)} \end{bmatrix}^T$$

yielding the following controller $K = \big(\widetilde{X} + MQ\big)\big(\widetilde{Y} - NQ\big)^{-1}$

$$K = \frac{1}{\lambda^3 + 19\lambda^2 + (103+1/3)\lambda + (146+2/3)} \times \\ \times \begin{bmatrix} -40(\lambda+2)(\lambda+6) \\ 0 \\ (\lambda+2)(\lambda+5)(\lambda+6) \end{bmatrix} \tag{30}$$

which has the desired sparsity pattern.

D. Parametrization of All Sparse, Stabilizing Controllers

In this subsection we present a particularly important corollary of Theorems 4.2 and 4.6. Given the plant G in the set $\mathbb{R}(\lambda)^{p\times m}$, suppose G stabilizable with a sparsity constrained controller K belonging to the set $\mathcal{S}$. We provide next the parametrization of *all* stabilizing controllers of G, belonging to the set $\mathcal{S}$. We achieve this parametrization, starting from a DCF (1) of G satisfying (21) and imposing additional constraints on the Youla parameter, constraints that guarantee that the resulted controller will belong to $\mathcal{S}$. The constraints are for the Youla parameter to lie in the set of all stable TFMs belonging to a certain linear subspace. Here comes the precise statement.

Corollary 4.10: Given a plant G in the set $\mathbb{R}(\lambda)^{p\times m}$ stabilizable with a sparsity constrained controller K belonging to the set $\mathcal{S}$, and consequently a DCF (1) of G satisfying the first relation in (21), the set of all stabilizing controllers of G belonging to the set $\mathcal{S}$ is given by $K = \big(\widetilde{X} + MQ\big)\big(\widetilde{Y} - NQ\big)^{-1}$ where the Youla parameter Q (*stable* TFM, belonging to the set $\mathbb{R}(\lambda)^{m\times p}$) is such that

$$\text{vec}(Q) \in \text{Null}\Big(\Phi\big(M^T \otimes \widetilde{M}\big)\Big), \tag{31}$$

where Φ is the matrix defined in (22). We make here the elementary observation that Q is stable if and only if $\text{vec}(Q)$ is stable.

Proof: The DCF we start with satisfies the first relation (21), meaning $\text{Pattern}(\widetilde{X}\widetilde{M}) \leq K^{\text{bin}}$ and equivalently $\text{vec}(\text{Pattern}(\widetilde{X}\widetilde{M})) \leq \text{vec}(K^{\text{bin}})$. Then for any Youla parameter Q, we get via Theorem 4.6 that $K = \big(\widetilde{X} + MQ\big)\big(\widetilde{Y} - NQ\big)^{-1}$ belongs to the set $\mathcal{S}$ if and only if $\Phi(M^T \otimes \widetilde{M})\text{vec}(Q) = -\Phi\ \text{vec}(\widetilde{X}\widetilde{M})$. Now, because $\text{vec}(\text{Pattern}(\widetilde{X}\widetilde{M})) \leq \text{vec}(K^{\text{bin}})$, due to the way the Φ matrix is defined in (22) and finally we get that $\Phi\,\text{vec}(\widetilde{X}\widetilde{M}) = 0^{n_G\times(mp)}$, hence the proof. ■

Remark 4.11: For an introduction to linear subspaces for TFMs and vector bases of such subspaces we refer to [22]. For a reliable, state–space algorithm capable of actually computing a basis of the null space of $\Phi\big(M^T \otimes \widetilde{M}\big)$, we refer to [23]. Note that main result from [23] allows for the computation of a basis having only stable poles, by performing a column compression of the normal rank of $\Phi\big(M^T\otimes\widetilde{M}\big)$ by post–multiplication with a unimodular matrix, (for more details see [23] and the following subsection).

E. Numerical Example – Continued

In this subsection we will illustrate numerically the result of Corollary 4.10. We start with the same data from Subsection IV-C but with a different DCF of the plant. The factors $\widetilde{M}$, $\widetilde{N}$ will still be as in (27) and M, N will be as in (26) but $\widetilde{X}$ and $\widetilde{Y}$ will be given by

$$\widetilde{X} = \begin{bmatrix} -\frac{40}{(\lambda+5)} & 0 & 1 \end{bmatrix}^T,$$

$$\widetilde{Y} = \frac{\lambda^3 + 19\lambda^2 + (103+1/3)\lambda + (146+2/3)}{(\lambda+2)(\lambda+5)(\lambda+6)}.$$

which is the DCF satisfying the first relation in (21) since it is the DCF for which the sparse controller given in (30) is the central controller. For the argument stated in Remark 4.5, the remaining factors X and Y of the DCF are not needed.

For this example (as well as for what is presented in Subsection IV-C), the Φ matrix defined in (22) is given by $\Phi = \begin{bmatrix} 0 & 1 & 0 \end{bmatrix}$. We compute an unimodular matrix Δ (using for instance the state–space techniques from [23])

$$\Delta(\lambda) = \begin{bmatrix} 0 & 1 & 0 \\ 0 & 0 & 1 \\ 1 & 0 & -\frac{\lambda+6}{\lambda+2} \end{bmatrix} \tag{32}$$

that by post multiplying $\big(\Phi(M^T \otimes \widetilde{M})\big)$ will perform a column normal–rank compression, such that

$$\big(\Phi(M^T \otimes \widetilde{M})\big)\Delta = \frac{(\lambda-1)(\lambda+2)}{(\lambda+3)(\lambda+4)}\begin{bmatrix} 1 & 0 & 0 \end{bmatrix}$$

It can be seen now that the two last columns of Δ make out for a stable basis of the $\mathrm{Null}\Big(\Phi\big(M^T \otimes \widetilde{M}\big)\Big)$ subspace, hence we define the set $\mathcal{Q}$ as

$$\mathcal{Q} \overset{def}{=} \Big\{ Q \in \mathbb{R}(\lambda)^{3\times 1} \Big| \mathrm{vec}\big(Q(\lambda)\big) = \begin{bmatrix} 1 & 0 \\ 0 & 1 \\ 0 & -\frac{\lambda+6}{\lambda+2} \end{bmatrix} \begin{bmatrix} \alpha(\lambda) \\ \beta(\lambda) \end{bmatrix}$$
$$\text{with } \alpha(\lambda), \beta(\lambda) \text{ stable, real}-\text{rational functions} \Big\} \tag{33}$$

The set of *all* stabilizing controllers of G, belonging to the set $\mathcal{S}$ is given by $K = \big(\widetilde{X} + MQ\big)\big(\widetilde{Y} - NQ\big)^{-1}$, with $Q \in \mathcal{Q}$.

V. A Meaningful, Particular Case

In this section we provide a reference to the solution of the same problem (stabilization via sparse controllers), under the scenario that the given plant G satisfies a particular criteria. Specifically, we look at the case when the plant G admits both a left coprime factorization $G = \widetilde{M}^{-1}\widetilde{N}$ over Ω and a right coprime factorization $G = NM^{-1}$ (see Subsection II-B) such that both "denominators" $\widetilde{M}$ and M are block–diagonal. As it turns out such a factorization is guaranteed to exist for *almost all* plants, meaning that it is a *generic* property. Furthermore, for any given plant it is quite easy to check if such a factorization exists and if this is the case, it is also easy to compute. Since we are not aware of any existing references we have dubbed this an *Input/Output Decoupled* DCF.

It is proved in detail in [24] that the advantages the Input/Output Decoupled DCF brings are important. Firstly it makes all the equivalent results presented in the previous section far less complicated, since now vectorization is not needed. Secondly, it makes possible to characterize the set of all decentralized stabilizing controllers via the Youla parametrization such that the sparsity constraints on the controller are recast as *sparsity* constraints on the Youla parameter.

VI. Conclusions

In this paper we have provided necessary and sufficient conditions for the stabilizability of a given plant, with a controller satisfying sparsity constraints that are quadratically invariant with respect to the plant. These conditions are formulated in terms of the existence of a specific doubly coprime factorization of the plant featuring additional sparsity constraints on certain factors . Along the way have obtained the set of all decentralized stabilizing controllers, characterized via the Youla parametrization. The sparsity constraints on the controller are recast as convex constraints on the Youla parameter. In order to achieve this, it is noteworthy that the constraints on the Youla parameter become linear subspace constraints on the Youla parameter, only from the particular coprime factorization having special sparsity constraints on certain factors. Solving the stabilization problem provides the missing link for fully exploiting the powerful optimal synthesis methods for sparse controllers from [10].

References

[1] H.S. Wittsenhausen "A Counterexample in Stochastic Optimum Control", *SIAM J. Control*, Vol.6, No.1, 1969. (pp 131-147)

[2] V.D. Blondel and J.N. Tsitsiklis 'A Survey of Computational Complexity Results in Systems and Control", *Automatica*, Vol.36, No.9, 2000. (pp. 1249-1274)

[3] C.H. Papadimitriou, J.N. Tsitsiklis "Intractable Problems in Control Theory ", *SIAM J. Control Optim.*. Vol.24, 1986. (pp. 639-654)

[4] M. Vidyasagar "Control System Synthesis: A Factorization Approach", *MIT Press*, Signal Processing, Optimization, and Control Series, 1985.

[5] M.Rotkowitz, N.C.Martins "On the Closest Quadratically Invariant Constraint", Proceedings of the48-th IEEE Conference on Decision and Control, Shanghai, P.R.China,2009.

[6] C. N. Nett, C. A. Jacobson, and M. J. Balas, "A connection between state-space and doubly coprime fractional representations" *IEEE Trans. Automat. Control*, Vol. AC-29, 1984. (pp. 831832)

[7] Alan J. Laub "Matrix Analysis for Scientists and Engineers" SIAM `www.ec-securehost.com/SIAM/ot91.html`

[8] B. Francis " A Course in $\mathcal{H}_\infty$ Control Theory", *Series Lecture Notes in Control and Information Sciences*, New York: Springer-Verlag, 1987, vol. 88.

[9] M. Rotkowitz "Tractable Problems in Optimal Decentralized Control", *Ph.D. disertation, Stanford Univ., Stanford CA*, Jun 2005.

[10] M. Rotkowitz, S. Lall "A Characterization of Convex Problems in Decentralized Control", *IEEE Trans. Aut. Control*, Vol.51, No.2, 2006. (pp. 274-286)

[11] Ş. Sabău, N.C.Martins and M. Rotkowitz "A coordinate-free parameterization of all stabilizing controllers subject to quadratically invariant sparsity constraints ", submitted to *IEEE Trans. Aut. Control*.

[12] P. G. Voulgaris Control under a hierarchical decision making structure, *Proc. of the Amer. Control Conf.*, San Diego, CA, June 1999.

[13] P. G. Voulgaris Control of nested systems, *Proc. of the Amer. Control Conf.*, Chicago, IL, June 2000.

[14] P.G. Voulgaris A convex characterization of classes of problems in control with sepcic interaction and communication structures, *Proc. of the American Control Conf.*, Arlington, VA, June 2001.

[15] X. Qi, M.V. Salapaka, P.G. Voulgaris and M. Khammash "Structured Optimal and Robust Control with Multiple Criteria: A Convex Solution", *IEEE Trans. Aut. Control*, Vol.49, No.10, 2004. (pp. 1623-1640)

[16] B. Bamieh, P. G. Voulgaris "A convex characterization of distributed control problems in spatially invariant systems with communication constraints" *Systems & Control Letters*, Vol. 54, 2005. (pp. 575 583)

[17] V. Yadav, M.V. Salapaka and P.G. Voulgaris "Architectures for Distributed Controller with Sub–Controller Communication Uncertainty", *IEEE Trans. Aut. Control*, Vol.55, No.8, 2010. (pp. 1765-1780)

[18] K. Zhou, J.C. Doyle and K. Glover " Robust and Optimal Control", *Upper Saddle River, NJ: Prentice Hall*, 1996.

[19] W.A. Wolovich "The Application of State Feedback Invariants to Exact Model–Matching", presented at *The 5th Annual Princeton Conf. on Inform Sci. Syst.*, Princeton, NJ, March 1971.

[20] Z. Gao, P.J. Antsaklis "On Stable Solutions of the One– and Two–Sided Model–Matching Problems", *IEEE Trans. Aut. Control*, Vol.34, No.9, 1989. (pp. 978-982)

[21] D.Chu, P.Van Dooren "A Novel Numerical Method for Exact Model–Matching Problem with Stability", *Automatica*, Vol. 42, 2006. (pp. 1697–1704)

[22] G.D. Forney "Minimal Bases of Rational Vector Spaces, with Applications to Multivariable Linear Systems", *SIAM Journal of Control*, Vol. 13, 1975. (pp. 493–520)

[23] C.Oară, Ş.Sabău "Minimal Indices Cancellation and Rank Revealing Factorizations for Rational Matrix Functions", *Linear Algebra and Its Applications*, Vol. 431, Issue 10, 2009. (pp. 1785 – 1814)

[24] Ş. Sabău "Optimal Control with Information Pattern Constraints", *Ph.D. Thesis*, University of Maryland at College Park, 2011.