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Equivalence and equivariants of mechanical control systems

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Abstract— **In this paper we characterise geodesically accessible mechanical control systems under state and mechanical state equivalence. To this end, we consider two families of structure functions which are equivariants for the mechanical state equivalence (and, more generally, for the state equivalence) of two mechanical control systems satisfying the geodesic accessibility property. Those structure functions, constructed with the help of symmetric product and Lie product, encode in a geometric way all invariant information about mechanical control systems.**

I. INTRODUCTION

Mechanical control systems form an important and rich class of control systems whose study has attracted the attention of many researchers, mainly during the last three decades. Indeed, this class of control systems has many applications in real life, thus being a very engaging subject of study and, also, it offers very interesting and challenging mathematical problems. Mechanical control systems, that form a natural bridge between mechanics and control theory, are studied for instance in [1], [2], [15].

A big number of works on mechanical control systems has emerged both on the Lagrangian and the Hamiltonian perspectives, leading to new insights in control and motion planning of mechanical control systems. Based on the Lagrangian point of view of mechanics, many works have been developed using the affine connection formulation (see for instance [2], [3], [8], [11], [12]). For an approach based on the Hamiltonian formalism we refer the reader to Chapter 12 of [13] and, e.g., the papers [4], [19]. In the present paper we shall rest on the Lagrangian point of view of mechanical control systems.

Our goal is to find a complete set of equivariants of mechanical control systems and, as consequence, to characterise the equivalence (under a diffeomorphism of configuration manifolds) of two mechanical control systems using these equivariants. Our equivariants also describe the equivalence of the corresponding control systems, defined on the tangent bundle of the configuration manifolds, under extended point transformations as well as under arbitrary diffeomorphisms (not necessarily fiber preserving) of the tangent bundle.

The outline of the paper is as follows. Section II provides some preliminary notions and notation. In Section III, we define families of structure functions, expressed via symmetric products and Lie products of input vector fields, and we use these families to geometrically characterise the mechanical state equivalence of two mechanical control systems that satisfy the geodesic accessibility property. This is a first version of our main result stated in terms of objects on the configuration manifold. In Section IV, a second version is given for control systems that admit a geodesically accessible mechanical structure and in Section V we present two examples.

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II. PRELIMINARIES

In our previous study [18], a geometric setting for studying general mechanical control systems is presented. Here we review tools and definitions necessary for the present paper.

A. Mechanical control systems

We define a *mechanical control system* $(\mathcal{MS})$ as a 4-tuple $(Q, \nabla, \mathfrak{g}_0, d)$, in which

(i) Q is an n-dimensional configuration manifold;
(ii) ∇ is a symmetric affine connection on Q;
(iii) $\mathfrak{g}_0 = (g_0, g_1, \ldots, g_m)$ is an $(m+1)$-tuple of smooth vector fields on Q;
(iv) $d : TQ \to TQ$ is a map sending the fiber T_qQ into the fiber T_qQ, for any $q \in Q$, linear on fibers.

A curve $\gamma : I \to Q,\ I \subset \mathbb{R}$, is a trajectory of $(\mathcal{MS})$ if it satisfies the equation

$$\nabla_{\dot\gamma(t)}\dot\gamma(t) = g_0(\gamma(t)) + d(\dot\gamma(t)) + \sum_{r=1}^{m} u_r g_r(\gamma(t)). \tag{1}$$

The term $d(\dot\gamma(t))$ corresponds to dissipative-type (or gyroscopic-type) forces acting on the system; the vector field g_0 results from an external uncontrolled force and the vector fields $g_1, \ldots, g_m$ result from controlled external forces. Finally, $u = (u_1, \ldots, u_m)$ are controls of the system. In local coordinates $(x^1, \ldots, x^n)$ on Q, this equation is equivalent to the second-order system of differential equations

$$\ddot x^i = -\Gamma^i_{jk}(x)\dot x^j \dot x^k + d^i_j(x)\dot x^j + g^i_0(x) + \sum_{r=1}^{m} u_r g^i_r(x), \tag{2}$$

with $1 \le i \le n$. Here, and in what follows, we use the summation convention except for terms involving controls. The above system is also equivalent to the first-order system of differential equations on TQ, equipped with coordinates $(x^1, \ldots, x^n, y^1, \ldots, y^n)$, which we will also denote $(\mathcal{MS})$:

$$\begin{aligned} \dot x^i &= y^i, \\ \dot y^i &= -\Gamma^i_{jk}(x)y^j y^k + d^i_j(x)y^j + g^i_0(x) + \sum_{r=1}^{m} u_r g^i_r(x). \end{aligned} \tag{3}$$

A particularly important role will be played by mechanical control systems for which $d = 0$ and $g_0 = 0$, i.e., mechanical control systems which are neither subject to dissipative-type (or gyroscopic-type) forces nor uncontrolled ones. These systems are known as *affine connection control systems* and are thus defined as a 3-tuple $(\mathcal{ACS}) = (Q, \nabla, \mathfrak{g})$, with Q and ∇ as before and $\mathfrak{g} = (g_1, \ldots, g_m)$ an m-tuple of smooth input vector fields on Q (see [8], [9], [10], [11], [12]).

B. Geodesically accessible mechanical control systems

Associated with an affine connection ∇ on Q, there is a symmetric real-bilinear operation called the *symmetric product*:

$$\langle X : Y \rangle = \nabla_X Y + \nabla_Y X,$$

with X, Y smooth vector fields on the configuration manifold Q. In local coordinates $(x^1, \ldots, x^n)$ on Q, we have

$$\langle X : Y \rangle = \left(\frac{\partial X^i}{\partial x^j} Y^j + \frac{\partial Y^i}{\partial x^j} X^j + \Gamma^i_{jk} X^j Y^k + \Gamma^i_{jk} Y^j X^k \right) \frac{\partial}{\partial x^i}.$$

The symmetric product was first introduced by Crouch in [5]. For a geometric interpretation see [8].

Let $\mathcal{SYM}(\mathfrak{g})$ be the smallest distribution on Q containing the input vector fields $g_1, \ldots, g_m$ and closed under the symmetric product defined by the connection ∇.

Definition 1 ([18]): The system $(\mathcal{MS})$ is called *geodesically accessible* at $x_0 \in Q$ if

$$\mathcal{SYM}(\mathfrak{g})(x_0) = T_{x_0} Q,$$

and geodesically accessible if the above equality holds for all $x_0 \in Q$.

A geodesically accessible mechanical control system will be denoted shortly by $(\mathcal{GAMS})$. If additionally, the system is affine connection (i.e., if we have a $(\mathcal{GAMS})$ with $d = 0$ and $g_0 = 0$), then it will be called *geodesically accessible affine connection system* and denoted shortly by $(\mathcal{GACS})$.

In order to use the above definition, we will provide an algorithmic way of constructing $\mathcal{SYM}(\mathfrak{g})$. Consider the following sequence of families of vector fields on Q :

$$\mathrm{Sym}^1(\mathfrak{g}) = \mathfrak{g} = \{ g_r \mid 1 \leq r \leq m \}$$

and, inductively,

$$\mathrm{Sym}^i(\mathfrak{g}) = \left\{ \langle X : Y \rangle \mid X \in \mathrm{Sym}^p(\mathfrak{g}), Y \in \mathrm{Sym}^l(\mathfrak{g}), p + l = i \right\}.$$

Define

$$\mathrm{Sym}(\mathfrak{g}) = \bigcup_{i=1}^{\infty} \mathrm{Sym}^i(\mathfrak{g}). \tag{4}$$

Then $\mathcal{SYM}(\mathfrak{g}) = \operatorname{span} \mathrm{Sym}(\mathfrak{g})$.

The geodesic accessibility property plays a crucial role in our approach because, as shown in [18], it guarantees the uniqueness of the mechanical structure.

C. Mechanical state equivalence

Given two mechanical control systems $(\mathcal{MS}) = (Q, \nabla, \mathfrak{g}_0, d)$ and $(\widetilde{\mathcal{MS}}) = (\tilde{Q}, \tilde{\nabla}, \tilde{\mathfrak{g}}_0, \tilde{d})$, we say that $(\mathcal{MS})$ and $(\widetilde{\mathcal{MS}})$ are mechanical state equivalent, shortly *MS-equivalent*, if they are related by a diffeomorphism. More precisely, $(\mathcal{MS})$ and $(\widetilde{\mathcal{MS}})$ are MS-equivalent (respectively, locally MS-equivalent at points $x_0 \in Q$ and $\tilde{x}_0 \in \tilde{Q}$) if there exists a diffeomorphism $\phi : Q \to \tilde{Q}$ (respectively, a local diffeomorphism $\phi : U \to \tilde{U}$, $\phi(x_0) = \tilde{x}_0$, with U a neighborhood of x_0 and $\tilde{U}$ a neighborhood of $\tilde{x}_0$) such that

$$\phi(\nabla) = \tilde{\nabla}, \quad \phi_* d = \tilde{d}, \quad \text{and} \quad \phi_* g_i = \tilde{g}_i, \tag{5}$$

for $0 \leq i \leq n$, where ϕ_* stands for the tangent map of ϕ and $\phi(\nabla)$ denotes the affine connection ∇ transformed via $\tilde{x} = \phi(x)$, that is, the affine connection $\bar{\nabla}$ whose Christoffel symbols are given by

$$\bar{\Gamma}^i_{jk}(\tilde{x}) = \Gamma^p_{qr} \frac{\partial x^q}{\partial \tilde{x}^j} \frac{\partial x^r}{\partial \tilde{x}^k} \frac{\partial \tilde{x}^i}{\partial x^p} + \frac{\partial^2 x^p}{\partial \tilde{x}^j \partial \tilde{x}^k} \frac{\partial \tilde{x}^i}{\partial x^p}.$$

III. Main results: first version

In this section we discuss the MS-equivalence of mechanical control systems satisfying the geodesic accessibility property, in terms of conditions involving two families of structure functions, which we will prove to be *equivariants* of the MS-equivalence, that is, to transform via the conjugating diffeomorphism.

Consider two geodesically accessible affine connection systems $(\mathcal{GACS}) = (Q, \nabla, \mathfrak{g})$ and $(\widetilde{\mathcal{GACS}}) = (\tilde{Q}, \tilde{\nabla}, \tilde{\mathfrak{g}})$ around points $x_0 \in Q$ and $\tilde{x}_0 \in \tilde{Q}$, respectively. We assume that $\dim Q = \dim \tilde{Q}$ and consider $\mathfrak{g} = (g_1, \ldots, g_m)$ and $\tilde{\mathfrak{g}} = (\tilde{g}_1, \ldots, \tilde{g}_m)$, m-tuples of smooth input vector fields on Q and $\tilde{Q}$, respectively.

The geodesic accessibility property (see Definition 1) guarantees the existence of independent vector fields $v_1, \ldots, v_n \in \mathrm{Sym}(\mathfrak{g})$ and $\tilde{v}_1, \ldots, \tilde{v}_n \in \mathrm{Sym}(\tilde{\mathfrak{g}})$. We assume that $g_1, \ldots, g_m$ are independent and we take $v_i = g_i$, $1 \leq i \leq m$ (and analogously, $\tilde{v}_i = \tilde{g}_i$, $1 \leq i \leq m$). A generalisation to the case of g_i that are dependent is straightforward (although technically more involved).

Any element v_j in the frame $(v_1, \ldots, v_n)$, being an element of a certain $\mathrm{Sym}^i(\mathfrak{g})$ (see (4)), is the symmetric product of two elements w_1 and w_2 of $\mathrm{Sym}^l(\mathfrak{g})$ and $\mathrm{Sym}^p(\mathfrak{g})$, respectively, $l + p = i$. Both w_1 and w_2 are successions of symmetric products, so vector fields belonging to $\mathrm{Sym}^i(\mathfrak{g})$ shall be referred to as symmetric products of length i. We shall choose the elements of the frame $(v_1, \ldots, v_n)$ to be vector fields of smallest possible length. Clearly, $v_1 = g_1, \ldots, v_m = g_m$ are of length one.

We will say that two frames $(v_1, \ldots, v_n)$ and $(\tilde{v}_1, \ldots, \tilde{v}_n)$ are conform if each $\tilde{v}_j$, $1 \leq j \leq n$, is constructed as an analogous successive symmetric product as that defining v_j (with the elements of $\mathrm{Sym}^p(\mathfrak{g})$ and $\mathrm{Sym}^l(\mathfrak{g})$ being replaced by the corresponding ones of $\mathrm{Sym}^p(\tilde{\mathfrak{g}})$ and $\mathrm{Sym}^l(\tilde{\mathfrak{g}})$).

Fix a frame $(v_1, \ldots, v_n)$ and consider the equalities

(LAR) $\quad \big[v_{i_q}, \ldots, [v_{i_3}, [v_{i_2}, v_{i_1}]] \ldots\big] = \alpha^s_{i_1 \ldots i_q} v_s$, and

(SAR) $\quad \langle v_{i_q} : \ldots \langle v_{i_3} : \langle v_{i_2} : v_{i_1} \rangle \rangle \ldots \rangle = \beta^s_{i_1 \ldots i_q} v_s$,

defining the *structure functions* $\alpha^s_{i_1 \ldots i_q}$ and $\beta^s_{i_1 \ldots i_q}$, where $q \geq 2$ and $1 \leq s, i_1, \ldots, i_q \leq n$. Equalities (LAR) and (SAR) give, respectively, information about the Lie algebraic relations and the symmetric algebraic relations of the system.

Analogously, we can derive the structure functions $\tilde{\alpha}^s_{i_1\dots i_q}$ and $\tilde{\beta}^s_{i_1\dots i_q}$ for $(\widetilde{\mathcal{GACS}})$. We consider the families of structure functions

$$\mathfrak{s} = \{\alpha^s_{i_1\dots i_q},\ \beta^s_{i_1\dots i_q}\} \quad \text{and} \quad \tilde{\mathfrak{s}} = \{\tilde{\alpha}^s_{i_1\dots i_q},\ \tilde{\beta}^s_{i_1\dots i_q}\}.$$

We will say that a family of smooth functions $\{\gamma^s_{i_1\dots i_q} \mid q \geq 2\}$ is of *constant rank* r, in an open neighborhood U of $x_0 \in Q$, if $\left\{d\gamma^s_{i_1\dots i_q}(x) \mid q \geq 2\right\}$ span an r-dimensional space at any $x \in U$. We call *order* of a family of constant rank r the minimal number ρ such that $\dim \operatorname{span}\left\{d\gamma^s_{i_1\dots i_q} \mid 2 \leq q \leq \rho\right\}(x_0) = r$.

Next result establishes the local MS-equivalence of systems $(\mathcal{GACS})$ and $(\widetilde{\mathcal{GACS}})$ in terms of conditions involving only the sets of structure functions $\mathfrak{s}$ and $\tilde{\mathfrak{s}}$ and proving that they are equivariants of MS-equivalence.

Theorem 1: Two geodesically accessible affine connection systems $(\mathcal{GACS}) = (Q, \nabla, \mathfrak{g})$ and $(\widetilde{\mathcal{GACS}}) = (\tilde{Q}, \tilde{\nabla}, \tilde{\mathfrak{g}})$, whose families of structure functions $\mathfrak{s}$ and $\tilde{\mathfrak{s}}$ are of constant rank in neighborhoods of $x_0 \in Q$ and $\tilde{x}_0 \in \tilde{Q}$, are MS-equivalent around x_0 and $\tilde{x}_0$, respectively, if and only if there exists a diffeomorphism $\varphi : U \to \tilde{U}$, where U and $\tilde{U}$ are neighborhoods of x_0 and $\tilde{x}_0$ in Q and $\tilde{Q}$, respectively, such that

$$\begin{array}{lrcl} \text{(LAC)} & \alpha^s_{i_1\dots i_q} & = & \tilde{\alpha}^s_{i_1\dots i_q} \circ \varphi, \\ \text{(SAC)} & \beta^s_{i_1\dots i_q} & = & \tilde{\beta}^s_{i_1\dots i_q} \circ \varphi, \end{array}$$

for $q \leq \rho + 1$, with ρ being the common order of families $\mathfrak{s}$ and $\tilde{\mathfrak{s}}$.

Remark 1: The above conditions have a clear meaning that justifies their names: (LAC) says that the Lie modules, generated by the symmetric vector fields of $(\mathcal{GACS})$ and $(\widetilde{\mathcal{GACS}})$, (i.e., vector fields from $\mathrm{Sym}(g_1, \dots, g_m)$ and $\mathrm{Sym}(\tilde{g}_1, \dots, \tilde{g}_m)$, respectively) coincide (up to conjugation by a diffeomorphism of the configuration manifolds Q and $\tilde{Q}$); (SAC) states that the symmetric modules, generated by all symmetric vector fields of $(\mathcal{GACS})$ and $(\widetilde{\mathcal{GACS}})$, coincide (up to conjugation by the same diffeomorphism).

Remark 2: If a diffeomorphism ϕ establishing the MS-equivalence of $(\mathcal{GACS})$ and $(\widetilde{\mathcal{GACS}})$ exists then it is unique. On the other hand, the diffeomorphism φ conjugating the structure functions is almost never unique. To discuss relations between ϕ and φ, let r denote the rank of the families $\mathfrak{s}$ and $\tilde{\mathfrak{s}}$. Clearly, r satisfies $0 \leq r \leq n$, where $\dim Q = n$. We can distinguish three cases: (i) if $r = n$, i.e., $\mathfrak{s}$ and $\tilde{\mathfrak{s}}$ are of maximal possible rank, then the diffeomorphism φ conjugating them is unique (and can be, implicitly, expressed via $\mathfrak{s}$ and $\tilde{\mathfrak{s}}$). In this case, the diffeomorphisms φ and ϕ coincide; (ii) if $r = 0$, i.e., $\mathfrak{s}$ and $\tilde{\mathfrak{s}}$ consists of constant functions only, then (LAC) and (SAC) imply that the structure functions have to be the same (see Corollary 1, below) and, if this is the case, any diffeomorphism φ conjugates them; (iii) if $0 < r < n$, then only an r-dimensional "part" of the diffeomorphism ϕ is determined by the diffeomorphism φ.

The case $r = 0$ leads to the following result for affine connection systems whose all structure functions $\alpha^s_{i_1\dots i_q}$ and $\beta^s_{i_1\dots i_q}$, $q \geq 2$ and $1 \leq s, i_1, \dots, i_q \leq n$, are constant.

Corollary 1: Consider a geodesically accessible affine connection system $(\mathcal{GACS})$ with constant structure functions $\alpha^s_{i_1\dots i_q}, \beta^s_{i_1\dots i_q} \in \mathbb{R}$. A geodesically accessible affine connection system $(\widetilde{\mathcal{GACS}})$ is locally MS-equivalent to $(\mathcal{GACS})$ if and only if their structure functions coincide, i.e., $\tilde{\alpha}^s_{i_1\dots i_q} = \alpha^s_{i_1\dots i_q}$ and $\tilde{\beta}^s_{i_1\dots i_q} = \beta^s_{i_1\dots i_q}$.

IV. S-Equivalence and MS-Equivalence of Control Systems That Admit a $(\mathcal{GAMS})$ Structure

Our result Theorem 1 is actually based on a more general result concerning the equivalence of general control-affine systems to mechanical systems. We shall discuss this issue in the present section. Again, for proofs of the results of this section, see [17].

A. S-equivalence and MS-equivalence of control systems

Let Σ and $\tilde{\Sigma}$ be control-affine systems evolving respectively on M and $\widetilde{M}$, smooth manifolds of dimension $2n$:

$$\Sigma: \quad \dot{z} = F(z) + \sum_{s=1}^{m} u_s G_s(z), \quad z \in M, \quad \text{and}$$
$$\tilde{\Sigma}: \quad \dot{\tilde{z}} = \tilde{F}(\tilde{z}) + \sum_{s=1}^{m} u_s \tilde{G}_s(\tilde{z}), \quad \tilde{z} \in \widetilde{M}.$$

We say that Σ and $\tilde{\Sigma}$ are state equivalent, shortly *S-equivalent*, if they are related by a diffeomorphism (and then also their trajectories, corresponding to the same controls, are related by that diffeomorphism) [6], [13], [16]. More precisely, Σ and $\tilde{\Sigma}$ are S-equivalent (respectively, locally S-equivalent at points $z_0 \in M$ and $\tilde{z}_0 \in \widetilde{M}$) if there exists a diffeomorphism $\Psi : M \to \widetilde{M}$ (respectively, a local diffeomorphism $\Psi : W \to \tilde{W}$, $\Psi(z_0) = \tilde{z}_0$, with W a neighborhood of z_0 and $\tilde{W}$ a neighborhood of $\tilde{z}_0$) such that $\Psi_* F = \tilde{F}$ and $\Psi_* G_r = \tilde{G}_r$, $1 \leq r \leq m$, where Ψ_* denotes the tangent map of Ψ.

Definition 2: We say that Σ is S-equivalent to a mechanical control system if there exists a mechanical system $(\mathcal{MS})$ of the form (3) such that Σ and $(\mathcal{MS})$ are S-equivalent. In this case, we also say that Σ admits a *mechanical structure*.

Analogously, we define local S-equivalence of Σ at z_0 to a mechanical system $(\mathcal{MS})$ at $(x_0, y_0) \in TQ$. In this case, we say that Σ admits locally a mechanical structure.

To distinguish the role of various diffeomorphisms that appear, we systematically use the following notation. By Ψ we will denote a diffeomorphism $\Psi : M \to \tilde{M}$ (respectively $\Psi : M \to TQ$) conjugating a system Σ on M with $\tilde{\Sigma}$ on $\tilde{M}$ (respectively with a mechanical system $(\mathcal{MS})$ on TQ) and a diffeomorphism conjugating their equivariants by ψ. By ϕ we denote a diffeomorphism $\phi : Q \to \tilde{Q}$, establishing the MS-equivalence, as defined in Section II-C, between two mechanical systems $(\mathcal{MS}) = (Q, \nabla, \mathfrak{g}_0, d)$ and $(\widetilde{\mathcal{MS}}) = (\tilde{Q}, \tilde{\nabla}, \tilde{\mathfrak{g}}_0, \tilde{d})$. A diffeomorphism conjugating their equivariants is denoted by φ. When considering the MS-equivalent systems $(\mathcal{MS})$ and $(\widetilde{\mathcal{MS}})$ as control systems of the form (3) written, respectively, in state spaces TQ and $T\tilde{Q}$, we will also call them MS-equivalent. Clearly, they are S-equivalent under the extended point transformation $\Phi : TQ \to T\tilde{Q}$ given by $\Phi = (\phi_1, \phi_2)$ with $\tilde{x} = \phi_1(x) = \phi(x)$ and $\tilde{y} = \phi_2(x, y) = \mathrm{D}\phi(x)y$, where (x, y) and $(\tilde{x}, \tilde{y})$ are local coordinates, respectively, on TQ and $T\tilde{Q}$. More generally, given two control systems Σ and $\tilde{\Sigma}$, that are S-equivalent to mechanical control systems, we will denote by $\Phi : M \to \tilde{M}$ any diffeomorphism establishing the S-equivalence of Σ and $\tilde{\Sigma}$ and preserving the bundle structures of the underlying manifolds (imposed by their mechanical structures).

B. S-equivalence to a mechanical control system

Given a control-affine system Σ as in Section IV-A, we denote by $\mathcal{V}$ the smallest vector space, over $\mathbb{R}$, containing the vector fields $G_1, \ldots, G_m$ and satisfying

$$[\mathcal{V}, \mathrm{ad}_F\mathcal{V}] = \{[V_i, \mathrm{ad}_F V_j] \mid V_i, V_j \in \mathcal{V}\} \subset \mathcal{V},$$

We define the sequence of families of vector fields on M :

$$\mathcal{V}_1 = \{G_r \mid 1 \le r \le m\}$$

and, inductively, $$\mathcal{V}_i = \bigcup_{p+l=i} [\mathcal{V}_p, \mathrm{ad}_F\mathcal{V}_l]\,,\ p, l \ge 1,$$

where $[\mathcal{V}_p, \mathrm{ad}_F\mathcal{V}_l]=\{[V_k, \mathrm{ad}_F V_j] \mid V_k \in \mathcal{V}_p,\ V_j \in \mathcal{V}_l\}$. Then we have

$$\mathcal{V} = \mathrm{Vect}_{\mathbb{R}} \bigcup_{i=1}^{\infty} \mathcal{V}_i. \tag{6}$$

Likewise, we define the sequence $\tilde{\mathcal{V}}_1, \ldots, \tilde{\mathcal{V}}_i$ and $\tilde{\mathcal{V}}$ on $\tilde{M}$.

Recall that a point $z_0 \in M$ is said to be an equilibrium point for the system Σ if $F(z_0) = 0$. We call a *zero-velocity point* for the mechanical control system $(\mathcal{MS})$ any point of the form $(x_0, y_0) = (x_0, 0)$, that is any point of the zero section of TQ. An equilibrium point for $(\mathcal{MS})$ is a zero-velocity point $(x_0, 0)$ such that $g_0(x_0) = 0$. Next result provides a geometric characterisation of $(\mathcal{GAMS})$'s and $(\mathcal{GACS})$'s, locally around zero-velocity points.

Theorem 2: [18] Let M be a smooth $2n$-dimensional manifold. A control system Σ is locally, at $z_0 \in M$, S-equivalent to a geodesically accessible mechanical system $(\mathcal{GAMS})$ around a zero-velocity point $(x_0, 0)$ if and only if

(MS0) $F(z_0) \in \mathcal{V}(z_0)$,
(MS1) $\dim \mathcal{V}(z) = n$ and $\dim (\mathcal{V} + [F, \mathcal{V}])(z) = 2n$,
(MS2) $[\mathcal{V}, \mathcal{V}](z) = 0$,

for any z in a neighborhood of z_0. Moreover, Σ is locally, at $z_0 \in M$, S-equivalent to a $(\mathcal{GACS})$ around an equilibrium point, if and only if it additionally satisfies

(MSND) $\mathrm{ad}_F\mathcal{V}(z_0) \subset T_{z_0}Q^{\Sigma}$,
(MSNU) $F|_{Q^{\Sigma}} = 0$, where $Q^{\Sigma} = \{z \in M \mid F(z) \in \mathcal{V}(z)\}$.

The conditions in this theorem have a clear geometric meaning: (MS0) implies that $z_0 \in M$ is mapped into a zero-velocity point; (MS1), together with (MS0), is equivalent to the geodesic accessibility property; and (MS2) is responsable for the mechanical structure of the system. The additional conditions (MSND) and (MSNU) correspond, respectively, to the absence of dissipative-type forces and the absence of external uncontrolled forces acting on the system, that is, respectively, $d = 0$ and $g_0 = 0$. Since a $(\mathcal{GACS})$ is a $(\mathcal{GAMS})$ for which we have additionally $d = 0$ and $g_0 = 0$, zero-velocity points for $(\mathcal{GACS})$'s coincide with equilibrium points and thus in that case (MS0) can be replaced by (MS0)' stating that $F(z_0) = 0$.

It has been shown in [18] that conditions (MS0)–(MS2) of Theorem 2 encode all the structure information of a $(\mathcal{GAMS})$–system to which Σ is S-equivalent. Therefore, under these three conditions, we are able to construct for the system Σ, locally around a point $z_0 \in M$, all canonical objects in the definition of a mechanical control system $(Q^{\Sigma},\ \nabla^{\Sigma},\ \mathfrak{g}_0^{\Sigma},\ d^{\Sigma})$. For the purposes of the present paper, we will define only the configuration manifold Q^{Σ} and the control vector fields $g_1^{\Sigma}, \ldots, g_m^{\Sigma}$. For the detailed construction of the connection ∇^{Σ}, the map d^{Σ}, and the uncontrolled vector field g_0^{Σ}, see [18]. The configuration manifold Q^{Σ} is given by $Q^{\Sigma} = \{z \in M \mid F(z) \in \mathcal{V}(z)\}$. The distribution $\mathrm{span}\mathcal{V}$ is involutive (because of (MS2)) and its integral leaves define a foliation denoted by $\mathcal{FV}$. We define the surjective submersion $\pi : M \to Q^{\Sigma}$ by attaching to any $z \in M$ the point $q = \pi(z)$ defined as $\{q\} = Q^{\Sigma} \cap L_z$, where L_z is the leaf of $\mathcal{FV}$ passing through z. Notice that the intersection consists of one point only since the manifold Q^{Σ} is transversal to the leaves of $\mathcal{FV}$. Any vector field $V \in \mathcal{V}$ gives rise to a vector field v on Q^{Σ}. Indeed, by definition of $\mathcal{V}$, the vector field $\mathrm{ad}_F V$ satisfies the condition $[\mathrm{ad}_F V, \mathcal{V}] \subset \mathcal{V}$, which implies that $\mathrm{ad}_F V$ projects to the vector field

$$v := -\pi_*(\mathrm{ad}_F V) \tag{7}$$

on Q^{Σ}. Since $G_r \in \mathcal{V},\ 1 \le r \le m$, equality (7) implies, in particular, that $g_r^{\Sigma} := -\pi_*(\mathrm{ad}_F G_r)$ are the projected input vector fields. In this way, given n locally independent vector fields $V_1, \ldots, V_n \in \mathcal{V}$ we obtain a local frame $v_1, \ldots, v_n$ on Q^{Σ}. Conversely, given a local frame $v_1, \ldots, v_n$ on Q^{Σ}, there exists a unique collection of independent vector fields $V_1, \ldots, V_n \in \mathrm{span}\mathcal{V}$ such that V_i is the vertical lift of v_i.

C. Structure functions for control systems that admit a mechanical structure

We assume that Σ and $\tilde{\Sigma}$ are locally S-equivalent, at $z_0 \in M$ and $\tilde{z}_0 \in \tilde{M}$, to geodesically accessible mechanical systems $(\mathcal{GAMS})$ and $(\widetilde{\mathcal{GAMS}})$, respectively, around zero-velocity points $(x_0, 0) \in TQ$ and $(\tilde{x}_0, 0) \in T\tilde{Q}$. Clearly, the systems Σ and $\tilde{\Sigma}$ satisfy conditions (MS0)-(MS2) of Theorem 2. Due to (MS1), there exist vector fields $V_1, \ldots, V_n \in \mathcal{V}$ and vector fields $\tilde{V}_1, \ldots, \tilde{V}_n \in \tilde{\mathcal{V}}$ such that the families

$$\{V_i, \mathrm{ad}_F V_i \mid 1 \le i \le n\} \text{ and } \left\{\tilde{V}_i, \mathrm{ad}_{\tilde{F}} \tilde{V}_i \mid 1 \le i \le n\right\}$$

consists of $2n$ independent vector fields. In this case we will call $(V_1, \ldots, V_n)$ to be a $\mathcal{V}$-frame and $(\tilde{V}_1, \ldots, \tilde{V}_n)$ to be a

$\widetilde{\mathcal{V}}$-frame. We assume that $G_1, \dots, G_m$ are independent and we take $V_i = G_i$, (and analogously, $\tilde{V}_i = \tilde{G}_i$), $1 \le i \le m$.

We can choose a $\mathcal{V}$-frame $(V_1, \dots, V_n)$ (and we will assume that throughout) such that any element V_j of $\mathcal{V}$ belongs to a certain $\mathcal{V}_i$ and thus it is the Lie brackets of elements of $\mathcal{V}_p$ and $\mathrm{ad}_F \mathcal{V}_l$, with $p+l \le i$. Similarly to what we have done in Section III, vector fields belonging to $\mathcal{V}_i$ shall be referred to as vector fields obtained as a succession of Lie brackets of length i and we shall choose the elements of the frame to be vector fields of smallest possible length.

We will say that two frames $(V_1, \dots, V_n)$ and $(\tilde{V}_1, \dots, \tilde{V}_n)$ are conform if each $\tilde{V}_j,\ 1 \le j \le n$, is constructed via an analogous succession of Lie brackets as that defining V_j (with the elements of $\mathcal{V}_p$ and $\mathrm{ad}_F \mathcal{V}_l$ being replaced by the corresponding ones of $\tilde{\mathcal{V}}_p$ and $\mathrm{ad}_{\tilde{F}} \tilde{\mathcal{V}}_l$).

Fix an $\mathcal{V}$-frame and define smooth functions $A^s_{i_1 \dots i_q}$ and $B^s_{i_1 \dots i_q}$ on M, called *structure functions*, via the fundamental equalities

$$\begin{aligned}\left[\mathrm{ad}_F V_{i_q}, \dots, [\mathrm{ad}_F V_{i_3}, [\mathrm{ad}_F V_{i_2}, \mathrm{ad}_F V_{i_1}]] \dots\right] = \\ (-1)^{q-1} A^s_{i_1 \dots i_q} \, \mathrm{ad}_F V_s \bmod \mathrm{span} \mathcal{V},\end{aligned} \tag{8}$$

$$\begin{aligned}\left[\mathrm{ad}_F V_{i_q}, \dots, [\mathrm{ad}_F V_{i_3}, [\mathrm{ad}_F V_{i_2}, V_{i_1}]] \dots\right] = \\ (-1)^{q-1} B^s_{i_1 \dots i_q} \, V_s\end{aligned} \tag{9}$$

$q \ge 2,\ 1 \le s, i_1, \dots, i_q \le n$. We observe that the absence of terms $\mathrm{ad}_F V_s$ on the right hand side of (9) results from the fact that the left hand side of this equality is an element of $\mathcal{V}$. Now we will show that the structure functions $A^s_{i_1 \dots i_q}$ and $B^s_{i_1 \dots i_q}$, defined by equalities (8) and (9), are very closely related to the functions $\alpha^s_{i_1 \dots i_q}$ and $\beta^s_{i_1 \dots i_q}$, defined by the fundamental equalities (LAR) and (SAR) in Section III. The relation is coming from two facts. First, the functions $A^s_{i_1 \dots i_q}$ and $B^s_{i_1 \dots i_q}$ are constant on the leaves of the canonical foliation $\mathcal{FV}$ and thus project well via π. Secondly, given a $\mathcal{V}$-frame $(V_1, \dots, V_n)$, equality (7) allows to obtain a frame $(v_1, \dots, v_n)$ on Q^Σ. We have:

Lemma 1 ([18]): Let v_i and v_j be the projections on Q^Σ of the vectors fields $\mathrm{ad}_F V_i$ and $\mathrm{ad}_F V_j$, respectively. Then:

(i) $[v_i, v_j] = \pi_* \left([\mathrm{ad}_F V_i, \mathrm{ad}_F V_j]\right)$,

(ii) $\langle v_i : v_j \rangle = \pi_* \left([F, [\mathrm{ad}_F V_i, V_j]]\right)$,

(iii) the functions $A^s_{i_1 \dots i_q}$ and $B^s_{i_1 \dots i_q}$ are constant on the leaves of the canonical foliation $\mathcal{FV}$.

(iv) The structure functions are related via

$$\begin{aligned} A^s_{i_1 \dots i_q}(z) &= \alpha^s_{i_1 \dots i_q}(\pi(z)) \\ B^s_{i_1 \dots i_q}(z) &= \beta^s_{i_1 \dots i_q}(\pi(z)). \end{aligned}$$

The above lemma explains that equality (8) projects, via π_*, to equality (LAR) on Q^Σ. Taking the Lie bracket with F of both sides of equality (9), we get

$$\begin{aligned}\left[F, \left[\mathrm{ad}_F V_{i_q}, \dots, [\mathrm{ad}_F V_{i_3}, [\mathrm{ad}_F V_{i_2}, V_{i_1}]] \dots\right]\right] = \\ (-1)^{q-1} B^s_{i_1 \dots i_q} \, \mathrm{ad}_F V_s \bmod \mathrm{span} \mathcal{V}\end{aligned}$$

which, in turn, projects via π_* to equality (SAR) on Q^Σ. Analogously, we derive $\tilde{A}^s_{i_1 \dots i_q}$ and $\tilde{B}^s_{i_1 \dots i_q}$ for the system $\tilde{\Sigma}$. As in Section III, we consider the families of structure functions for the systems Σ and $\tilde{\Sigma}$, respectively,

$$\mathfrak{S} = \left\{ A^s_{i_1 \dots i_q} \ ,\ B^s_{i_1 \dots i_q} \right\} \text{ and } \tilde{\mathfrak{S}} = \left\{ \tilde{A}^s_{i_1 \dots i_q} \ ,\ \tilde{B}^s_{i_1 \dots i_q} \right\}.$$

D. Main results: second version

Next we state the second main result of this paper, that is, a generalisation of Theorem 1 to control affine systems locally S-equivalent to $(\mathcal{GACS})$.

Theorem 3: Let Σ and $\tilde{\Sigma}$ be control systems locally S-equivalent, around equilibrium points $z_0 \in M$ and $\tilde{z}_0 \in \tilde{M}$, to geodesically affine connection systems $(\mathcal{GACS})$ and $(\widetilde{\mathcal{GACS}})$, respectively. Assume that the families $\mathfrak{S}$ and $\tilde{\mathfrak{S}}$ of structure functions, of a given conform pair of an $\mathcal{V}$-frame and an $\widetilde{\mathcal{V}}$-frame, have constant rank in neighborhoods of z_0 and $\tilde{z}_0$, respectively. The following conditions are equivalent:

(i) Σ and $\tilde{\Sigma}$ are S-equivalent, locally around z_0 and $\tilde{z}_0$;

(ii) Σ and $\tilde{\Sigma}$ are MS-equivalent, locally around z_0 and $\tilde{z}_0$;

(iii) The families $\mathfrak{S}$ and $\tilde{\mathfrak{S}}$ have the same order ρ and there exists a diffeomorphism $\psi :\ W \ \to \tilde{W}$, with W and $\tilde{W}$ neighborhoods of z_0 and $\tilde{z}_0$, respectively, such that

$$\begin{aligned} \text{(LAC)} \quad A^s_{i_1 \dots i_q} &= \tilde{A}^s_{i_1 \dots i_q} \circ \psi, \quad q \le \rho + 1, \\ \text{(SAC)} \quad B^s_{i_1 \dots i_q} &= \tilde{B}^s_{i_1 \dots i_q} \circ \psi, \quad q \le \rho + 1. \end{aligned}$$

V. Examples

We will illustrate our results with two examples.

Example 1: Let $Q = \mathbb{R}_+ \times \mathbb{S}^1$ be the configuration manifold of the following affine connection system:

$$(\mathcal{GACS}): \quad \begin{aligned} \ddot{x}^1 &= x^1 (\dot{x}^2)^2 + \cos(x^2) u_1 + \sin(x^2) u_2, \\ \ddot{x}^2 &= -\tfrac{2}{x^1} \dot{x}^1 \dot{x}^2 - \tfrac{\sin x^2}{x^1} u_1 + \tfrac{\cos x^2}{x^1} u_2. \end{aligned}$$

Clearly, the system is geodesically accessible since

$$\mathcal{SYM}(g_1, g_2)(q) = \mathrm{Vect}_{\mathbb{R}}\{g_1, g_2\}(q) = T_q Q, \text{ for all } q \in Q.$$

Writing the system as a control system on the tangent bundle TQ, we easily identify the vector fields

$$\begin{aligned} F &= y^i \frac{\partial}{\partial x^i} + x^1 (y^2)^2 \frac{\partial}{\partial y^1} - \frac{2}{x^1} y^1 y^2 \frac{\partial}{\partial y^2}, \ i = 1, 2, \\ G_1 &= \cos(x^2) \frac{\partial}{\partial y^1} - \frac{\sin x^2}{x^1} \frac{\partial}{\partial y^2}, \\ G_2 &= \sin(x^2) \frac{\partial}{\partial y^1} + \frac{\cos x^2}{x^1} \frac{\partial}{\partial y^2}, \end{aligned}$$

Direct Lie bracket computations show that

$$[\mathrm{ad}_F G_1, \mathrm{ad}_F G_2] = 0 \text{ and } [\mathrm{ad}_F G_j, G_i] = 0, \quad i, j = 1, 2.$$

Therefore the set $\mathfrak{S}$ of structure functions of the system $(\mathcal{GACS})$ has constant rank equal zero. Consider also

$$(\widetilde{\mathcal{GACS}}): \quad \begin{aligned} \ddot{\tilde{x}}^1 &= u_1, \\ \ddot{\tilde{x}}^2 &= u_2. \end{aligned}$$

Also, for this system all structure functions are identically zero, and so, the set of of structure functions $\tilde{\mathfrak{S}}$ has also constant rank zero. Any diffeomorphism ψ conjugates the structure functions of $\mathfrak{S}$ with those belonging to $\tilde{\mathfrak{S}}$, since both sets contain zero functions only. Therefore, condition

(iii) of Theorem 3 is trivially satisfied and, it follows that, the systems $(\mathcal{GACS})$ and $(\widetilde{\mathcal{GACS}})$ are MS-equivalent. In fact, we can check easily that the diffeomorphism given by

$$\begin{array}{rclrcl}\tilde{x}^1 &=& x^1\cos x^2, & \tilde{y}^1 &=& y^1\cos x^2 - x^1y^2\sin x^2,\\ \tilde{x}^2 &=& x^1\sin x^2, & \tilde{y}^2 &=& y^1\sin x^2 + x^1y^2\cos x^2,\end{array}$$

transforms $(\mathcal{GACS})$ into $(\widetilde{\mathcal{GACS}})$, showing that $(\mathcal{GACS})$ is the system $(\widetilde{\mathcal{GACS}})$, represented in polar coordinates.

Example 2: Consider a simple model of a planar rigid body, see Figure 1, whose equations of motion are

$$\begin{aligned}\ddot{\theta} &= -\frac{h}{J}u_2,\\ \ddot{x} &= \frac{\cos\theta}{m}u_1 - \frac{\sin\theta}{m}u_2,\\ \ddot{y} &= \frac{\sin\theta}{m}u_1 + \frac{\cos\theta}{m}u_2,\end{aligned}$$

with $q = (\theta, x, y) \in Q = \mathbb{S}^1 \times \mathbb{R}^2$ the configuration of the system, where θ describes the relative orientation of the body reference frame $\mathcal{R}_{\text{body}}$ with respect to the inertial (spatial) frame $\mathcal{R}_{\text{spatial}}$, the vector (x, y) denotes the position of the center of mass with respect to $\mathcal{R}_{\text{spatial}}$, m is the mass of the body and J its moment of inertia about the center of mass, $h > 0$ is the distance between a point in which is applied a force F and the center of mass O_{body}, u_1 is the component of F in the body b_1-direction and u_2 be the component in the b_2-direction (see also [2], [18]).

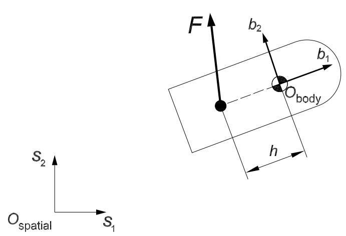

Fig. 1. The planar rigid body.

The Christoffel symbols of the corresponding Levi-Civita connection ∇ (see e.g., [2], [14]) are $\Gamma^i_{jk} = 0$. We have $g_0 = 0,\ d = 0$ and the input vector fields

$$\begin{aligned}g_1 &= \frac{\cos\theta}{m}\frac{\partial}{\partial x} + \frac{\sin\theta}{m}\frac{\partial}{\partial y} \quad \text{and}\\ g_2 &= -\frac{h}{J}\frac{\partial}{\partial\theta} - \frac{\sin\theta}{m}\frac{\partial}{\partial x} + \frac{\cos\theta}{m}\frac{\partial}{\partial y}.\end{aligned}$$

The system is geodesically accessible since

$$\mathcal{SYM}(g_1, g_2)(q) = \mathrm{Vect}_{\mathbb{R}}\{g_1(q), g_2(q), \langle g_1 : g_2\rangle(q)\} = \mathbb{R}^3,$$

for all $q \in Q$. Lie brackets and symmetric brackets computations show that all structure functions are constant and equal either one or (a function of) h/J. Therefore, the family of structure functions $\mathfrak{s}$ has rank and order zero and any system MS-equivalent to the planar rigid body must have the same structure functions as those obtained for this system (see Corollary 1). It follows that the mass m is not an invariant and we can normalize it to one (by taking $\tilde{x} = mx$ and $\tilde{y} = my$) and that the distance h and the moment of inertia J are not invariant either but their ratio is.

VI. Conclusions

The S-equivalence and MS-equivalence of geodesically accessible affine connection systems $(\mathcal{GACS})$ are studied in this paper, leading to the first version of our main results, stated in terms of objects on the configuration manifold. These results are then generalised for control affine systems locally S-equivalent to $(\mathcal{GACS})$. Our main results give necessary and sufficient conditions for the local S-equivalence and local MS-equivalence in terms of two collections of equivariants, called structure functions. Among structure functions we distinguish those that are related to the Lie algebra, generated by the input vector fields and their symmetric products, and those related to the symmetric algebra, generated by the input vector fields and their symmetric products.

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