2011 50th IEEE Conference on Decision and Control and
European Control Conference (CDC-ECC)
Orlando, FL, USA, December 12-15, 2011

Optimal L_2-gain estimator design for distributed parameter systems

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Abstract— **This paper considers the optimal induced L_2-gain estimator design problem for infinite dimensional systems with finite dimensional outputs. It is shown that this problem is equivalent to a dual regulator design problem. Moreover the relationship between the dual problem and a two player differential game is established. A solution to the latter problem is derived which, in turn solves the dual regulator design problem and the optimal estimator problem.**

I. INTRODUCTION

Distributed parameter systems occur in numerous engineering applications. Estimation of non-measured outputs of distributed systems based on measurements or observed outputs is of key importance to infer information of system variables from partial information. One typically distinguishes estimation from filtering problems. Estimation problems are concerned with the (optimal) approximation of non-observed variables from measurements, filtering problems deal with the estimation of state variables. Estimators and filters infer estimates of variables in a causal manner from observed data. The estimation problem is depicted in Fig. 1 and typically involves a given dynamical system that is affected by noise and that produces noise-corrupted measurements y, which are subsequently used to estimate a non-observed signal z. The estimator to be designed is a causal system that processes measurements y to estimates $\hat{z}$.

In this paper we present a complete solution to the design of a deterministic optimal L_2-gain output estimator for linear distributed parameter systems. Optimal L_2-gain estimators arise in finite time estimation problems and are analogues to the H_∞ estimators, which involve infinite time horizons of the to be estimated signals. The optimal L_2-gain estimator design problem is of relevance, since its solution enables the design of estimators in a robust estimation setting. In this way estimators for uncertain systems can be designed.

For finite dimensional systems it is known that there is a strong relation between estimation and control problems, which is usually evidenced using arguments in duality theory. This relation is studied for instance in [7] for finite dimensional systems and in [8] for an H_2-optimal estimator design for distributed parameter systems. In this paper, we will generalize this result to the optimal L_2-gain estimator design problem in an infinite dimensional setting. In the work of Van Keulen and Curtain, for instance see [3], the design of H_∞ optimal output feedback controllers has been studied in an infinite dimensional setting. This work presents methods for coupled estimator and controller design, which are inherent for the H_∞ framework. Estimator design in the absence of a controller can be treated as a special case of the approach in [3] and the resulting estimator will coincide with the estimator derived in this work. However, the derivation of a complete solution to the optimal L_2-gain estimator design problem presented here is of independent interest as we consider this problem from different perspectives, including a game theoretic analysis. The solution provides an intuitive interpretation to the problem, since it is based on a completion of the squares arguments. This method has been used in [6] to solve the optimal H_∞ regulator design problem for distributed parameter systems.

The paper is organized as follows. In the first section we formalize the optimal L_2-gain estimator design problem for distributed parameter systems. In the second section we introduce notion of dual systems. In the third section we introduce an L_2-gain regulator design problem for a dual system. We show the equivalence of the L_2-gain estimator design problem and the previously mentioned regulator design problem. Subsequently, we will provide a solution to the regulator design problem which is based on game theory and a completion of the squares argument. Using this result, an explicit state space realization of an optimal L_2-gain estimator is given in section four. In the last section conclusions of the presented estimator design procedure are drawn and recommendations for future work are given.

A. Notation:

We denote the inner product associated with a Hilbert space $\mathcal{X}$ by $\langle\,,\,\rangle$. The induced inner product on $L_2(\mathbb{T},\mathcal{X})$ is denoted by $\langle\langle\,,\,\rangle\rangle$, such that $\langle\langle x, x'\rangle\rangle = \int_{\mathbb{T}}\langle x(\tau), x'(\tau)\rangle \mathrm{d}\tau$. We use $||\cdot||_2$ to indicate the 2-norm on $\mathcal{X}$ as well as $L_2(\mathbb{T},\mathcal{X})$. For a operator $A : L_2(\mathbb{T},\mathbb{X}) \to L_2(\mathbb{T},\mathbb{X})$, its induced $2,2$-norm is the smallest number α for which $||Ax|| \le \alpha||x||$. The inner product on elements of $\mathbb{R}^{n\times m}$ is denoted by $\langle M_1, M_2\rangle = \operatorname{tr} M_1^* M_2$ where tr represents the trace. Let $\mathcal{M}$ denote the function space $L_2(\mathbb{T}\times\mathbb{T},\mathbb{R}^{n\times m})$.

II. PROBLEM STATEMENT

Let $\mathcal{X}$ be a Hilbert space, and let $\mathcal{Y} = \mathbb{R}^m$, $\mathcal{Z} = \mathbb{R}^n$, $\mathcal{D}_1 = \mathbb{R}^{d_1}$ and $\mathcal{D}_2 = \mathcal{Y}$ be Euclidean spaces equipped with the standard inner product. Consider the system Σ_p with states $x(t) \in \mathcal{X}$, outputs $z(t) \in \mathcal{Z}$, measurements $y(t) \in \mathcal{Y}$ and

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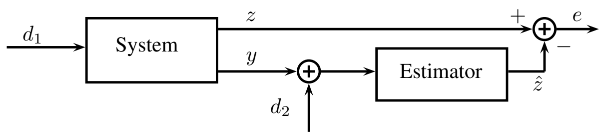

Fig. 1. Interconnection of plant and estimator.

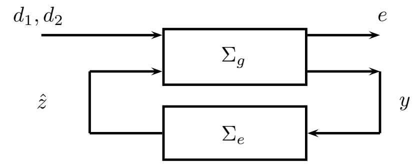

Fig. 2. Estimator design problem reformulated as controller design problem for a generalized plant Σ_g.

disturbances $d_1(t) \in \mathcal{D}_1$, $d_2(t) \in \mathcal{D}_2$, given by:

$$\Sigma_p : \begin{cases} \dot{x} = Ax + Gd_1 \\ y = Cx + Sd_2 \\ z = Hx \end{cases} \tag{1}$$

The operator $A : D(A) \to \mathcal{X}$ is a linear (possible unbounded) operator and A is the generator of a strongly continuous semi-group operator $T(t) : \mathcal{X} \to \mathcal{X}$. In this work we consider time instants $t \in \mathbb{T} = [0, t_e]$ with $t_e \in [0, \infty)$. It is assumed that $T(t)$ is exponentially stable, i.e. there exists a positive constant α such that for all $x_0 \in \mathcal{X}$, there exists an M such that $||T(t)x_0|| \leq Me^{-\alpha t}$ for all $t \in \mathbb{T}$. Moreover, we assume that the pair (A, C) is $\mathbb{T}$-observable, which means that $x_0 = 0$ whenever $CT(t)x_0 = 0$ for all $t \in \mathbb{T}$. Let $d_1 \in L_2(\mathbb{T}, \mathcal{D}_1)$ and $d_2 \in L_2(\mathbb{T}, \mathcal{D}_2)$.

We introduce a second system Σ_e, called the estimator. We demand that Σ_e is a, possibly time variant, linear mapping $L_2(\mathbb{T}, \mathcal{Y}) \to L_2(\mathbb{T}, \mathcal{Z})$ which is causal and which can be represented by the input/output-map:

$$\Sigma_e(M) : \hat{z}(t) := \int_0^{t_e} M(\tau, t) y(\tau) \mathrm{d}\tau \tag{2}$$

with $M \in \mathcal{M}$, called the convolution kernel of the estimator. The convolution kernel M is required to be in the class $\mathcal{M} = L_2(\mathbb{T} \times \mathbb{T}, \mathbb{R}^{n \times m})$, with $M(\tau, t) = 0$ whenever $\tau > t$. We indicate the parametrization of the estimator with respect to M by $\Sigma_e(M)$ when this is convenient. The estimation error is defined as $e = z - \hat{z}$. The estimator is connected to Σ_p, as shown in Figure 1. The interconnection of Σ_p with Σ_e, represents the transfer between the disturbances d_1 and d_2 and the error e. The composite system is defined by the mapping $\Sigma_p \wedge \Sigma_e : L_2(\mathbb{T}, \mathcal{D}_1 \times \mathcal{D}_2) \to L_2(\mathbb{T}, \mathcal{Z})$. The induced L_2-gain of the system after interconnection is given by:

$$||\Sigma_p \wedge \Sigma_e(M)||_{2,2} = \sup_{\substack{d_1 \in L_2(\mathbb{T}, \mathcal{D}_1) \\ d_2 \in L_2(\mathbb{T}, \mathcal{D}_2) \\ x_0 \in \mathcal{X}}} \frac{||e||_2}{\sqrt{||x_0||_2^2 + ||d_1||_2^2 + ||d_2||_2^2}}$$

We consider the following problem:

Problem 1: Determine γ^* such that?

$$\gamma^* := \inf_{M \in \mathcal{M}} ||\Sigma_p \wedge \Sigma_e(M)||_{2,2}.$$

Moreover, find the optimal estimator $M_{opt} \in \mathcal{M}$ such that $||\Sigma_p \wedge \Sigma_e(M_{opt})||_{2,2} = \gamma^*$ if it exists or, alternatively, find for all $\epsilon > 0$ the almost optimal estimator $M_\epsilon \in \mathcal{M}$ such that $\gamma^* \leq ||\Sigma_p \wedge \Sigma_e(M_\epsilon)||_{2,2} \leq \gamma^* + \epsilon$.

Note that this estimation problem involves an optimization over a finite horizon. Also note that in this problem the disturbances (d_1, d_2) as well as the initial condition x_0 are assumed to be unknown in (1). This means that the L_2-gain depends on an optimization that involves the initial condition as well. The estimation problem where the initial condition x_0 is known is different from the one considered here. For known initial conditions the estimation involves the following problem:

Problem 2: Determine $\gamma^*(x_0)$ together with an (almost) optimal estimator M'_ϵ where:

$$\gamma^*(x_0) := \inf_{M \in \mathcal{M}} \sup_{\substack{d_1 \in L_2(\mathbb{T}, \mathcal{D}_1) \\ d_2 \in L_2(\mathbb{T}, \mathcal{D}_2)}} \frac{||e||_2}{\sqrt{||x_0||_2^2 + ||d_1||_2^2 + ||d_2||_2^2}}.$$

We stress that this problem demands a different treatment.

In the next section we will show that the interconnection of the system and the estimator can be represented as a generalized plant. It will turn out that we can solve the estimator design problem for the generalized plant using duality theory.

III. Generalized plant

In the remainder of this paper we will study the following alternative representation of the problem. We introduce the system $\Sigma_g : (d_1, d_2, \hat{z}) \to (e, y)$, the generalized plant associated with the estimator design problem for Σ_p, which is given by:

$$\Sigma_g : \begin{cases} \dot{x} = Ax + \begin{bmatrix} G & 0 & 0 \end{bmatrix} \begin{bmatrix} d_1 \\ d_2 \\ \hat{z} \end{bmatrix} \\ \begin{bmatrix} e \\ y \end{bmatrix} = \begin{bmatrix} H \\ C \end{bmatrix} x + \begin{bmatrix} 0 & 0 & -I \\ 0 & S & 0 \end{bmatrix} \begin{bmatrix} d_1 \\ d_2 \\ \hat{z} \end{bmatrix} \end{cases} \tag{3}$$

The interconnection of the generalized plan Σ_g and the estimator Σ_e is realized by sharing of the variables y and $\hat{z}$ between Σ_g and Σ_e as shown in Figure 2. This interconnection defines the operator $\Sigma_g \wedge \Sigma_e : L_2(\mathbb{T}, \mathcal{D}_1 \times \mathcal{D}_2) \to L_2(\mathbb{T}, \mathcal{Z})$. Observe that the following holds.

Lemma 1: $\Sigma_p \wedge \Sigma_e(M) = \Sigma_g \wedge \Sigma_e(M)$.

The optimal L_2-gain estimator Σ_e to be designed, is the estimator which minimizes the L_2-gain of the system $\Sigma_g \wedge \Sigma_e$. Hence, the estimator design problem can be equivalently phrased as a regulator design problem.

A. Duality

We define the notion of a dual system for infinite dimensional systems, as done in [4] for finite dimensional systems. We will characterize the dual system of Σ_g, i.e.

we will provide a state space realization and use this to establish a series of problems equivalent to the estimator design problem. Consider the system Σ:

$$\Sigma : \begin{cases} \dot{x} = & Ax + Bu \\ y = & Cx + Du \end{cases} \tag{4}$$

defined on the interval $t \in \mathbb{T}$. Assume $x(0) = x_0 \in \mathcal{X}$ and let $A : D(A) \to \mathcal{X}$ be the infinitesimal generator of an exponentially stable semi group $S(t)$ for $t \in \mathbb{T}$. It follows from [2] that the unique mild solution is given by:

$$\begin{aligned} y(t) =& CS(t)x_0 + \int_0^t CS(t-\tau)Bu(\tau)\mathrm{d}\tau + Du(t) \\ =:& G(x_0, u)(t) \end{aligned} \tag{5}$$

where G is the system operator mapping from $(x_0, u) \in \mathcal{X} \times L_2(\mathbb{T},\mathcal{U})$ to $y \in L_2(\mathbb{T},\mathcal{Y})$. This mapping is well defined under the assumption that A is exponentially stable.

Definition 1: Let Σ be a system with system operator G defined as above. The dual system of Σ, denoted by Σ^*, is the system with system operator $G^* : L_2(\mathbb{T},\mathcal{Y}) \to \mathcal{X} \times L_2(\mathbb{T},\mathcal{U})$, and is defined as the Hilbert adjoint of G. Hence, G^* is the operator for which $\langle\langle G(x_0, u), \tilde{y}\rangle\rangle = \langle\langle (x_0, u), G^*\tilde{y}\rangle\rangle$ for all $(x_0, u) \in \mathcal{X} \times L_2(\mathbb{T},\mathcal{U})$ and $\tilde{y} \in L_2(\mathbb{T},\mathcal{U})$, where $\langle\langle (x_0, u), (\tilde{x}_0, \tilde{u})\rangle\rangle := \langle x_0, \tilde{x}_0\rangle_{\mathcal{X}} + \langle\langle u, \tilde{u}\rangle\rangle$. The following theorem relates the state space realizations of a system to the state space realization of its dual system. In order to prove the theorem, the following lemma is needed.

Lemma 2: [5, theorem 10.8] The infinitesimal generator of the adjoint semi group is the adjoint of the infinitesimal generator of the original semi group.

Theorem 1: Let the system operator G be defined by equation (5). Then the dual operator G^* is given by $G^*\tilde{y} := (\tilde{x}_0, \tilde{u})$ with:

$$\begin{aligned} \tilde{x}_0 =& \int_t^{t_e} S^*(\tau)C^*\tilde{y}(\tau)\mathrm{d}\tau \\ \tilde{u}(t) =& \int_t^{t_e} B^*S^*(\tau - t)C^*\tilde{y}(\tau)\mathrm{d}\tau + D^*\tilde{y}(t) \end{aligned}$$

where $t \in \mathbb{T}$ and $\tilde{S}(t) : \mathcal{X} \to \mathcal{X}$ is semi-group operator with generator $-A^*$ for $t \in \mathbb{T}$.

Proof: Consider the following differential equation with endpoint condition $p(t_e) = 0$:

$$\tilde{\Sigma} : \begin{cases} \dot{p} = & -A^*p - C^*\tilde{y} \\ \tilde{u} = & B^*p + D^*\tilde{y} \end{cases}$$

Given that A is the infinitesimal generator of semigroup $S(t)$ for $t \in \mathbb{T}$, it follows from Lemma 2 that A^* is the infinitesimal generator of $S^*(t)$. Suppose $x_0, p_0 \in \mathcal{X}$, $u, \tilde{u} \in \mathcal{U}$ and $y, \tilde{y} \in \mathcal{Y}$. We observe that the following relation holds:

$$\frac{\mathrm{d}}{\mathrm{d}t}\langle x(t), p(t)\rangle + \langle y(t), \tilde{y}(t)\rangle = \langle u(t), \tilde{u}(t)\rangle$$

since

$$\begin{aligned} &\frac{\mathrm{d}}{\mathrm{d}t}\langle x(t), p(t)\rangle + \langle y(t), \tilde{y}(t)\rangle = \\ &\quad \langle Ax(t) + Bu(t), p(t)\rangle + \langle Cx(t) + Du(t), \tilde{y}(t)\rangle + \\ &\quad \langle x(t), -A^*p(t) - C^*\tilde{y}(t)\rangle = \langle u(t), \tilde{u}(t)\rangle \end{aligned}$$

Hence after integration over $\mathbb{T} = [0, t_e]$, we infer

$$\langle x(t_e), p(t_e)\rangle - \langle x_0, p(0)\rangle + \langle\langle G(x_0, u), \tilde{y}\rangle\rangle = \langle\langle u, \tilde{u}\rangle\rangle$$

In particular, with the end-condition $p(t_e)$ we find

$$\langle\langle G(x_0, u), \tilde{y}\rangle\rangle = \langle x_0, p(0)\rangle + \langle\langle u, \tilde{u}\rangle\rangle$$

By definition, the left hand side equals $\langle\langle (x_0, u), G^*\tilde{y}\rangle\rangle$ so we infer that:

$$\langle\langle (x_0, u), G^*\tilde{y}\rangle\rangle = \langle x_0, p(0)\rangle + \langle\langle u, \tilde{u}\rangle\rangle.$$

This shows that $G^*\tilde{y} = (\tilde{x}_0, \tilde{u})$, where $\tilde{x}_0 = p(0)$ with $p(t)$ the solution of (1) given by:

$$\begin{cases} p(t) = & S^*(t_e - t)p(t_e) + \int_t^{t_e} S^*(\tau - t)C^*\tilde{y}(\tau)\mathrm{d}\tau \\ \tilde{u}(t) = & B^*p(t) + D^*\tilde{y}(t) \end{cases}$$

This concludes the proof. ∎

The theorem above enables to determine the dual system once system Σ and the initial condition is given.

The following lemma indicates that the L_2-gain of a system given by (4) and its dual system are equal.

Lemma 3: The L_2 gain of a system Σ and its dual system Σ^* are equal, i.e $||\Sigma||_{2,2} = ||\Sigma^*||_{2,2}$. This is a standard result for operators on a Hilbert space, which can be found in e.g. [1, thm 3.9-2].

Using Theorem 1, it follows that Σ_g^* is characterized by:

$$\Sigma_g^* : \begin{cases} \dot{p} = -A^*p - \begin{bmatrix} H^* & C^* \end{bmatrix} \begin{bmatrix} \tilde{e} \\ \tilde{y} \end{bmatrix} \\ \begin{bmatrix} \tilde{d}_1 \\ \tilde{d}_2 \\ \hat{\tilde{z}} \end{bmatrix} = \begin{bmatrix} G^* \\ 0 \\ 0 \end{bmatrix} p + \begin{bmatrix} 0 & 0 \\ 0 & S^* \\ -I & 0 \end{bmatrix} \begin{bmatrix} \tilde{e} \\ \tilde{y} \end{bmatrix} \end{cases} \tag{6}$$

From Lemma 3 it directly follows that the L_2-gain Σ_g and Σ_g^* are equal, i.e. $||\Sigma_g||_{2,2} = ||\Sigma_g^*||_{2,2}$.

We define an output feedback regulator $\Sigma_c : L_2(\mathbb{T}, \mathcal{Z}) \to L_2(\mathbb{T}, \mathcal{Y})$ with interconnection variables z and y. The interconnection of Σ_g^* and an output feedback regulator Σ_c is realized by interconnection of the variables $\tilde{y}$ and y resp. $\hat{\tilde{z}}$ and z as shown in Figure 3. This interconnection defines the operator $\Sigma_g^* \wedge \Sigma_c^* : L_2(\mathbb{T}, \mathcal{Z}) \to L_2(\mathbb{T}, \mathcal{D}_1 \times \mathcal{D}_2)$. We introduce the following lemma:

Lemma 4: Let Σ_g be given by (3) and let Σ_c^* be a output feedback regulator for Σ_g^* and let the interconnections $\Sigma_c \wedge \Sigma_g$ and $\Sigma_c^* \wedge \Sigma_g^*$ be as defined. Then following equality holds:

$$(\Sigma_c \wedge \Sigma_g)^* = \Sigma_c^* \wedge \Sigma_g^*.$$

The equality can be derived by straight forward calculation of $(\Sigma_c \wedge \Sigma_g)^*$ and $\Sigma_c^* \wedge \Sigma_g^*$ using their state space representations.

We remark that the system Σ_e^* can be interpreted Σ_e^* as output feedback regulator for Σ_g^*. Moreover we define a state feedback regulator $\Sigma_s : L_2(\mathbb{T}, \mathcal{X}) \to L_2(\mathbb{T}, \mathcal{Y})$ with interconnection variables q and y. The interconnection of Σ_g^* and a state feedback regulator Σ_s is realized by interconnection of the variables p and q resp. $\tilde{y}$ and y. This interconnection defines the operator $\Sigma_g^* \wedge \Sigma_s : L_2(\mathbb{T}, \mathcal{Z}) \to L_2(\mathbb{T}, \mathcal{D}_1 \times \mathcal{D}_2)$.

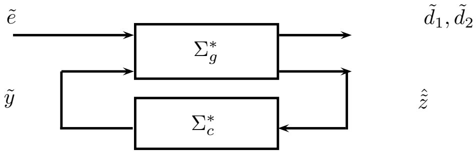

Fig. 3. Controller design problem for the dual system of the generalized plant Σ_g^*.

We are now in the position to present the main contribution of this paper. We will show that the estimator design problem presented above can be formulated as regulator design problem for the dual system, interconnected as in figure 3. It will turn out that due to the specific structure of the equivalent regulator design problem, this problem admits an insightful solution based on a completion of the squares argument.

IV. Main result

The main result of this work will be to show the relation between four problems associated to Σ_g, Σ_g^* and a two player zero sum differential game. We will show that for one of the problems a solution can be found, which can be used to solve the related problems and finally provide a solution to the estimator design problem defined in problem 1. Associated to Σ_g^*, we introduce the linear functional $J_\gamma(\tilde{y}, \tilde{e}, p_{t_e})$, which is, for fixed γ, defined as:

$$J_\gamma(\tilde{y}, \tilde{e}, p_{t_e}) = ||p(0)||_2^2 + ||\tilde{d_1}||_2^2 + ||\tilde{d_2}||_2^2 - \gamma^2 ||\tilde{e}||_2^2$$

and where $(\tilde{d_1}, \tilde{d_2}, \tilde{e})$ satisfy the system evolution of Σ_g^*, given by the differential equation in (6). Secondly, we introduce the following problems:

- P. 1 Find an estimator $\Sigma_e : L_2(\mathbb{T}, \mathcal{Y}) \to L_2(\mathbb{T}, \mathcal{Z})$ for Σ_g such that the L_2-gain of the interconnection is less than γ, as shown in Figure 2,i.e. $||\Sigma_g \wedge \Sigma_e|| < \gamma$
- P. 2 Find an output feedback regulator $\Sigma_c : L_2(\mathbb{T}, \mathcal{Z}) \to L_2(\mathbb{T}, \mathcal{Y})$ such that the L_2-gain of the interconnection of Σ_g^* and Σ_c, as shown in Figure 3, is less than γ, i.e. $||\Sigma_g^* \wedge \Sigma_c|| < \gamma$
- P. 3 Find a state feedback regulator $\Sigma_s : L_2(\mathbb{T}, \mathcal{X}) \to L_2(\mathbb{T}, \mathcal{Y})$, with realization $y_s(t) = F(t)p(t)$ and $F(t) \in \mathcal{L}(\mathcal{X}, \mathcal{Y})$, such that the L_2-gain of the interconnection of Σ_g^* and Σ_s is less than γ, i.e. $||\Sigma_g^* \wedge \Sigma_s|| < \gamma$
- P. 4 Two-player zero-sum differential game with full state information feedback and value function $J_\gamma(\tilde{y}, \tilde{e}, p_{t_e})$. Find a strategy $(y_N \in L^2(\mathbb{T}, \mathbb{R}^y)$, $(e_N \in L^2(\mathbb{T}, \mathbb{R}^z)$ such that a Nash equilibrium is established. This is characterized by:

  $$J_\gamma(y_N, e, p_{t_e}) \leq J_\gamma(y_N, e_N, p_{t_e}) \leq J_\gamma(y, e_N, p_{t_e})$$

  for all $y \in L^2(\mathbb{T}, \mathbb{R}^y)$ and $e \in L^2(\mathbb{T}, \mathbb{R}^z)$.
- P. 5 Find a strategy $(y_o(e_o), e_o)$ that establishes a sup-inf equilibrium for J_γ in the sense that:

  $$J_\gamma(y_o(e), e, p_{t_e}) = \inf_y J_\gamma(y, e, p_{t_e}) \quad \text{for all } e$$

  and:

  $$J_\gamma(y_o(e_o), e_o, p_{t_e}) = \sup_e \inf_y J_\gamma(y, e, p_{t_e}).$$

In the remainder of this section we will show that the solution of problem 5 provides a solution to all of the problems stated above. This follows from the following theorems. The proofs of the theorems are given at the end of this section. First the relation between problem $P1$ and problem $P2$ is expressed by the following theorem.

Theorem 2 (relation $P1$ and $P2$): The estimator Σ_e solves problem $P1$ if and only if the controller $\Sigma_c = \Sigma_e^*$ solves problem $P2$.

Due to the special structure of the generalized plant, a solution to the state feedback regulator design problem $P3$ provides a solution to the output feedback regulator design problem $P2$.

Theorem 3 (Relation between $P3$ and $P2$): Suppose that the state feedback regulator Σ_s implements a feedback law $\tilde{y}_s(t) = Ftpt$ such that it solves $P3$. Then $y_s(t)$ provides a solution to problem $P2$.

Next, we state the relation between problem $P4$ and problem $P3$. It will turn out that the Nash equilibrium strategy provides a solution to the state feedback regulator design problem.

Theorem 4 (relation $P4$ and $P3$): If the zero sum strategy (y_N, e_N) establishes a Nash equilibrium for $J_\gamma(y, e, p_{t_e})$, then the strategy y_N can be interpreted as a state feedback regulator which solves $P3$.

Theorem 5 (relation $P5$ and $P4$): Suppose that a solution to the sup-inf problem $P5$ is given by the strategy $(y_o(e_o), e_o)$. Then $(y_o(e_o), e_o)$ is a Nash equilibrium strategy and provides a solution to problem $P4$.

The theorems show that the solution to problem $P5$ provides a solution to $P1$ since $P5$ solves $P4$, $P4$ solves $P3$, $P3$ solves $P2$ and $P2$ solves $P1$. Before we proceed with the proof theorems above, a number of lemmas regarding the solution of $P4$ and $P5$ will be introduced. We show that under full information feedback the two player zero sum differential game has an unique Nash-equilibrium. To do so, we first show that the value function J_γ can be rewritten in a more convenient form.

Lemma 5: Suppose there exists a symmetric operator $P(t)$, which is for all $p_1, p_2 \in D(A)$ a solution of differential equation:

$$\begin{aligned} \langle p_1, \dot{P} p_2 \rangle =& \langle PA^* p_1, p_2 \rangle + \langle p_1, PA^* p_2 \rangle + \langle G^* p_1, G^* p_2 \rangle \\ & - \langle P(C^*(SS^*)^{-1} C - \gamma^{-2} H^* H) P p_1, p_2 \rangle. \end{aligned} \tag{7}$$

with $P(0) = I$. Then for all $(\tilde{d_1}, \tilde{d_2}, \tilde{e}, \tilde{y}, \hat{\tilde{z}})$ that satisfy (6) we have:

$$\begin{aligned} J_\gamma(y, e, p(t_e)) =& \langle p_{t_e}, P(t_e) p_{t_e} \rangle + ||\gamma^{-1} HPp + \gamma e||_2^2 \\ & + ||(SS^*)^{-\frac{1}{2}} CPp + (SS^*)^{\frac{1}{2}} y||_2^2 \end{aligned} \tag{8}$$

Proof: We introduce the identity $V(p(t)) = \langle p(t), P(t) p(t) \rangle$ and differentiate V with respect to t. We omit the time index for brevity.

$$\dot{V}(p) = \langle p, P\dot{p}\rangle + \langle \dot{p}, Pp\rangle + \langle p, \dot{P}p\rangle.$$

Then we substitute the dynamics of Σ_g^* for $\dot{p}$. Given that P solves the Ricatti equation for all $p \in D(A^*)$, substitute $\dot{P}(t)$ with the Ricatti equation with $p_1 = p_2 = p(t)$:

$$\begin{aligned}\dot{V}(p) =& \langle p, P(-A^*p - C^*y - H^*e)\rangle \\ &+ \langle -A^*p - C^*y - H^*e, Pp\rangle \\ &+ \langle PA^*p, p\rangle + \langle p, PA^*p\rangle + \langle G^*p, G^*p\rangle \\ &+ \gamma^{-2}\langle HPp, HPp\rangle - \langle S^{-1}CPp, S^{-1}CPp\rangle\end{aligned}$$

Subsequently, one can reformulate this using a completion of the squares argument. We use $d_1 = G^*p$ and $d_2 = S^*y$ and after integration of the right- and left-hand side from $t = 0$ to $t = t_e$ one obtains an expression that equals the definition of $J_\gamma(y, e, p_{t_e})$:

$$\begin{aligned}&\langle p(0), P(0)p(0)\rangle + \int_0^{t_e} ||d_1(t)||_2^2 + ||d_2(t)||_2^2 - \gamma^2||e(t)||_2^2 \mathrm{d}t \\ &= \langle p_{t_e}, P(t_e)p_{t_e}\rangle + \int_0^{t_e} ||(SS^*)^{-\frac{1}{2}}CPp(t) + (SS^*)^{\frac{1}{2}}y(t)||_2^2 \\ &- ||\gamma e(t) - \gamma^{-1}HPp(t)||_2^2 \,\mathrm{d}t.\end{aligned}$$

This concludes the proof. ■

From the result above a Nash equilibrium strategy that provides a solution to $P4$ follows intermediately.

Theorem 6 (Solution to problem $P4$.): The strategy (y_N, e_N) which is definede as:

$$e_N(t) = +\gamma^{-2}HPp(t) \tag{9a}$$

$$y_N(t) = -(SS^*)^{-1}CPp(t). \tag{9b}$$

establishes a unique Nash equilibrium. The value of the game under the equilibrium strategy is $J_\gamma(y_N, e_N, p_{t_e}) = \langle p_{t_e}, P(t_e)p_{t_e}\rangle$. Under the Nash equilibrium strategy the closed loop dynamics is given by:

$$\dot{p}_N(t) = (-A^* + (C^*(SS^*)^{-1}C - \gamma^{-2}H^*H)P)p_N(t).$$

Proof: Using Lemma 5 we rewrite the value function $J_\gamma(y, e, p(t_e))$ as done in equation (8). When the value function is evaluated at (9) the quadratic terms vanish. By convexity of the norms uniqueness of the Nash Equilibrium strategy follows and we have that $J_\gamma(y_N, e_N, p_{t_e}) \leq J_\gamma(y, e_N, p_{t_e})$ resp. $J_\gamma(y_N, e, p_{t_e}) \leq J_\gamma(y_N, e_N, p_{t_e})$. Therefore, the following inequality holds for all $y \in L^2(\mathbb{T}, \mathbb{R}^y)$ and $e \in L^2(\mathbb{T}, \mathbb{R}^z)$

$$J_\gamma(y_N, e, p_{t_e}) \leq J_\gamma(y_N, e_N, p_{t_e}) \leq J_\gamma(y, e_N, p_{t_e}),$$

which shows that y_N, e_N establishes a Nash equilibrium with value $J_\gamma(y_N, e_N, p_{t_e}) = \langle p_{t_e}, P(t_e)p_{t_e}\rangle$. The closed loop dynamics follow immediately by substitution of the Nash equilibrium strategy into (6). ■

A sup-inf strategy to solve problem $P5$ can be derived by solution of two linear quadratic optimization problems. The first problem solves the inf-problem parameterized by a fixed but arbitrary disturbance strategy e to determine the strategy $y_o(e)$ which minimizes $J_\gamma(y_o(e), e, p(t_e))$. The second problem solves the sup-problem to determine the strategy e_o, which maximizes the value function $J_\gamma(y_o(e), e, p(t_e))$.

Lemma 6 (Solution to $P5$): A strategy which establishes a sup-inf strategy to J_γ is given by:

$$\begin{aligned}y_o(t) =& -(SS^*)^{-1}C(\Gamma + \Lambda)p(t) \\ e_o(t) =& \gamma^2 H(\Gamma + \Lambda)p(t)\end{aligned}$$

where Γ and Λ solves the Ricatti equations given by (10) resp. (11), with boundary values $\Gamma(0) = I$ resp. $\Lambda(0) = 0$.

$$\begin{aligned}\langle p_1, \dot{\Gamma}p_2\rangle =& \langle p_1, A\Gamma p_2\rangle + \langle p_1, \Gamma A^* p_2\rangle + \langle p_1, GG^* p_2\rangle \\ &- \langle p_1, \Gamma C^*(SS^*)^{-1}C\Gamma p_2\rangle \qquad (10) \\ \langle p_1, \dot{\Lambda}p_2\rangle =& \langle p_1, \tilde{A}\Lambda p_2\rangle + \langle p_1, \Lambda\tilde{A}^* p_2\rangle + \langle p_1, \Gamma H^* H\Gamma p_2\rangle \\ &- \langle p_1, \Lambda(C^*(SS^*)^{-1}C - \gamma^{-2}H^*H)\Lambda p_2\rangle \quad (11)\end{aligned}$$

where $\tilde{A} = (A - \Gamma(C^*(SS^*)^{-1}C - \gamma^{-2}H^*H))$. The proof of this lemma is omitted for brevity. The solution can be derived along the line of section 3 in [6].

A. Proofs of the main result

In next section we will provide a proof to theorems stated before.

Proof: [Proof of theorem 2, relation $P1$ and $P2$.] Let the solution to problem $P1$ be given by Σ_e. Then the system $\Sigma_g \wedge \Sigma_e$ is defined by the interconnection of Σ_g with Σ_e and satisfies $||\Sigma_g \wedge \Sigma_e||_{2,2} \leq \gamma$. We define $\Sigma_e^* : L_2(\mathbb{T}, \mathcal{Z}) \to L_2(\mathbb{T}, \mathcal{Y})$ as the dual system of Σ_e. From Lemma 3 it follows that $||(\Sigma_g \wedge \Sigma_e)^*||_{2,2} \leq \gamma$. Moreover, from lemma 4 it follows that $(\Sigma_g \wedge \Sigma_e)^* = (\Sigma_g^* \wedge \Sigma_e^*)$, such that it follows that $||(\Sigma_g^* \wedge \Sigma_e^*||_{2,2} \leq \gamma$. Therefore $\Sigma_c = \Sigma_e^*$ provides a solution to problem $P2$. The converse holds on the basis of the same arguments. ■

Proof: [Proof of theorem 3, relation $P3$ and $P2$.] Given that the regulator $\Sigma_s : y_s(t) = F(t)p(t)$ solves problem $P3$, it follows that the interconnection of Σ_s and system Σ_g^* has L_2-gain less then or equal to γ. After interconnection of Σ_g^* with Σ_s the system admits the following dynamics.

$$\Sigma_g^* \wedge \Sigma_s = \begin{cases}\dot{p}(t) = & (-A^* - C^*F(t))p(t) - H^*\tilde{e}(t) \\ y_s(t) = & Fp(t), \quad \text{with: } p(t_e) = 0.\end{cases}$$

From the generalized plant, equation (6) specifically, it follows that $\tilde{e} = -\tilde{\tilde{z}}$. Therefore $y_s(t)$ can be implemented by a operator $\tilde{\tilde{z}} \to y$, by replacement of $\tilde{e}$ by $-\tilde{\tilde{z}}$ in the system above. This operator can be interpreted as an output feedback regulator which solves problem $P2$. ■

Next we will show that a solution to $P4$ provides a solution to $P3$.

Proof: [Proof of theorem 4, relation $P4$ and $P3$.] In Theorem 6 it is shown that the equilibrium strategy can be implemented as a state feedback law for system Σ_g^*. Given that (y_N, e_N) establishes a Nash equilibrium, for all $e \in Z$ we have $J_\gamma(y_N, e, p_{t_e}) \leq J_\gamma(y_N, e_N, p_{t_e})$. Moreover we have that $J_\gamma(y_N, e_N, p_{t_e}) = \langle p_{t_e}, P(t_e)p_{t_e}\rangle$, such that it follows that $J_\gamma(y_N, e, p_{t_e}) \leq \langle p_{t_e}, P(t_e)p_{t_e}\rangle$ for all $e \in$

$L_2(\mathbb{T}, \mathcal{Z})$. For system Σ_g^*, we have defined $p_{t_e} = 0$. Therefore, we infer that if y_N is applied to the system Σ_g^* then $||p(0)||_2^2 + ||d_1||_2^2 + ||d_2||_2^2 - \gamma^2 ||e||_2^2 \leq 0$ for all $e \in L_2(\mathbb{T}, \mathcal{X})$. This is equivalent to:

$$\sup_{e \in L_2(\mathbb{T}, \mathcal{X})} \frac{||p(0)||_2^2 + ||d_1||_2^2 + ||d_2||_2^2}{||e||_2^2} \leq \gamma^2.$$

Therefore we conclude that if the system is driven with $\tilde{y} = y_N$, the L_2-gain of the system is less or equal to γ, which concludes the proof. ■

When we compare the solutions to problem $P4$ and $P5$, the relation between the Nash equilibrium strategy and the sup-inf equilibrium strategy follows.

Proof: [Proof of theorem 5, relation $P5$ and $P4$] First we make the following observation. Assume that solutions to the Ricatti equations given by equations (7), (10) and (11) exist with boundary values $P(0) = I, \Gamma(0) = I$ and $\Lambda(0) = 0$. Calculation shows that $\langle p_1, (\dot{P} - (\dot{\Gamma} + \dot{\Lambda}))p_2 \rangle = 0$ for all $p_1, p_2 \in D(A)$. Moreover it follows from the boundary conditions that $P(0) = \Gamma(0) + \Lambda(0)$. We conclude that $P(t) = \Gamma(t) + \Lambda(t)$. The equivalence of Nash equilibrium strategy and the sup-inf strategy follows now follows from the realizations of y_o and y_N resp. e_o and e_N. ■

In this section we have shown the equivalence between the problems $P1$ and $P2$ and the relationship between the problems $P2$ and $P3$, problem $P3$ and $P4$ and problem $P4$ and $P5$. This enables to solves the optimal L_2-gain estimator design problem. The solution to this problem will presented in the next section.

V. Solution to estimator design problem.

In this section we will derive the solution to the optimal L_2-gain estimator design problem with use of the relations between the problems as established in the previous section. Given the relation between problem $P4$ and $P3$ as stated before, it follows that the Nash equilibrium strategy (y_N, e_N) given by (9), also provides a state feedback law which implements a state regulator that solves problem $P3$. Therefore, we infer that the feedback law

$$\tilde{y}_s(t) = -(SS^*)^{-1} C P(t) p(t),$$

provides a solution to $P3$. With use of Theorem 3 an output feedback regulator which solves $P2$ can be derived from the solution to $P3$. We find that the output feedback regulator with the following realization:

$$\Sigma_c : \begin{cases} \dot{p}(t) = (-A^* + C^*(SS^*)^{-1} C P(t)) p(t) + H^* \tilde{\hat{z}}(t) \\ \tilde{y}(t) = -(SS^*)^{-1} C P(t) p(t) \end{cases}$$

with $p(t_e) = 0$ solves $P2$. In this realization, $P(t)$ is the positive solution of the operator Ricatti equation (7) and satisfies the boundary condition $P(0) = I$. The system obtained after interconnection of the regulator Σ_c and the plant Σ_g^* has L_2-gain less then or equal to γ. With use of the equivalence of Problem $P1$ and Problem $P2$ it now follows that an estimator which solves problem $P1$ can be obtained from dualization of the regulator Σ_c which solves problem $p2$. Using the duality theory introduced in section III-A, it now follows that an estimator to solve problem $P1$ is realized $\Sigma_e = \Sigma_c^*$:

$$\Sigma_e : \begin{cases} \dot{\xi} = (A - PC(S^*S)^{-1} C^*) \xi + PC^*(SS^*)^{-1} y \\ \hat{z} = H \xi \quad \text{with: } \xi(0) = 0, \end{cases} \tag{12}$$

where $P(t)$ is again the solution to Ricatti equation given by equation (7) with initial condition $P(0) = I$. We conclude with the observation that the estimator solving problem $P1$ realizes the input/output mapping:

$$\hat{z}(t) = \int_0^t H T_{P(t)C(S^*S)^{-1}C^*}(t - \tau) P(t) C^* (SS^*)^{-1} y(\tau) \mathrm{d}\tau,$$

where $t \in \mathbb{T}$ and $T_{P(t)C(S^*S)^{-1}C^*}$ is the mild evolution operator with infinitesimal generator $A - P(t)C(S^*S)^{-1}C^*$.

Remark 1: It is interesting to remark that the solution to problem 2 proceeds along almost identical lines. That is, for $x_0 = 0$ one can shown that the solution to problem 2 is equal to (12) with the initial condition to the differential equation (7) set to $P(0) = 0$.

VI. Conclusions

A method for the design of optimal L_2-gain estimators for distributed parameter systems is presented in this paper. The method is based on duality theory for distributed parameter systems with finite dimensional input and outputs. The method exploits the special structure in the estimation problem, which enables to reformulate the estimation problem as a regulation problem. It has been shown that there exist a two player differential game problem for which the value function is quadratic in its decision variables. The quadratic structure enables one to apply a completion of the squares argument to obtain the equilibrium strategy for the game. It has been shown that the equilibrium strategy provides a solution to the equivalent optimal L_2-gain regulator design problem and indirectly enables to solve the optimal L_2-gain estimator design problem.

References

[1] E. Kreyszig, Introductory functional analysis with applications, 1989, John Wiley & Sons New York.

[2] Curtain R.F. and Zwart H.J. An introduction to infinite-dimensional linear systems theory, 1995 , Springer.

[3] Van Keulen, B. $H - \infty$-Control for Distributed Parameter Systems: a state space approach, 1993, Birkhuser Boston.

[4] Van der Schaft A.J. Duality for linear systems: External and state space characterizations of the adjoint system, In Analysis of controlled dynamical systems: proceedings of a conference held in Lyon, France, July 1990, Volume 8 van Progress in systems and control theory,P 393-403

[5] Fuhrmann, P.A. Linear systems and operators in Hilbert space, 1981, McGraw-Hill New York

[6] Bensoussan, A. and Bernhard, P. On the standard problem of H_∞-optimal control for Infinite Dimensional Systems, 1992, Chapter 9 in Identification and control in systems governed by partial differential equations

[7] Mutsaers, M.E.C and Weiland, S. Reduced-Order Observer Design using a Lagrangian Method, 2009, Conference on Decision and Control.

[8] Vissers, J.A.W. and Weiland, S. Design of optimal deterministic output estimators for distributed parameter systems, 2010, Conference on Mathmatical Theory of Networks and Systems.