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# Game Couplings: Learning Dynamics and Applications

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***Abstract*— Modern engineering systems (such as the Internet) consist of multiple coupled subsystems. Such subsystems are designed with local (possibly conflicting) goals, with little or no knowledge of the implementation details of other subsystems. Despite the ubiquitous nature of such systems very little is formally known about their properties and global dynamics.**

**We investigate such distributed systems by introducing a novel game-theoretic construct, that we call game-coupling. Game coupling intuitively allows us to stitch together the payoff structures of two or more games into a new game. In order to study efficiency issues, we extend the price of anarchy framework to this setting, where we now care about local and global performance. Such concerns give rise to a new notion of equilibrium, as well as a new learning paradigm. We prove matching welfare guarantees for both, both for individual subsystems as well as for the global system, using a generalization of the $(\lambda, \mu)$-smoothness framework [17].**

**In the second part of the paper, we establish conditions leading to advantageous couplings that preserve or enhance desirable properties of the original games, such as convergence of best response dynamics and low price of anarchy.**

## I. INTRODUCTION

Game-theoretic approaches are successful in designing and analyzing distributed systems and have recently been used for distributed control [9], [10], [14], [20]. A natural stable state of a system of agents corresponds to the game-theoretic concept of equilibrium, at which each agent minimizes its own cost. Basic properties of equilibria, such as existence, reachability via learning dynamics, and total cost are quite well-understood if the entire population of agents is homogeneous [6], [8], [10], [13], [16]. In many systems, especially large ones, such homogeneity is only encountered locally at a group level while overall the system is heterogeneous. For example in routing of Internet traffic the same time delay results in much higher costs for low-latency applications such as Voice-over-IP than for other applications such as email. Similarly, in load balancing (assignment of jobs to machines) the delay of a job mostly depends on jobs that use heavily the same component of the machine (e.g. processors or input/output). What are the prerequisites or the tradeoffs for local guarantees to carry over to the global level when *coupling* different groups? We formally study such questions (largely overlooked thus far) by introducing *game-couplings*.

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We consider games (systems) with $J \geq 2$ disjoint groups $N^1 \dots N^J$ of players (agents) such that agents in group $N^j$ take part in a game with a desirable property $\mathcal{P}$ for any fixed behavior of all agents from other groups. We define such settings as couplings of $\mathcal{P}$-games. We examine the effect of different localized classes of properties $\mathcal{P}$ on the performance of each group. We also look for necessary or sufficient conditions for couplings to preserve (to some degree) $\mathcal{P}$.

We first study the property of $(\lambda, \mu)$-smoothness of cost changes to individual strategy updates. $(\lambda, \mu)$-smoothness provably implies tight bounds on the inefficiency of standard equilibrium concepts in several game classes [15], [17], [18]. We show tight guarantees on group cost when each subgame $N^j$ exhibits local $(\lambda^j, \mu^j)$-smoothness. These performance guarantees carry over to a wide set of equilibrium notions as well as for no-regret learning algorithms. Using LP-duality arguments, we identify a new equilibrium notion for which all these bounds extend automatically for *all* groups.

Furthermore, we introduce a novel learning framework modeling interactions of competing groups/institutions (e.g. Internet providers). In our framework each group has a center that provides public advertisement [2], [3] of (possibly different) strategies to each player in the group. We analyze centers whose advertised behaviors exhibits vanishing average regret with hindsight. This is a rather natural benchmark, since several simple learning algorithms offer such guarantees [5]. On the side of the individual agents, we make a similarly weak assumption. We assume that the average performance of each agent will eventually be roughly as high as that of her advertised strategy. We prove tight welfare guarantees for each group in this framework as well as global guarantees which match those of our proposed equilibrium.

We give conditions for couplings to preserve the existence of types of potential functions. Specific learning dynamics are known to converge [9], [13] given a potential, i.e. a strong measure of cost alignment among players. Our necessary and sufficient condition for preserving an exact potential leverages a condition in [13]. We also provide conditions for game couplings to preserve the existence of a weak potential (equivalently, weak acyclicity), a more general notion guaranteeing weaker convergence than an exact potential.

We apply our techniques to solve in the affirmative an open question regarding the convergence of Nash dynamics of a heterogeneous population in aggregation games [11].

RELATED WORK. Structural properties of games often follow from analysis of their sub-games. Sandholm [19] considers sub-games defined by all subsets of players (as opposed to a specific partition as we do here) and shows that a game has an exact potential if and only if all active players in a sub-game have identical utility functions. Fabrikant et al. [6] show that the uniqueness of a Nash equilibrium in any sub-game is sufficient, but not necessary, for a game to be weakly acyclic. Monderer [12] defines the classes of $J$-potential games and $J$-congestion games for $J \in \mathbb{N}$ and shows they are isomorphic. In a $J$-congestion game, players' mappings of costs to delay can belong in $J$ classes. The case $J = 1$ is treated in Monderer and Shapley's seminal paper [13]. An instance captured by our framework is $J = 2$ and $N^1, N^2$ being the groups of players with delays from the same class.

Our view of a population divided into groups is often adopted in distributed adaptive control, where a center can only control a local group of agents, e.g. in the collective intelligence [20] and probability collectives [21] frameworks.

## II. Preliminaries

We model distributed systems as games $G$ with simultaneous moves. For a game $G$ we denote by $\{1, \dots, n\}$ the players (agents), by $\Sigma_i$ player $i$'s set of (pure) strategies, and by $\Delta(\Sigma)$ all distributions over outcomes $\Sigma = \times_{i=1}^{n} \Sigma_i$. Any player $i$ aims to minimize her *cost*[1] $C_i(\sigma_1, \dots, \sigma_n)$ with $\sigma_h \in \Sigma_h$ chosen by player $h = 1..n$. Individual costs are aggregated by the *social cost* $C(\sigma) = \sum_i C_i(\sigma)$. We denote by $v_{-i} = (v_1, \dots, v_{i-1}, v_{i+1}, \dots, v_r)$ a vector $v = (v_1, \dots, v_r)$ without its $i$th entry, for $1 \leq i \leq r$.

A stable state (equilibrium) is an outcome in which each player minimizes its own cost in some sense. A pure Nash equilibrium is a basic stable outcome in which for any player, its strategy minimizes its cost given others' strategies.

*Definition* 1*: A strategy vector* $(\sigma_1, \dots, \sigma_n) \in \Sigma$ *is a* pure Nash equilibrium *(PNE) if any player i minimizes its cost by playing* $\sigma_i$*, i.e.* $C_i(\sigma_i, \sigma_{-i}) \leq C_i(\sigma'_i, \sigma_{-i}), \forall i, \forall \sigma'_i \in \Sigma_i$

While PNE do not exist in some games, all other equilibria we study in Section III exist in any game. PNE may not be efficient from a social perspective. A standard measure of distributed inefficiency is the *price of anarchy* (PoA) [8], defined as the ratio of the social cost of the worst PNE to the optimum: $PoA(G) = \dfrac{\max_{\sigma \in \text{PNE}} C(\sigma)}{\min_{\sigma^* \in \Sigma} C(\sigma^*)}$

In hierarchical systems, group-level stability is a prerequisite for global stability. We assume that groups are $\mathcal{P}$-games i.e. they satisfy a property $\mathcal{P}$ e.g. the existence of PNE or constant PoA. Our goal is to understand whether this local $\mathcal{P}$ is preserved (to some extent) globally. To this goal we introduce and study the novel concept of game *coupling*.

*Definition* 2*: G is a* $(N^1, \dots, N^J)$-coupling of $\mathcal{P}$-games *if*

- $N^1, \dots, N^J$ *are* groups *partitioning* $\{1 \dots n\}$*, i.e.* $N^j \cap N^{j'} = \emptyset$*,* $\forall 1 \leq j < j' \leq J$ *and* $N^1 \cup \dots \cup N^J = \{1 \dots n\}$*. The groups are fixed: no player can choose its group.*
- *Let*[2] $\Sigma^j = \times_{i \in N^j} \Sigma_i$ *and* $\Sigma^{-j} = \times_{i \notin N^j} \Sigma_i$*. For any j and any fixed vector* $\sigma^{-j} \in \Sigma^{-j}$ *of players in* $N^{-j}$*, the sub-game* $G|_{N^{-j} \leftarrow \sigma^{-j}}$ *(played by* $N^j$*) has property* $\mathcal{P}$*.*

[1] In Section V we use utility maximization instead of cost minimization.
[2] Subscripts refer to players, while superscripts refer to sets of players.

*Coupling Example*. In a load balancing game (LBG), each agent (job) chooses a machine. Each machine $e$ has a specific cost function $c_e$ which depends only on $e$'s load, i.e. number of jobs on it. PNE always exist in such games [13]. Jobs are rarely this homogeneous; instead, there are often groups of jobs, e.g. computation-intensive or memory-intensive. In this case, each machine has a cost function $c_e^j$ for each type $j$ of jobs. When fixing the strategies of other jobs, the game experienced by jobs of any type $j$ is a standard LBG and hence admits a PNE. Thus, a LBG with heterogeneous jobs is a coupling of games that admit PNE. The question is then when does the global (heterogeneous) LBG also admit PNE.

We start by analyzing costs of PNE and other equilibria within each group and consequences for coupling efficiency.

## III. Price of Anarchy within Groups

Roughgarden [17] identified $(\lambda, \mu)$-smoothness, a canonical property of games that yields tight PoA bounds.

*Definition* 3*:[17] Game G is* $(\lambda, \mu)$*-smooth if for all* $\sigma, \sigma' \in \Sigma$

$$\sum_{i=1}^{n} C_i(\sigma'_i, \sigma_{-i}) \leq \lambda \cdot C(\sigma') + \mu \cdot C(\sigma)$$

If $G$ is $(\lambda, \mu)$-smooth (with $\lambda \geq 0$ and $\mu \in (0, 1)$), then [17] each of $G$'s PNE has cost at most $\lambda/(1-\mu)$ times that of a socially optimal outcome, i.e. $PoA(G) \leq \lambda/(1-\mu)$.

PoA bounds based on $(\lambda, \mu)$-smoothness extend [17] to three other standard equilibrium concepts that we review now. A mixed Nash equilibrium (MNE) is a product probability distribution in $\Delta(\Sigma)$ in which each player minimizes its (expected) cost given others' strategies. For any correlated equilibrium (CE) $\pi \in \Delta(\Sigma)$, if a mediator draws $\sigma$ from $\pi$ and reveals to each player $i$ only its strategy $\sigma_i$ then $i$ minimizes cost by playing $\sigma_i$, assuming others also follow $\sigma_{-i}$. A coarse correlated equilibrium (CCE or equivalently Hannan-consistent strategy [5]) is more general than a CE in that the deviation $\sigma'_i$ cannot depend on the draw $\sigma_i$. Average coarse correlated equilibria with respect to a socially optimal $\sigma^* \in \Sigma$ ($ACCE^*$ [15]) comprise the class of distributions for which the social cost is lower than the sum of costs when each agent $i$ unilaterally deviated to $\sigma^*_i$. $ACCE^*$ is the class for which the best $PoA$ bound via $(\lambda, \mu)$-smoothness is tight.

*Definition* 4*: A* correlated equilibrium *(CE)* $\pi \in \Delta(\Sigma)$ *is a distribution such that for all i* $\sum_{\sigma_{-i} \in \Sigma_{-i}} C_i(\sigma_i, \sigma_{-i}) \pi(\sigma_i, \sigma_{-i}) \leq \sum_{\sigma_{-i} \in \Sigma_{-i}} C_i(\sigma'_i, \sigma_{-i}) \pi(\sigma_i, \sigma_{-i}), \forall \sigma_i, \sigma'_i \in \Sigma_i$.

*At a* coarse correlated equilibrium *(CCE)* $\pi \in \Delta(\Sigma)$*,* $\forall i$*,* $\sum_{\sigma \in \Sigma} C_i(\sigma_i, \sigma_{-i}) \pi(\sigma_i, \sigma_{-i}) \leq \sum_{\sigma_{-i} \in \Sigma_{-i}} C_i(\sigma'_i, \sigma_{-i}) \pi_i(\sigma_{-i})$ *for all* $\sigma'_i \in \Sigma_i$ *where* $\pi_i(\sigma_{-i}) = \sum_{\tau_i \in \Sigma_i} \pi(\tau_i, \sigma_{-i})$ *is the marginal probability that vector* $\sigma_{-i} \in \Sigma_{-i}$ *will be played.*

*At an* $ACCE^*$ $\pi$ *for some socially optimal* $\sigma^*$ *(i.e.* $C(\sigma^*) \leq C(\sigma), \forall \sigma$*) we have* $\sum_{\sigma \in \Sigma} C_i(\sigma_i, \sigma_{-i}) \pi(\sigma_i, \sigma_{-i}) \leq \sum_{\sigma \in \Sigma} \sum_i C_i(\sigma^*_i, \sigma_{-i}) \pi(\sigma_i, \sigma_{-i})$

We reviewed equilibria in increasing order of generality.

$$PNE \subseteq MNE \subseteq CE \subseteq CCE \subseteq ACCE^*$$

We denote the ratio of the social cost[3] of the worst equilibrium in class $\mathcal{C} \subseteq \Delta(\Sigma)$ to the optimum by $PoA_{\mathcal{C}}(G) =$

[3] For distribution $\pi$ we use the expected social cost $C(\pi) = \mathbb{E}_{\sigma \sim \pi}[C(\sigma)]$

$\frac{\max_{\sigma\in\mathcal{C}} C(\sigma)}{\min_{\sigma^*\in\Sigma} C(\sigma^*)}$. For any $G$, $PoA(G) = PoA_{\text{PNE}}(G) \leq PoA_{\text{MNE}}(G) \leq PoA_{\text{CE}}(G) \leq PoA_{\text{CCE}}(G)$ [4] $\leq PoA_{\text{ACCE}^*}(G) = \inf\{\frac{\lambda}{1-\mu} : G \text{ is } (\lambda,\mu)\text{-smooth}\}$

We present localized $(\lambda, \mu)$-smoothness arguments for game couplings. Analogously to the social cost, we define group $j$'s total cost at $\sigma \in \Sigma$ by $C^j(\sigma) = \sum_{i\in N^j} C_i(\sigma)$.

*Definition 5 ($(\vec{\lambda}, \vec{\mu})$-smoothness): A coupling is $(\vec{\lambda}, \vec{\mu})$-smooth (with $\vec{\lambda} \in \mathbb{R}^J, \vec{\mu} \in [0,1)^J$) if each sub-game $j$ has the local $(\lambda^j, \mu^j)$-smoothness property. That is, if for all sub-games $j$, for every two outcomes $(\sigma^j, \sigma^{-j})$ and $(\sigma', \sigma^{-j})$:*

$$\sum_{i\in N^j} C_i(\sigma'_i, \sigma^j_{-i}, \sigma^{-j}) \leq \lambda^j \cdot C^j(\sigma', \sigma^{-j}) + \mu^j \cdot C^j(\sigma^j, \sigma^{-j})$$

*where vectors $\sigma', \sigma^j \in \times_{k\in N^j} \Sigma_k, \sigma^{-j} \in \times_{k\notin N^j} \Sigma_k$.*

We define the local PoA for global equilibria $\mathcal{C}$, comparing group $j$'s costs at $\sigma \in \mathcal{C}$ to its minimal group cost given $\sigma^{-j}$.

*Definition 6: The* local price of anarchy *of group $j$ in $G$ for $\mathcal{C} \subseteq \Delta(\Sigma)$ is $PoA^j_{\mathcal{C}}(G) = \max_{\sigma\in\mathcal{C}} \frac{C^j(\sigma)}{C^j(\text{OPT}^j, \sigma^{-j})}$ where $\text{OPT}^j$ minimizes group $j$'s cost (i.e. $C^j(\text{OPT}^j, \sigma^{-j}) \leq C^j(\sigma^j, \sigma^{-j})$, $\forall \sigma^j \in \Sigma^j$) given the behavior $\sigma^{-j}$ of the outside agents.*

$PoA^j_{\mathcal{C}}(G)$ measures the inefficiency of equilibria $\mathcal{C}$ (e.g. PNE) of the coupling $G$ instead of equilibria within group $j$ only for $\sigma^{-j}$. For a single group ($J=1$), $PoA^1_{\mathcal{C}} = PoA_{\mathcal{C}}$.

Local smoothness leads to a bound on the local PoA for the most general equilibrium concept we introduced thus far that holds even with adversarial behavior of the other groups.

*Theorem 1: $PoA^j_{\text{ACCE}^*}(G) \leq \frac{\lambda^j}{1-\mu^j}$ for all $(\vec{\lambda}, \vec{\mu})$-smooth $G$.*

### *A. Dual Equilibrium Notions*

We use LP duality to characterize the distributions for which $PoA^j$ bounds derived via local smoothness arguments are tight. For $\sigma' \in \Sigma^j$ we finding the best such $PoA^j$ bound, formulated below as a linear fractional problem (LP):

$$\begin{aligned} \min\ & \lambda^j \ / \ (1-\mu^j) \\ \text{s.t. } & \textstyle\sum_{i\in N^j} C_i(\sigma'_i, \sigma_{-i}) \leq \lambda^j C^j(\sigma', \sigma^{-j}) + \mu^j C^j(\sigma), \forall \sigma \in \Sigma \\ & 0 < \mu^j < 1, \lambda^j > 0 \end{aligned}$$

Introducing $p^j = \frac{\lambda^j}{1-\mu^j} > 0$ and $z^j = \frac{1}{1-\mu^j} > 0$ yields the LP

$$\begin{aligned} \min\ & p^j \\ \text{s.t. } & p^j C^j(\sigma', \sigma^{-j}) + z^j (C^j(\sigma) - \textstyle\sum_{i\in N^j} C_i(\sigma'_i, \sigma_{-i})) \geq C^j(\sigma) \end{aligned}$$

the corresponding dual to which has as follows:

$$\begin{aligned} \max\ & \textstyle\sum_{\sigma\in\Sigma} s_\sigma C^j(\sigma) \\ \text{s.t. } & \textstyle\sum_{\sigma\in\Sigma} s_\sigma C^j(\sigma', \sigma^{-j}) \leq 1 \ \text{ and } \ s_\sigma \geq 0 \ \forall \sigma \in \Sigma \\ & \textstyle\sum_{\sigma\in\Sigma} s_\sigma \big(\sum_{i\in N^j} C_i(\sigma'_i, \sigma_{-i}) - C^j(\sigma)\big) \geq 0 \end{aligned}$$

Since the social costs are positive, we can replace the first inequality with an equality. Furthermore, since this quantity is a constant (and furthermore equal to 1), we can divide the objective by it without having any effects on the system:

$$\begin{aligned} \max\ & \textstyle\sum_{\sigma\in\Sigma} s_\sigma C^j(\sigma) \ / \ \sum_{\sigma\in\Sigma} s_\sigma C^j(\sigma', \sigma^{-j}) \\ \text{s.t. } & \textstyle\sum_{\sigma\in\Sigma} s_\sigma C^j(\sigma', \sigma^{-j}) = 1 \ \text{ and } \ s_\sigma \geq 0 \ \forall \sigma \in \Sigma \\ & \textstyle\sum_{\sigma\in\Sigma} s_\sigma \big(\sum_{i\in N^j} C_i(\sigma'_i, \sigma_{-i}) - C^j(\sigma)\big) \geq 0 \end{aligned}$$

[4] Blum et al. [4] call $PoA_{\text{CCE}}(G)$ the price of total anarchy in $G$.

Finally, due to scaling invariance the normalization $\sum_{\sigma\in\Sigma} s_\sigma C^j(\sigma', \sigma^{-j}) = 1$ can be replaced by $\sum_{\sigma\in\Sigma} s_\sigma = 1$.

$$\begin{aligned} \max\ & \textstyle\sum_{\sigma\in\Sigma} s_\sigma C^j(\sigma) \ / \ \sum_{\sigma\in\Sigma} s_\sigma C^j(\sigma', \sigma^{-j}) \\ \text{s.t. } & \textstyle\sum_{\sigma\in\Sigma} s_\sigma = 1 \ \text{ and } \ s_\sigma \geq 0 \ \forall \sigma \in \Sigma \\ & \textstyle\sum_{\sigma\in\Sigma} s_\sigma \big(\sum_{i\in N^j} C_i(\sigma'_i, \sigma_{-i}) - C^j(\sigma)\big) \geq 0 \end{aligned}$$

Hence, we can define group $j$'s *average coarse correlated equilibria with respect to localized optima* ($ACCEL^j$) as distributions for which local $PoA^j$ bounds via $(\lambda^j, \mu^j)$-smoothness are always tight; $ACCEL = \bigcap_{j\in J} ACCEL^j$.

*Definition 7: $ACCEL^j = \{s : \exists r^j \in \min_{\sigma^j} C^j(\sigma^j, s^{-j})$ s.t. $\sum_{\sigma\in\Sigma} C^j(\sigma) s(\sigma) \leq \sum_{\sigma\in\Sigma} \sum_{i\in N^j} C_i(r_i, \sigma_{-i}) s(\sigma)\}$.*

*Theorem 2: $PoA^j_{\text{ACCEL}^j}(G) = \min\{\frac{\lambda^j}{1-\mu^j} : G \ (\vec{\lambda}, \vec{\mu})$-smooth$\}$*

ACCEL are the distributions for which the average regret of players within each group for not following the prerogative of a group optimal strategy is non-positive. If $G$ is an organization with hierarchical structure, ACCEL correspond to policies that are plausible given competent management that guides a competent-on-average population.

### *B. ACCEL and Public Advertising*

We introduce a novel learning procedure that incorporates public advertising and which offers welfare guarantees analogous to those of $ACCEL$. Intuitively, the setting has as follows. Within each group $j$, there exists a broadcasting center that can communicate with all agents within the group. On each day $t = 1 \ldots T$, the center of group $j$ computes a strategy vector $\text{ADV}^j(t)$ for the group and advertises to each agent $i$ her respective strategy $\text{ADV}^j_i(t)$. There exist two high level issues in any such model: first, how does the center decide on which vector to advertise and second, how do the individual agents respond to the recommendations.

In terms of center actions, prior public advertising models [2], [3] assumed that there existed a single center with full information over the whole game that was able to broadcast to all agents. In such settings the center can easily broadcast a global optimum solution or the best Nash equilibrium. Here, we are moving towards a more restricted and realistic model where each center only controls a *local* neighborhood of agents. Many real life settings share this structure (e.g. competing Internet providers, or more generally competing institutions/organizations). In such settings, the managing centers have a high incentive in employing sophisticated online algorithms in order to effectively calibrate their predictions. Here, we will analyze centers whose advertised behaviors exhibits vanishing average regret with hindsight. This is a rather natural benchmark, since several simple learning algorithms can offer such guarantees[5].

On the side of the individual agents, we make a similarly weak assumption. We assume that the average performance

[5] A policy (sequence of strategies) satisfies no-regret if its cumulated payoffs are almost as good as ones of the best fixed (time-invariant) strategy given the history of play. CCE are limit points of time-averages of no-regret policies. Generally, no-regret algorithms offer guarantees in expectation over their randomized strategies. For ease of notation, we consider pure strategy outcomes. The analysis trivially extends to the case of randomized strategies.

of each agent $i$ (of group $j$) will eventually be (almost) as high as that of his advertised strategy $\text{ADV}_i^j(t)$. Any dummy agent can meet this benchmark merely by following the recommended strategy. A more realistic agent could still achieve such guarantees by interpolating between his innate learning strategy and the provided advice. We will show that advertising-guided learning offers guarantees analogous to those of $ACCEL$. We start by bounding the possible negative effects of agents' experimentation.

*Lemma 1: If in group $j$ the time-average[6] cost of each $i$ is (almost) as low as that of $i$'s advertised strategies $\text{ADV}_i^j$,*

$$\tfrac{1}{T}\sum_t C_i\big(\sigma(t)\big) \le \tfrac{1}{T}\sum_t C_i\big(\text{ADV}_i^j(t), \sigma^{-i}(t)\big) + o(1)$$

*then advertising-guided learning only incurs a $\lambda^j/(1-\mu^j)$ overhead when compared to the advertised strategy.*

$$\tfrac{1}{T}\sum_t C^j(\sigma(t)) \le \tfrac{\lambda^j}{1-\mu^j}\tfrac{1}{T}\sum_t C^j\big(\text{ADV}^j(t), \sigma^{-j}(t)\big) + o(1)$$

Given a history of play $\sigma(1), \dots, \sigma(T)$, we denote the best group response of group $j$ with hindsight as $\text{OPT}^j(T)$:

$$\text{OPT}^j(T) = \text{argmin}_{s^j \in \Sigma^j} \tfrac{1}{T}\sum_t C^j\big(s^j, \sigma^{-j}(t)\big)$$

Given a game coupling $(N^1, N^2, \dots, N^J)$, we define its *super-game* as follows: it is a game with $J$ agents, the available strategies to each super-agent $j$ correspond to strategy tuples for all agents in group $j$, i.e. $\sigma^j \in \times_{i\in N^j}\Sigma_i$. Finally, the cost of the super-agent $j$ is the group cost for all agents in group $j$, i.e. $C^j(\sigma) = \sum_{i\in N^j} C_i(\sigma)$. We let $\lambda^{sup} \in \mathbb{R}_+, \mu^{sup} \in [0,1)$ such that the super-game is $(\lambda^{sup}, \mu^{sup})$-smooth. Finally, we define a socially optimal strategy vector as $global\_\text{OPT} \in \text{argmin}_\sigma C(\sigma)$.

We now prove cost bounds for advertising-guided learning.

*Theorem 3: If each agent's time-average cost is (almost) as low as that of her advertised strategy, and the advertised strategy for each group $j$ has vanishing time-average regret:*

$$\tfrac{1}{T}\sum_t C^j\big(\text{ADV}^j(t), \sigma^{-j}(t)\big) \le \tfrac{1}{T}\sum_t C^j\big(\text{OPT}^j(T), \sigma^{-j}(t)\big) + o(1)$$

*then for advertising-guided learning, the* group *cost satisfies*

$$\frac{1/T\sum_t C^j\big(\sigma(t)\big)}{1/T\sum_t C^j\big(\text{OPT}^j(T), \sigma^{-j}(t)\big)} \le \frac{\lambda^j}{1-\mu^j} + o(1)$$

*and for $\min_j \frac{1-\mu^j}{\lambda^j} > \mu^{sup}$, the* social *cost satisfies*

$$\frac{1/T\sum_t C\big(\sigma(t)\big)}{\min_{\sigma'} C(\sigma')} \le \frac{\lambda^{sup}}{\min_j \frac{1-\mu^j}{\lambda^j} - \mu^{sup}} + o(1)$$

An identical PoA bound of $(\lambda^{sup})/(\min_j \frac{1-\mu^j}{\lambda^j} - \mu^{sup})$ for social cost can be derived for all $ACCEL$ distributions.

We will now shift focus from costs and PoA to the existence of PNE and their reachability via learning dynamics – we have seen learning via advertising in this section. We will relate these topics to bounds on PoA in Section V.

[6]In this section, whenever we write $\sum_t$, we mean $\sum_{t=1}^T$.

## IV. Couplings and potentials

In this section we identify structural properties of games within each group that are preserved in the global game when augmented with conditions on the interplay among groups (for clarity, we only present results for two groups). We consider two properties of the natural Nash dynamics, namely existence of an exact potential and weak acyclicity.

### A. Potential functions review

A potential function $\Phi : \Sigma \to \mathbb{R}$ simultaneously encodes improvement opportunities and is closely linked to PNE.

*Definition 8: [13] Game $G$ has an* exact potential *$\Phi(\cdot)$ if $C_i(\sigma_i, \sigma_{-i}) - C_i(\sigma_i', \sigma_{-i}) = \Phi(\sigma_i, \sigma_{-i}) - \Phi(\sigma_i', \sigma_{-i})$ $\forall i$, $\forall \sigma_i, \sigma_i' \in \Sigma_i, \sigma_{-i} \in \Sigma_{-i}$. Game $G$ has an* ordinal potential *$\Phi(\cdot)$ if $C_i(\sigma_i, \sigma_{-i}) < C_i(\sigma_i', \sigma_{-i}) \Leftrightarrow \Phi(\sigma_i, \sigma_{-i}) < \Phi(\sigma_i', \sigma_{-i})$*

*$G$ has a* weak potential *$\Phi(\cdot)$ [10] if and only if at any strategy vector $(\sigma_1, \dots, \sigma_n)$ that is not a (pure) Nash equilibrium there exists a player $i$ that can simultaneously lower both her cost and $\Phi$ by switching to some strategy $\sigma_i'$: $C_i(\sigma_i, \sigma_{-i}) > C_i(\sigma_i', \sigma_{-i})$ and $\Phi(\sigma_i, \sigma_{-i}) > \Phi(\sigma_i', \sigma_{-i})$*

*If $G$ has an exact potential (respectively ordinal potential, or weak potential) then $G$ is called an* exact potential *(respectively* ordinal potential*, or* weakly acyclic*) game.*

Def. 8 reviews potentials in increasing order of generality.

exact potentials $\subseteq$ ordinal potentials $\subseteq$ weak potentials

An ordinal potential $\Phi$ is also a weak one, since it suffices that *one* player increases $\Phi$ upon improving her utility. The existence of an ordinal potential is equivalent [13] to the convergence of *Nash dynamics*, i.e. asynchronous updates by each player to a better strategy given others' current strategies. Any weakly acyclic game has at least one PNE, for example the global optimum of the weak potential.

### B. Exact potential games

The compelling property of existence of an exact potential function implies, among others, convergence of distributed Nash dynamics. In distributed control of multi-agent systems, exact potentials arise for example in the "wonderful life utility" scheme [20], by which a planner can ensure that individual agents will act in accordance to common welfare.

We give a necessary and sufficient condition for a coupling of exact potential games to also have an exact potential. Our condition leverages a well-known characterization [13].

*Lemma 2: [13] Game $G$ has an exact potential if and only if for any $i, k$, any strategies $\sigma_i, \sigma_i'$ of $i$ and $\sigma_k, \sigma_k'$ of $k$ and for any $\sigma_{-ik}$ of the other players we have $d_{\sigma_i\sigma_i'\sigma_k\sigma_k'}(\sigma_{-ik}) = 0$ where $d_{\sigma_i\sigma_i'\sigma_k\sigma_k'}(\sigma_{-ik}) := \Delta_{\sigma_i\sigma_k}(\sigma_{-ik}) - \Delta_{\sigma_i\sigma_k'}(\sigma_{-ik}) - \Delta_{\sigma_i'\sigma_k}(\sigma_{-ik}) + \Delta_{\sigma_i'\sigma_k'}(\sigma_{-ik})$ and $\Delta_{s_is_k}(\sigma_{-ik}) := C_i(s_i, s_k, \sigma_{-ik}) - C_k(s_i, s_k, \sigma_{-ik}), \forall s_i, s_k$*

That is, the differences in $i$'s and $k$'s utilities sum up to 0 on any 4-cycle of strategy updates. Our condition on couplings abstracts away the individuals in a group and considers an auxiliary game using group potentials.

*Theorem 4: Let $G$ be a $(N^1, N^2)$-coupling of exact potential games: $\forall \sigma^j \in \Sigma^j$, the sub-game $G|_{N^j \leftarrow \sigma^j}$ induced by group*

*$N^j$ playing strategy vector $\sigma^j$ has exact potential $\Phi_{\sigma^j}(\cdot)$, for $j = 1, 2$. Define a game $\Gamma$ with players $\{1, 2\}$ in which player $j$'s strategy space is $\Sigma^j$ and the utilities from playing $(\sigma^1, \sigma^2)$ are $(\Phi_{\sigma^2}(\sigma^1), \Phi_{\sigma^1}(\sigma^2))$. Then $G$ is an exact potential game if and only if $\Gamma$ is an exact potential game.*

An exact potential is preserved when some players' strategies are fixed. Theorem 4's condition that each induced sub-game has an exact potential is tight. Also, if $\Gamma$ is not an exact potential game, then $G$ may even lack a PNE: any two-player game $G$ satisfies Theorem 4's condition since $G$ is trivially a coupling of potential games.

### C. Weakly acyclic games

We present a second instantiation of coupling with useful structural properties. It requires and aims for significantly less cohesion among groups of players. Specifically, Lemma 3 below provides a sufficient condition for a game-coupling to preserve weak acyclicity. We then use it to identify a model of malice in congestion games that guarantees, unlike other similar models, the existence of a PNE.

Weak acyclicity, a natural property of engineered systems (e.g. in the *consensus* problem [10]), is defined as the existence of a better-response path from any strategy vector to a PNE. In better-response dynamics, players asynchronously update to a better strategy given others' current strategies. Even though defined as a property of global convergence of coordinated Nash dynamics, simple *distributed* dynamics converge to PNE [10] in any weakly acyclic game.

Recall Def. 8: weak acyclicity amounts to existence of a weak potential. Our sufficient condition is quite unrestrictive – it only concerns the change of the sub-game weak potential at a sub-game *equilibrium*. Intuitively, this requires that the progress (towards convergence) of one group is not impeded by the progress of the other group. Interestingly, we only require this condition *upon* (local) convergence.

*Lemma 3: If a coupling $G$ of weakly acyclic games satisfies*

- *$G|_{N^j \leftarrow \sigma^j}$ has weak potential $\Phi_{\sigma^j}$ for $\forall \sigma^j \in \Sigma^j, j = 1, 2$.*
- *if $\sigma^2$ is PNE in $G|_{N^1 \leftarrow \sigma^1}$ then any better-response $\bar{\sigma}_i^1$ to $\sigma_{-i}^1$ (and $\sigma^2$) by any $i$ in sub-game $G|_{N^2 \leftarrow \sigma^2}$ does not reduce the weak potential: $\Phi^2_{\sigma^1}(\sigma^2) \leq \Phi^2_{\bar{\sigma}_i^1, \sigma_{-i}^1}(\sigma^2)$*

*then $G$ is a weakly acyclic game with weak potential $\Phi(\sigma^1, \sigma^2) = C \cdot \Phi_{\sigma^1}(\sigma^2) + \Phi_{\sigma^2}(\sigma^1)$ for $C > 0$ with $C \cdot \min(\Phi_{\sigma^1}(\sigma^2) - \Phi_{\sigma^1}(\bar{\sigma}_i^2, \sigma_{-i}^2)) > \max |\Phi_{(\sigma_i^2, \sigma_{-i}^2)}(\sigma^1) - \Phi_{(\bar{\sigma}_i^2, \sigma_{-i}^2)}(\sigma^1)|$ where the $\min$ and $\max$ are both over $i \in N^2$, $\sigma_i^2, \bar{\sigma}_i^2 \in \Sigma_i, \sigma_{-i}^2 \in \Sigma_{-i}^2, \sigma^1 \in \Sigma^1$ and, for the $\min$ only, the $\Phi^2$ difference in the argument must be strictly positive.*

One can show that the weak acyclicity of each induced sub-game for *any* fixed vectors $\sigma^1, \sigma^2$ is necessary. Lemma 3's second assumption relates sub-games. In general, weak acyclicity, unlike the existence of an exact or ordinal potential, is not preserved when a player's strategy is fixed.

*Congestion-seeking malice*: We apply Lemma 3 to the well-studied [13], [16] *congestion* games. These games arise in many settings with joint usage of resources and are isomorphic to exact potential games. In many congestion games, especially ones modeling routing applications through fixed networks, players have higher costs for higher congestion, because resource delays are increasing. Such players are naturally vulnerable to malicious players that seek congestion. Leveraging Lemma 3 with $N^2$ as the malicious congestion-seeking players, we can establish that a model of malice preserves some structure in several congestion game classes, unlike other models with a similar scope [1].

*Corollary 1: A $(N^1, N^2)$-coupling $G$ with $|N^2| \geq 2$ preserves weak acyclicity assuming that players in $N^2$ benefit from using resources $r$ used by others (in $N^1$ or $N^2$), namely*

***(A)** for any $i_2 \in N^2$ if $Supp(\sigma) \setminus \sigma_{i_2} \neq \emptyset$ then $i_2$ can BR by also using some $r \in Supp(\sigma) \setminus \sigma_{i_2}$, i.e. $\sigma_{i_2} := \sigma_{i_2} \cup \{r\}$.*

*where $Supp(\sigma)$ denotes resources used (by players) in $\sigma \in \Sigma$.*

Such malicious players (in $N^2$) are "lone wolves" since they act independently; they may also increase the congestion of other malicious players. We can show that Corollary 1 applies to several classes of congestion games (in particular assumption **(A)** holds): *market-sharing games* [17] (a natural model of uniform competition on resources), *facility location games* [17] (a game-theoretic distributed allocation problem) and *load-balancing games*, for which Corollary 1 is tight in several ways. Weak acyclicity is preserved even if each malicious player also minimizes delay (like a regular player) on some resources as long as assumption **(A)** holds.

Our existence guarantees for PNE in game couplings with malicious players imply that the (local) PoA is well-defined. Load-balancing games with affine delays are $(\frac{5}{3}, \frac{1}{3})$-smooth [17], yielding a $\frac{5}{2} = \frac{\frac{5}{3}}{1-\frac{1}{3}}$ upper bound on their local $PoA$.

## V. Aggregation via heterogeneity

Convergence guarantees for dynamics are made more relevant by quantitative statements about the quality of equilibria (or equilibria that such dynamics can reach)[7].

Mol et al. [11] show that heterogeneous systems in aggregation games (AG, defined below), have significantly lower PoA than homogeneous AGs. They only prove convergence of Nash dynamics and hence existence of PNE for homogeneous systems and one instance of a heterogeneous system (we review their results below after introducing the model).

We prove convergence of Nash dynamics for *any* heterogeneous AG, thus solving a salient open question in their work. For this we exhibit a new global potential that explicitly couples the potentials of each homogeneous sub-system. We thus show that heterogeneity is useful even as a utility design criterion (to reduce PoA) while it preserves global stability.

*Aggregation games* model systems aiming for high internal connectivity. Specifically, consider an undirected graph $Gr = (\{1, \ldots, N\}, E)$ without self-loops. There are $n \leq N$ players, that must each choose a different vertex in $1..N$; denote by $H$ the set of all $n$ players' vertices: $|H| = n$. Each player $i$ has a parameter $\beta_i \in [0, 1]$, inducing its utility function[8] $u_{\beta_i}$ it aims to maximize, where if $v_i$ is $i$'s vertex

$$u_{\beta_i}(v_i, H \setminus \{v_i\}) = E_{v_i, H} + \beta_i E_{v_i, \{1, \ldots, N\} \setminus H} \qquad (1)$$

[7]Profiles during learning dynamics, even non-convergent ones, may however be much better than any PNE [7].

[8]Utilities are more natural than costs for evaluating connectivity.

and $E_{I_1,I_2} = |\{e \in E : e = (i_1, i_2), i_1 \in I_1, i_2 \in I_2\}|$ denotes the number of edges between vertex sets $I_1$ and $I_2$. We write $G(Gr, \beta_1, \ldots, \beta_n)$ for the corresponding AG $G$.

Players with $\beta = 0$ are called *followers* as they maximize $u_0 = E_{v,H}$ i.e. the number of edges to $H$ (the other players' vertices). In contrast, players with $\beta = 1$ are called *leaders* because they maximize $u_1 = E_{v,\{1,\ldots,N\}}$ i.e. the degree of $v$ in the hope that other players, in particular followers, will be drawn to the adjacent vertices. A player with a general $\beta$ is called a $\beta$-leader; note $u_\beta(\cdot) = \beta u_1(\cdot) + (1-\beta)u_0(\cdot)$. We call an AG $G(Gr, \beta_1, \ldots, \beta_n)$ *homogeneous* if $\beta_i = \beta, \forall i$.

The *social welfare* (the counterpart of social cost) is the number $E_H$ of internal edges, *for any* $\beta_1, \ldots, \beta_n$. As $E_H$ should be maximized (distributively), $PoA(G) = \frac{\max_{H^*} E_{H^*}}{\min_{H \text{ PNE}} E_H}$.

We now consider issues regarding PNE, PoA and convergence of dynamics in AGs. For $n = \Theta(N)$ players there exist [11] graphs $Gr$ for which any uniform $\beta$ leads to high $PoA = \Theta(N)$. A balanced mix of $\beta$-leaders and followers has constant PoA for constant $\beta$, but existence of PNE had only been established for $\beta = 1$.

*Theorem 5: [11] There exist connected Gr such that for any $\beta$ and homogeneous AG $G = G(Gr, \beta, \ldots, \beta)$, $PoA(G) \geq n$.*

*For any graph Gr and AG $G(Gr, 0, \ldots, 0, \beta, \ldots, \beta)$ with $\lambda n$ $\beta$-leaders ($\beta \geq \frac{1}{n}$) and $(1-\lambda)n$ followers, we have $PoA(G) = O(\frac{1}{1-\lambda} \min(n, \frac{1}{\beta\lambda}))$. Hence, $PoA(G)$ is constant for constant $\lambda$ (i.e. a balanced mix) and constant $\beta$.*

Given their high PoA, AGs cannot be $(\lambda, \mu)$-smooth for small $(\lambda, \mu)$. We instead prove their stability via coupling.

Any homogeneous AG has an exact potential (implicitly shown in [11]). The form of this potential implies that any Nash dynamics converges to a PNE in polynomial time.

*Theorem 6: A homogeneous AG G, i.e. $\beta_i = \beta \in [0, 1]\ \forall i$ has exact potential $\Phi_\beta(H) = (1 + \beta)E_H + \beta E_{H,\{1,\ldots,N\}\setminus H}$*

In contrast, the only known structural result for a heterogeneous system is that when all $\beta_i$ are either 0 or 1, i.e. a mix of leaders and followers, the game has an ordinal potential. Structural results are however critical to Theorem 5 since it bounds the quality of PNE without proving that they exist.

We significantly generalize these results, using a weighted potential function. This is always an ordinal potential and it is an exact potential if and only if all weights are 1.

*Definition 9: A game has a* weighted potential *function [13] $\Phi : \times_{i=1}^n \Sigma_i \to \mathbb{R}$ with (positive) weights $w_1, \ldots, w_n$ if $u_i(\sigma_i, \sigma_{-i}) - u_i(\sigma'_i, \sigma_{-i}) = w_i \cdot (\Phi(\sigma_i, \sigma_{-i}) - \Phi(\sigma'_i, \sigma_{-i}))$ for any player $i$ and any strategies $\sigma_i, \sigma'_i \in \Sigma_i, \sigma_{-i} \in \Sigma_{-i}$.*

We are ready now for this section's main result: any set of players, with arbitrary $\beta_i < 1$ parameters, leads to a weighted potential function. The weighted potential is an explicit mapping of potentials in each sub-game. An analogous result when some $\beta$'s equal 1 follows easily. Thus Nash dynamics converge (and PNE exist) in any aggregation system.

*Theorem 7: Fix an AG G with $H = H^1 \cup \ldots \cup H^J$ where $H^j$ are vertices occupied by all $\beta^j$-leaders. Then G has weighted potential (with weights $1 - \beta^j > 0$ for each player $i \in H^j$)*

$$\Phi(H) = E_H + \sum_{j=1}^{J} \frac{\Phi_{\beta^j}(H^j, H \setminus H^j) - E_H}{1 - \beta^j}$$

*where $\Phi_{\beta^j}(H^j, H \setminus H^j) = (1 + \beta^j)E_H + \beta^j E_{H^j,\{1,\ldots,N\}\setminus H}$ is an exact potential of the aggregation sub-system over the (homogeneous) $H^j$ given fixed vertices of others (in $H \setminus H^j$).*

In a homogeneous system ($J = 1$, i.e. same $\beta$ for all), this weighted potential reduces to the exact one in Theorem 6.

The only AGs not covered by Theorem 7 are ones containing leaders ($\beta = 1$). Dealing with all leaders separately, one can easily identify an ordinal potential. Hence

*Corollary 2: A AG has an ordinal potential and thus a PNE.*

## VI. Concluding remarks

We introduced and studied game couplings, a concept that encapsulates globally heterogeneous systems exhibiting local homogeneity. We gave several applications of this framework to learning in games, quality of equilibria (PoA) and structural properties. An exciting research direction suggested by our work is to design groups of players and their couplings with desirable properties, in particular considering the effect of natural local types of behavior on global performance.

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