2011 50th IEEE Conference on Decision and Control and
European Control Conference (CDC-ECC)
Orlando, FL, USA, December 12-15, 2011

Line-of-sight path-following along regularly parametrized curves solved as a generic maneuvering problem

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***Abstract*— In the traditional maneuvering problem, the objective has been to solve a geometric task and a dynamic task, where the former is to converge to and follow a 1-dimensional manifold, a path, in the output space of the system, and the latter is to satisfy a desired dynamic behavior along the path. In this paper the objective is to generalize this problem statement, by rather stabilizing more general manifolds of higher dimension. With the system output constrained to the desired manifold, the dynamic task becomes to satisfy a dynamic assignment that ensures that the underlying control objective is solved with sufficient performance.**

In order to exemplify the theory, a case study is performed where a line-of-sight (LOS) algorithm is used to steer a simplified vessel to and along a desired parametrized path. In this case the desired manifold, which is of dimension 3, is defined as the set in which the LOS method is effectuated. The LOS algorithm then ensures, as the dynamic task, that the vessel steers correctly towards and along the path. A simulation study is provided to illustrate the effectiveness and properties of the resulting dynamic control law.

I. INTRODUCTION

Controlling a dynamic system to a path is a standard control problem, especially within vehicle control and robotics; see for instance [1], [2], [3], and references therein. This has further been refined in the specification of the maneuvering problem, which is to control the dynamical system under consideration according to desired maneuvers in the output space [4]. In [5] and more generally in [1], the maneuvering problem was broken down into two tasks: the Geometric Task, which was to converge to and follow a desired path parametrized by a scalar variable θ, and the Dynamic Task, which was to satisfy a dynamic assignment along the path. The dynamic assignment was further specified typically as a speed assignment for $\dot{\theta}$, but it could also be a time assignment for $\theta(t)$ or an acceleration assignment for $\ddot{\theta}$.

While the *geometric task* is equivalent to stabilizing a desired 1-dimensional manifold, the *dynamic task* specifies the reduced dynamics, the desired motion of the system, when the states are constrained to this manifold. Since the manifold was parametrized by a sufficiently smooth map $\theta \mapsto h_d(\theta)$, and the dynamic assignment was typically given by a speed assignment $\dot{\theta} = \upsilon_s(\theta, t)$, an interpretation is that this 1-dimensional system forms a guidance or reference system,

$$\begin{aligned} \dot{\theta} &= \upsilon_s(\theta, t) \\ y_d &= h_d(\theta) \end{aligned}$$

for the dynamical system under consideration. Moreover, in the presented control designs it was shown how this reference system was allowed to be corrected by feedback from the states of the plant in order to reduce transients and achieve a favorable overall behavior.

An obvious applications for this theory is position control of vehicles and robotic manipulators, and especially fully actuated systems as exemplified in various cases in [1]. Control of underactuated systems has also been addressed in the same framework, but with customized modifications according to the application at hand; see [6], [2], [7], and [8]. Even though maneuvering as presented is typically solved by tracking a point that traces the desired path, it can also, as elaborated in this paper, achieve the inherent properties of a path-following design where set stability of the desired path is directly aimed for [9]. Other challenges have also been addressed, for instance systems with unstable zero dynamics [10], where the extra degrees of freedom introduced in a maneuvering control problem are used to stabilize the zero dynamics.

In this paper, the objective is to make the maneuvering problem statement more generic such that it can be used to solve a wider class of control problems, while still retaining the constructive methodology of breaking the design problem down into the geometric and dynamic tasks.

Notation: In GS, LAS, LES, UGAS, UGES, etc., stands G for Global, L for Local, S for Stable, U for Uniform, A for Asymptotic, and E for Exponential. Total time derivatives of $x(t)$ are denoted $\dot{x}$, $\ddot{x}$, $x^{(3)}, \dots, x^{(n)}$, while a superscript denotes partial differentiation: $\alpha^t(x, \theta, t) := \frac{\partial \alpha}{\partial t}, \alpha^{x^2}(x, \theta, t) := \frac{\partial^2 \alpha}{\partial x^2}$, and $\alpha^{\theta^n}(x, \theta, t) := \frac{\partial^n \alpha}{\partial \theta^n}$, etc. The Euclidean vector norm is $|x| := (x^\top x)^{1/2}$, and the distance to a set $\mathcal{M}$ is $|x|_{\mathcal{M}} := \inf\{|x - y| : y \in \mathcal{M}\}$. Stacking several vectors into one is denoted $\mathrm{col}(x, y, z) := [x^\top, y^\top, z^\top]^\top$, similarly $\mathrm{row}(x, y, z)$ is a row vector, and whenever convenient, $|(x, y, z)| = |\mathrm{col}(x, y, z)|$. See also [11] for definitions of class-$\mathcal{K}$, $\mathcal{K}_\infty$, and $\mathcal{KL}$ functions.

II. THE MANEUVERING PROBLEM

In some control problems the task may be to control the output of the system to a manifold of higher dimension than 1, and then on the manifold satisfy some behavior defining desired maneuvers for the system. Consequently, the maneuvering problem can be generalized by consider as the

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978-1-61284-799-3/11/$26.00 ©2011 IEEE

geometric task to converge to a q-dimensional manifold, $q \geq 1$. The reduced dynamics for the system when constrained to this manifold should then be further specified by a dynamic assignment.

An example is the design in [12], where a formation of r vessels was controlled to an r-dimensional manifold, parametrized by the scalar path parameters for the individual vessels. On this manifold each vessel satisfied its individual path-following objective, while the assigned dynamics on the manifold ensured that all path variables were asymptotically synchronized in order to achieve group coordination.

A. Generic maneuvering problem statement

For a system output $y = h(x)$ where $h : \mathbb{R}^n \to \mathbb{R}^m$, the desired manifold is all points represented by the set

$$\mathcal{Q} := \{x \in \mathbb{R}^n : \ \exists \xi \in \mathbb{R}^q \text{ s.t. } h(x) = h_d(\xi)\} \tag{1}$$

where $q \leq m$ and the map $\xi \mapsto h_d(\xi)$ is sufficiently smooth. Given the parametrization $h_d(\xi)$ of the manifold and a dynamics assignment on the manifold, the **Maneuvering Problem** is comprised of the two tasks:

1) **Geometric task:** For some absolutely continuous function $\xi(t)$, force the output y to converge to the desired manifold $h_d(\xi)$,

$$\lim_{t\to\infty} |y(t) - h_d(\xi(t))| = 0. \tag{2}$$

2) **Dynamic task:** Force $\dot{\xi}$ to converge to a desired dynamic assignment $f_d(\xi, y, t)$,

$$\lim_{t\to\infty} \left|\dot{\xi}(t) - f_d(\xi(t), y(t), t)\right| = 0. \tag{3}$$

The main generalization in this problem statement is to consider a q-dimensional manifold instead of a 1-dimensional path. In addition, we allow the dynamic assignment f_d to incorporate feedback from the system output directly.

When addressing the geometric task, this is solved by stabilizing the noncompact set

$$\mathcal{A} = \{(\xi, x, \tau) \in \mathbb{R}^q \times \mathbb{R}^n \times \mathbb{R}_{\geq 0} : h(x) = h_d(\xi)\} \tag{4}$$

where as proposed by [11] and [1], τ is included to represent a possible time-variation in $f_d(\xi, y, t)$ with dynamics $\dot{\tau} = 1$, $\tau(0) = t_0$. Typically, this task is solved by tracking control or extensions of this, for instance those presented in [1]. For the dynamic assignment, giving the dynamic behavior of the solutions of the closed-loop system when constrained to the noncompact set $\mathcal{A}$, there is an underlying assumption that $\dot{\xi} = f_d(\xi, y(t), t)$ is well-behaved and satisfies necessary stability and performance properties. Thus, it is perhaps in the design of the dynamic task that the engineering skills are most important. An example of this is shown later in this paper where the line-of-sight algorithm is specified as the dynamic task to steer an underactuated vessel to and along a desired path.

B. Maneuvering control design

Consider the nonlinear system

$$\dot{x} = f(x, u, t), \quad y = h(x) \tag{5}$$

where for each $t \geq t_0$, $x(t) \in \mathbb{R}^n$ is the state vector, $u(t) \in \mathbb{R}^p$ is the control, $y(t) \in \mathbb{R}^m$ is the output, and $f : \mathbb{R}^n \times \mathbb{R}^p \times \mathbb{R}_{\geq 0} \to \mathbb{R}^n$ and $h : \mathbb{R}^n \to \mathbb{R}^m$ are smooth functions.

There are many techniques for control design of various classes of nonlinear systems, such as backstepping, feedback linearization, sliding-mode, linear control, CLF-based methods, etc. If by some appropriate method a control law for (5) can be constructed for our problem, then we have the result:

Proposition 1: Suppose there exist a control law

$$u = \alpha(\xi, x, t), \tag{6}$$

a smooth Lyapunov function $V : \mathbb{R}^q \times \mathbb{R}^n \times \mathbb{R}_{\geq 0} \to \mathbb{R}_{\geq 0}$, $\mathcal{K}_\infty$-functions α_1, α_2, and a continuous positive definite function α_3, such that for all $(\xi, x, \tau) \in \mathbb{R}^q \times \mathbb{R}^n \times \mathbb{R}_{\geq 0}$,

$$\alpha_1\left(|(\xi, x, \tau)|_{\mathcal{A}}\right) \leq V(\xi, x, \tau) \leq \alpha_2\left(|(\xi, x, \tau)|_{\mathcal{A}}\right) \tag{7a}$$

$$\begin{aligned} V^\xi(\xi, x, \tau) f_d(\xi, h(x), \tau) + V^x(\xi, x, \tau) f(x, \alpha(\xi, x, \tau), \tau) \\ + V^\tau(\xi, x, \tau) \leq -\alpha_3\left(|(\xi, x, \tau)|_{\mathcal{A}}\right). \end{aligned} \tag{7b}$$

Then, under the assumption that the closed-loop system

$$\dot{\xi} = f_d(\xi, h(x), t) \tag{8}$$

$$\dot{x} = f(x, \alpha(\xi, x, t), t) \tag{9}$$

is forward complete, the noncompact set (4) is UGAS, and this solves the maneuvering problem. $\lozenge$

The proof follows from standard Lyapunov arguments for noncompact sets; see [13] and [1]. In particular, let β_V be a class-$\mathcal{KL}$ function such that

$$V(\xi(t), x(t), t) \leq \beta_V\left(V(\xi_0, x_0, t_0),\ t - t_0\right) \tag{10}$$

which follows from (7), where $x_0 = x(t_0)$ and $\xi_0 = \xi(t_0)$. Letting $\beta_{\mathcal{A}}(s, t) := \alpha_1^{-1}\left(\beta_V\left(\alpha_2(s),\ t\right)\right)$ this gives $\forall t \geq t_0$,

$$|(\xi(t), x(t), t)|_{\mathcal{A}} \leq \beta_{\mathcal{A}}(|(\xi_0, x_0, t_0)|_{\mathcal{A}},\ t - t_0). \tag{11}$$

In Proposition 1, the dynamic assignment (3) is satisfied identically. However, since the dynamic assignment needs to be satisfied only in the limit, more possibilities exist. One option is to use the gradient tuning function first reported in [14] for the scalar case. In general we have:

Proposition 2: Suppose the conditions of Proposition 1 are satisfied. Every continuous tuning function $\omega : \mathbb{R}^q \times \mathbb{R}^n \times \mathbb{R}_{\geq 0} \mapsto \mathbb{R}^q$ that for all $(\xi, x, \tau) \in \mathbb{R}^q \times \mathbb{R}^n \times \mathbb{R}_{\geq 0}$ satisfies:

1) there exists a class-$\mathcal{K}$ function α_4 such that $|\omega(\xi, x, \tau)| \leq \alpha_4\left(|(\xi, x, \tau)|_{\mathcal{A}}\right)$, and
2) $V^\xi(\xi, x, \tau)\omega(\xi, x, \tau) \geq 0$,

and under the assumption that the closed-loop system (9) and

$$\dot{\xi} = f_d(\xi, h(x), t) - \mu\omega(\xi, x, t), \qquad \mu \geq 0 \tag{12}$$

is forward complete, renders the set (4) UGAS and solves the maneuvering problem. $\lozenge$

This proof also follows from standard Lyapunov arguments for noncompact sets where it is noticed that the bound on $\omega(\xi, x, \tau)$ by α_4 ensures that the dynamic task (3) is satisfied in the limit.

The above tuning function design enables a dynamic assignment for $\dot{\xi}$ corrected by feedback to give improved overall performance of the system, for instance by a rapid minimization of the distance to the desired manifold $\mathcal{Q}$ through a gradient algorithm. Another option, not explored here, is to use a filtered version of the gradient feedback as presented in [1]. In this case ω becomes the output of a filter that can be used to improve performance due to, for instance, noise or disturbances in the measurements. A further extension is presented in [15] where passivity-based techniques are used to construct such filters.

C. Gradient tuning function

In this section we will show that by choosing $\omega(\xi, x, \tau)$ as a gradient tuning function, we will achieve set stability of the desired manifold $\mathcal{Q} \times \mathbb{R}_{\geq 0}$ within the accuracy of an arbitrarily small offset $\delta > 0$, despite the fact that the maneuvering problem is solved as a tracking problem of the point $h_d(\xi(t))$. To this end, assume that $\sigma(\xi)V^{\xi}(\xi, x, \tau)$ satisfies the conditions for $\omega(\xi, x, \tau)$ in Proposition 2, where $\sigma(\xi) > 0$ is a bounded correction gain chosen by design. Then

$$\omega(\xi, x, \tau) := \sigma(\xi)V^{\xi}(\xi, x, \tau), \tag{13}$$

called the *gradient tuning function*, is one design alternative. The performance of the gradient tuning function follows from the results in [14] where it is shown for the 1-dimensional case that by choosing $\mu \gg 1$, a separation of time-scales is induced. In the fast time-scale, the dynamics (12) and (13) will make the solution $h_d(\xi(t))$ rapidly converge to a favorable position on the manifold, while the control law $u = \alpha(\xi, x, t)$ in the slow time-scale will drive the output $y(t)$ towards $y_d(t) = h_d(\xi(t))$.

To make this precise, we define $\mathcal{Q}_t := \mathcal{Q} \times \mathbb{R}_{\geq 0}$ and make the assumption:

Assumption 3: The set $\mathcal{Q}$ in (1) is compact and $\exists \rho > 0$ such that for every fixed pair (x, τ) in the set $\mathcal{H}_x(\rho) \times \mathbb{R}_{\geq 0}$, with $\mathcal{H}_x(\rho) : = \{x : |x|_{\mathcal{Q}} \leq \rho\}$, the function $\xi \mapsto V(\xi, x, \tau)$ has a global minimizer denoted $\xi_V(x, \tau)$ which, for (x, τ) fixed, is a LAS equilibrium for

$$\dot{\xi} = -\sigma(\xi)V^{\xi}(\xi, x, \tau), \tag{14}$$

with region of convergence $\mathcal{H}_{roc}(x, \tau)$, where the function $(x, \tau) \mapsto \xi_V(x, \tau)$ is locally Lipschitz on $\mathcal{H}_x(\rho)$, uniformly in τ. It is further assumed that $\exists k > 0$ such that $\mathcal{H}_{roc}(x, \tau)$ contains the ball $\mathcal{B}_{\xi}(k) := \{\xi : \ |\xi - \xi_V(x, \tau)| \leq k\}$ for all $(x, \tau) \in \mathcal{H}_x(\rho) \times \mathbb{R}_{\geq 0}$. $\Diamond$

A rapid minimization of the distance $|(\xi, x, \tau)|_{\mathcal{A}}$ is of interest. A measure of this minimum distance is

$$|(x, \tau)|_{\mathcal{Q}_t} = |x|_{\mathcal{Q}} := \inf_{\varkappa \in \mathcal{Q}} |x - \varkappa| = |x - \bar{\varkappa}(x)|, \tag{15}$$

where $\bar{\varkappa}(x)$ is a minimizer for (15), and for which the following bound holds,

$$\begin{aligned} |(x,t)|_{\mathcal{Q}_t} &= \inf_{\{\varkappa:\ \exists z: h(\varkappa) = h_d(z)\}} |x - \varkappa| \\ &= \inf_{\{\varkappa:\ \exists (z\ \tau):(z,\varkappa,\tau) \in \mathcal{A}\}} |x - \varkappa| \\ &\leq \inf_{\{(z,\varkappa,\tau) \in \mathcal{A}\}} |(\xi, x, t) - (z, \varkappa, \tau)| \\ &= |(\xi, x, t)|_{\mathcal{A}}, \quad \forall \xi \in \mathbb{R}^q. \end{aligned} \tag{16}$$

Moreover, for $\varkappa \in \mathcal{Q}$ let

$$\mathcal{P}(\varkappa) := \{z \in \mathbb{R}^q : \ |h(\varkappa) - h_d(z)| = 0\}. \tag{17}$$

Then for each pair $(\xi_o, \varkappa_o)$ such that $\xi_o \in \mathcal{P}(\varkappa_o)$ the following bound also holds,

$$\begin{aligned} |(\xi_o, x, t)|_{\mathcal{A}} &= \inf_{\{(z,\varkappa,\tau) \in \mathcal{A}\}} |(\xi_o, x, t) - (z, \varkappa, \tau)| \\ &= \inf_{\{(z,\varkappa): z \in \mathcal{P}(\varkappa)\}} |(\xi_o, x) - (z, \varkappa)| \leq |x - \varkappa_o|. \end{aligned} \tag{18}$$

For the closed loop system (9) and (12) we get $\forall t \geq t_0$,

$$\begin{aligned} |(x(t),t)|_{\mathcal{Q}_t} &\leq |(\xi(t), x(t), t)|_{\mathcal{A}} \\ &\leq \beta_{\mathcal{A}}\left(|(\xi_0, x_0, t_0)|_{\mathcal{A}}, t - t_0\right), \end{aligned} \tag{19}$$

which for each $r > 0$ shows attractivity of x to $\mathcal{Q}$ that is uniform over sets of initial conditions having $|(\xi_0, x_0, t_0)|_{\mathcal{A}} \leq r$. Define the set

$$\mathcal{H}(k, \rho) := \{(\xi, x, \tau) : x \in \mathcal{H}_x(\rho), \quad \xi \in \mathcal{B}_{\xi}(k)\} \tag{20}$$

in which we are guaranteed by Assumption 3 that there exists a global minimizer with a non-vanishing region of convergence. Letting $\varepsilon := \frac{1}{\mu}$ then in the fast time scale $t_f := \frac{1}{\varepsilon}t$ we get with $z' := \frac{dz}{dt_f} = \varepsilon \dot{z}$ that the motion of ξ is approximately described by the boundary layer system in the limit as $\varepsilon = 0$,

$$(x', \tau') = (0, 0), \qquad \xi' = -\sigma(\xi)V^{\xi}(\xi, x, \tau) \tag{21}$$

for which it follows from Assumption 3 that the set $\{(\xi, x, \tau) : \xi = \xi_V(x, \tau)\}$ is LAS for $(\xi(0), x(0), \tau(0)) \in \mathcal{H}(k, \rho)$. The fast state ξ will therefore rapidly converge to a small neighborhood of the manifold defined by $V^{\xi}(\xi, x, \tau) = 0$ where $V(\xi, x, \tau)$ is minimized. With $\xi \equiv \xi_V(x, \tau)$ we get the reduced system

$$\dot{x} = f(x, \alpha(\xi_V(x, t), x, t), t) \tag{22}$$

for which we consider the Lyapunov function $W(x, \tau) := V(\xi_V(x, \tau), x, \tau)$. Since (7) holds for all (ξ, x, τ) and $V^{\xi}(\xi_V(x, t), x, \tau) = 0$ by the first order optimality condition, the time derivative of $W(x, \tau)$ along the solutions of the reduced system becomes

$$\dot{W}(x, \tau) \leq -\alpha_3\left(\alpha_2^{-1}\left(W(x, \tau)\right)\right). \tag{23}$$

This implies there exists $\beta_W \in \mathcal{KL}$ such that $W(x(t), t) \leq \beta_W\left(W(x_0, t_0),\ t - t_0\right)$, which gives

$$\begin{aligned} |(x(t),t)|_{\mathcal{Q}_t} &\leq |(\xi_V(x(t),t), x(t), t)|_{\mathcal{A}} \\ &\leq \alpha_1^{-1}\left(W(x(t),t)\right) \\ &\leq \alpha_1^{-1}\left(\beta_W\left(W(x_0,t_0),\ t - t_0\right)\right) \\ &\leq \alpha_1^{-1}\left(\beta_W\left(V(\bar{\xi}(\bar{\varkappa}(x_0)), x_0, t_0),\ t - t_0\right)\right) \\ &\leq \alpha_1^{-1}\left(\beta_W\left(\alpha_2\left(|(\bar{\xi}(\bar{\varkappa}(x_0)), x_0, t_0)|_{\mathcal{A}}\right), t - t_0\right)\right) \\ &\leq \alpha_1^{-1}\left(\beta_W\left(\alpha_2\left(|x_0 - \bar{\varkappa}(x_0)|\right), t - t_0\right)\right) \\ &= \beta_1\left(|(x_0, t_0)|_{\mathcal{Q}_t},\ t - t_0\right), \quad \forall t \geq t_0, \end{aligned} \tag{24}$$

where $\beta_1(s,t) := \alpha_1^{-1}(\beta_W(\alpha_2(s),t)$, and (18) with $\bar{\xi}(\bar{\varkappa}(x_0)) \in \mathcal{P}(\bar{\varkappa}(x_0))$ was used. This shows that the set $\mathcal{Q}_t$ is UGAS for the reduced system. By the derivation above and the results in [16], the following theorem follows:

Theorem 4: Suppose the conditions of Proposition 2 are satisfied with $\omega(\xi,x,\tau) := \sigma(\xi)V^{\xi}(\xi,x,\tau)$. Then by (19) the set $\mathcal{Q}_t$ is uniformly globally attractive. By Assumption 3 there exists a class-$\mathcal{KL}$ function β such that for each $\delta > 0$ there exist $\mu^* > 0$ for which $\mu \geq \mu^*$ and $(\xi_0, x_0, t_0) \in \mathcal{H}(k,\rho)$, then

$$|(x(t),t)|_{\mathcal{Q}_t} \leq \beta\left(|(x_0,t_0)|_{\mathcal{Q}_t},\ t-t_0\right) + \delta. \quad \Diamond \tag{25}$$

This theorem shows that output invariance of the desired manifold is "nearly" achieved with a maximum offset bound δ. The disadvantage of not achieving perfect output invariance is, on the other hand, compensated by other advantages of the design, such as the ability to follow curves not being regularly parametrized or even self-intersecting curves [17]. Perhaps the most practical advantage, however, is that the desired parametrized manifold and the desired dynamics on the manifold can typically be constructed in a separate guidance system without redesign of the control law. This will be illustrated in the following example.

III. Maneuvering a vessel by the LOS method

Path-following problems for vessels are often addressed by using so-called Line-Of-Sight (LOS) algorithms. Contrary to direct position control, where the vessel may be driven both in the longitudinal and transversal directions to converge to the path, the LOS methods give more natural motions in the longitudinal direction by using the heading to steer the vessel to the path. An advantage is that the LOS algorithms work both in the fully actuated case and for underactuated vessels that only possess steering capability during forward motion.

A. Problem setup

In this case we consider the LOS methodology for controlling a vessel to a path being a regular curve in $\mathbb{R}^2$ according to the design presented by [3]. where we approach the control problem as a maneuvering problem.

For the sake of illustration, consider a vessel represented simply by the kinematic equation

$$\dot{\eta} = R(\psi)\nu \tag{26}$$

where $\eta = \mathrm{col}(x,y,\psi)$ is the position and heading with respect to an inertial reference frame, $\nu = \mathrm{col}(u,v,r)$ is the velocity vector in the vessel's body frame, and $R(\psi)$ is the corresponding 3×3 rotation matrix; see [18] for details. We assume the vessel has no sideslip and set correspondingly $v = 0$. Similarly, we assume the vessel moves forward at a desired speed $u(t) = U_d(t) \geq U_0 > 0$. The underlying maneuvering problem, according to [1], is then to steer the vessel onto and along a path given by the set

$$\mathcal{P} = \left\{\eta \in \mathbb{R}^2 \times \mathcal{S}^1 :\ \exists\theta \in \mathbb{R} \text{ s.t. } \eta = \eta_d(\theta)\right\} \tag{27}$$

where $\theta \mapsto \eta_d(\theta)$ is a continuous parametrization of the path. Accordingly, let $p = \mathrm{col}(x,y)$ be the vessel's position and $p_d(\theta)$ the parametrized path in $\mathbb{R}^2$ such that $\eta = \mathrm{col}(p,\psi)$ and $\eta_d(\theta) = \mathrm{col}(p_d(\theta), \psi_d(\theta))$, where

$$\psi_d(\theta) := \arctan\left(\frac{y_d^{\theta}(\theta)}{x_d^{\theta}(\theta)}\right) \tag{28}$$

is in the direction of the tangent vector along the path. The dynamic task is $\lim_{t\to\infty}\left|\dot{\theta}(t) - \upsilon_s(\theta(t),t)\right| = 0$, where the speed assignment υ_s corresponds to the desired surge speed, that is,

$$\upsilon_s(\theta,t) := \frac{U_d(t)}{\left|p_d^{\theta}(\theta)\right|}. \tag{29}$$

The following assumption is made to the path.

Assumption 5: The parametrization $\theta \mapsto p_d(\theta)$ is absolutely continuous, bounded, and $\exists(p_1,p_2) > 0$ such that $\forall\theta \in \mathbb{R}$, $p_1 \leq \left|p_d^{\theta}(\theta)\right| \leq p_2$. $\Diamond$

If the objective was direct position control to converge to $\mathcal{P}$, the traditional maneuvering design would continue by directly stabilizing the set

$$\mathcal{A} = \{(\eta,\theta) : \eta = \eta_d(\theta)\}, \tag{30}$$

which is a subset of $\mathcal{P}\times\mathbb{R}$. However, since the vessel in this case has steering capability only by using the yaw rate r as control input, the dynamic system is redefined into a SISO system where the vessel heading ψ will be controlled to a desired heading ψ_{los} given by the LOS algorithm, which in turn will steer the vessel towards the path. The LOS algorithm is generally given by a map $\psi_{los} = \bar{\alpha}_{los}(x,y)$ where $\bar{\alpha}_{los} : \mathbb{R}^2 \to \mathcal{S}^1$. Different methods for constructing this map exist in the literature; see for instance [6], [19], [7], [8] for straight-line paths. In this example we adopt the algorithm by [3] for path-following of regularly parametrized curves.

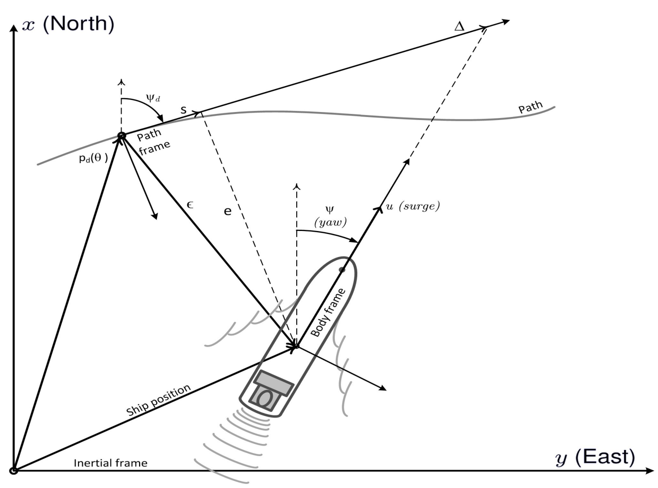

Fig. 1. Reference frames for the LOS path following setup.

1) LOS algorithm: Given a value θ, then $p_d(\theta)$ is an auxiliary point on the path. Similarly, let $q \in \mathbb{R}^2$ be a virtual point representing the position of a vessel. Define a moving path reference frame centered at $p_d(\theta)$ with its x-axis along the tangent vector at $p_d(\theta)$ and y-axis in the orthogonal direction according to the right-hand-rule. Then

$$\epsilon(q,\theta) = R_2(\psi_d(\theta))^{\top}(q - p_d(\theta)) =: \mathrm{col}(s,e) \tag{31}$$

is the position offset of the virtual point q relative to the path reference frame, where

$$R_2(\psi_d) = \begin{bmatrix} \cos(\psi_d) & -\sin(\psi_d) \\ \sin(\psi_d) & \cos(\psi_d) \end{bmatrix}$$

is the rotation matrix, $s(q, \theta)$ is the along-track error, and $e(q, \theta)$ is the cross-track error. In the path reference frame, the LOS algorithm is constructed to point the vessel at a lookahead distance Δ along the tangent vector to the path; see Figure 1.

With the definitions

$$V(q, \theta) := \frac{1}{2}\epsilon(q, \theta)^\top \epsilon(q, \theta), \tag{32}$$

$$\omega_s(q, \theta) := \frac{\mu}{|p_d^\theta(\theta)|} V^\theta(q, \theta) = -\mu \frac{p_d^\theta(\theta)^\top}{|p_d^\theta(\theta)|} (q - p_d(\theta)) \tag{33}$$

$$\psi_r(q, \theta) := -\arctan\left(\frac{e(q, \theta)}{\Delta}\right), \tag{34}$$

where $\mu > 0$ is a constant gain, the LOS heading for the vessel is given by the dynamic algorithm

$$\dot{\theta} = f_\theta(q, \theta, t) := \frac{\Delta}{\sqrt{e(q, \theta)^2 + \Delta^2}} \upsilon_s(\theta, t) - \omega_s(q, \theta) \tag{35a}$$

$$\psi_{los} = \alpha_{los}(q, \theta) := \psi_d(\theta) + \psi_r(q, \theta). \tag{35b}$$

The update law for $\dot{\theta}$ consists of two terms. The first term is the feedforward speed assignment term corresponding to the surge speed, but modified by a gain dependent on the cross-track error. The second term, ω_s, is a gradient algorithm that according to Section II makes $p_d(\theta)$ converge rapidly to a point that minimizes $\theta \mapsto V(q, \theta)$ for $\mu \gg 1$. Assuming the path is constructed such that for a given position q, this map has a global minimum and that $\theta(0)$ is within the region of convergence of this minimum, then Figure 1 indicates that $V(q, \theta)$ is minimum where the along-track error is zero and the cross-track error is minimized. It follows that ω_s is a stabilizing term for $s(q, \theta) = 0$. The following proposition shows the effectiveness of the method.

Proposition 6: For (26) with motion constrained to $u(t) \equiv U_d(t)$, $v(t) \equiv 0$, $q := p$, and $\psi(t) \equiv \alpha_{los}(p(t), \theta(t))$, with $p_d(\theta)$ satisfying Assumption 5, then the dynamic LOS algorithm (35) with $\upsilon_s(\theta, t)$ assigned as in (29), renders the set (30) UGAS and ULES. $\Diamond$

Proof: The proof is sketched in [3] but stated here in detail for completeness. Let $V(p, \theta)$ be a Lyapunov function and note the equivalence relations

$$|(\eta, \theta)|_{\mathcal{A}} \leq |\eta - \eta_d(\theta)| \leq \sqrt{2} \max\{1, L\} |(\eta, \theta)|_{\mathcal{A}} \tag{36a}$$

$$\frac{1}{(1 + 1/\Delta)} |\eta - \eta_d(\theta)| \leq |\epsilon(p, \theta)| \leq |\eta - \eta_d(\theta)|, \tag{36b}$$

where $L > 0$ is the global Lipschitz constant that follows from the absolute continuity property and boundedness of $|p_d^\theta(\theta)|$. The result is the bounds

$$c_1 |(\eta, \theta)|_{\mathcal{A}}^2 \leq V(p, \theta) \leq c_2 |(\eta, \theta)|_{\mathcal{A}}^2, \tag{37}$$

where $c_1 = \frac{1}{2(1+1/\Delta)^2}$ and $c_2 = \max\{1, L^2\}$. Notice also the relationships

$$V = \frac{1}{2}\epsilon^\top \epsilon = \frac{1}{2}(p - p_d(\theta))^\top (p - p_d(\theta)) \tag{38}$$

$$p_d^\theta(\theta) = |p_d^\theta(\theta)| \begin{bmatrix} \cos(\psi_d(\theta)) \\ \sin(\psi_d(\theta)) \end{bmatrix} \tag{39}$$

$$\dot{p} = U_d \begin{bmatrix} \cos(\psi) \\ \sin(\psi) \end{bmatrix} \tag{40}$$

$$R_2(\psi_d(\theta))^\top p_d^\theta(\theta) = |p_d^\theta(\theta)| \operatorname{col}(1, 0) \tag{41}$$

where $\operatorname{col}(\cos(\psi_d(\theta)),\ \sin(\psi_d(\theta)))$ is the unit tangent vector along the path at $p_d(\theta)$. Differentiating V along the solutions of (40) and (35a), this gives

$$\begin{aligned} \dot{V} &= -\frac{U_d(t)}{\sqrt{e^2 + \Delta^2}} e^2 - \frac{\mu}{|p_d^\theta|} V^\theta(p, \theta)^2 \\ &= -\frac{U_d(t)}{\sqrt{e^2 + \Delta^2}} e^2 - \frac{\mu}{|p_d^\theta|} (p - p_d)^\top p_d^\theta \left(p_d^\theta\right)^\top (p - p_d) \\ &= -\frac{U_d(t)}{\sqrt{e^2 + \Delta^2}} e^2 - \frac{\mu}{|p_d^\theta|} \epsilon^\top |p_d^\theta| \begin{bmatrix} 1 \\ 0 \end{bmatrix} \begin{bmatrix} 1 \\ 0 \end{bmatrix}^\top |p_d^\theta| \epsilon \\ &\leq -\frac{U_0}{\sqrt{e^2 + \Delta^2}} e^2 - \mu p_1 s^2 =: -\varphi_3(|\epsilon|) \end{aligned} \tag{42}$$

where $\varphi_3(\cdot)$ is a class-$\mathcal{K}_\infty$ function and we used $\sin(-\arctan(x)) = -\frac{x}{\sqrt{1+x^2}}$ and $\cos(-\arctan(x)) = \frac{1}{\sqrt{1+x^2}}$. Using (36), this implies that there exists a class-$\mathcal{KL}$ function β such that for all t in the maximal interval of existence $[0, T)$,

$$|(\eta(t), \theta(t))|_{\mathcal{A}} \leq \beta\left(|(\eta(0), \theta(0))|_{\mathcal{A}}, t\right). \tag{43}$$

From this and Assumption 5 it follows that the right-hand side of $(\dot{p}, \dot{\theta})$ is bounded on $[0, T)$, which rules out finite escape time giving $T = \infty$, and thus $\mathcal{A}$ is UGAS. To show ULES, notice for $|\epsilon| \leq M < \infty$, then $\exists c_3 > 0$ such that

$$\begin{aligned} -\frac{U_0}{\sqrt{e^2 + \Delta^2}} e^2 - \mu p_1 s^2 &\leq -\frac{U_0}{\sqrt{M^2 + \Delta^2}} e^2 - \mu p_1 s^2 \\ &\leq -c_3 |\epsilon|^2, \end{aligned} \tag{44}$$

which for each $M > 0$ such that $|(\eta, \theta)|_{\mathcal{A}} \leq (1 + 1/\Delta)M$ gives the necessary quadratic bound on $\dot{V}$. ■

2) Problem statement: Having shown the effectiveness of the LOS algorithm, the control problem becomes to stabilize the set in which the LOS algorithm is activated. For the vessel (26), this set is given by

$$\mathcal{Q} = \{(p, \psi, \theta) : \psi = \alpha_{los}(p, \theta)\}$$

which due to the constraint $\psi = \alpha_{los}$ is a 3-dimensional manifold in the state-space $\mathbb{R}^2 \times \mathcal{S}^1 \times \mathbb{R}$. At this point it would be straightforward to develop a control law for r to stabilize $\{\psi - \alpha_{los}(p, \theta) = 0\}$ with $\dot{\theta} = f_\theta(p, \theta, t)$. However, in order to exemplify the maneuvering methodology, we propose to parametrize the 3-dimensional manifold by the variables $(q, \alpha) \in \mathbb{R}^2 \times \mathcal{S}^1$ according to

$$\mathcal{Q} = \{(p, \psi, \theta) : \exists (q, \alpha) \text{ s.t. } p = q,\ \psi = \alpha,\ \alpha = \alpha_{los}(q, \theta)\}$$

Next, we include (q,α) as dynamic variables for (p,ψ) to track and propose as the geometric task to stabilize the set

$$\mathcal{B}=\{(p,\psi,q,\alpha,\theta):\ p=q,\ \psi=\alpha,\ \alpha=\alpha_{los}(q,\theta)\}, \tag{45}$$

which is a subset of $\mathcal{Q}\times\mathbb{R}^2\times\mathcal{S}^1$. In doing so the result becomes a reference filter that we later will show to have certain optimizing properties, taking the vessel states as input and providing the LOS heading as output.

The dynamic task should represent the desired behavior of the system when constrained to the set $\mathcal{B}$. First of all, using the parametrization variable q as a position state, the dynamics of θ should be governed by $\dot\theta=f_\theta(q,\theta,t)$, as defined in (35a), and $\psi_{los}=\alpha_{los}(q,\theta)$ according to (35b). In addition, for $q(t)\equiv p(t)$ and $\alpha(t)\equiv\psi(t)$ in $\mathcal{B}$, the dynamic tasks become

$$\lim_{t\to\infty}|\dot q(t)-\upsilon_q(\psi(t),t)|=0 \tag{46}$$

$$\lim_{t\to\infty}|\dot\alpha(t)-\upsilon_\alpha(q(t),\psi(t),\alpha(t),\theta(t),t)|=0, \tag{47}$$

where the dynamic assignments υ_q and υ_α are designed as

$$\upsilon_q(\psi,t)=\begin{bmatrix}\cos(\psi)\\ \sin(\psi)\end{bmatrix}U_d(t), \tag{48}$$

$$\begin{aligned}\upsilon_\alpha(q,\psi,\alpha,\theta,t)=&-K_\alpha(\alpha-\alpha_{los}(q,\theta))\\&+\alpha^q_{los}(q,\theta)\upsilon_q(\psi,t)+\alpha^\theta_{los}(q,\theta)f_\theta(q,\theta,t),\end{aligned} \tag{49}$$

$K_\alpha>0$, adhering to the motion of $\dot p$ and the desired motion of $\dot\alpha_{los}$ (including a stabilizing feedback term).

A remark is here necessary on the choice of the dynamic assignment for $\dot q$. In the fully actuated case, meaning that there are enough control inputs for the vessel (26) such that the vector $\nu=\mathrm{col}(u,v,r)$ can be assigned a control law in all degrees-of-freedom, then the preferable dynamic assignments would be $\dot q=\upsilon_q(\alpha,t)$ and $\dot\alpha=\upsilon_\alpha(q,\alpha,\theta,t)$ making (q,α) behave like a virtual vessel, and then use ν to control p to q and ψ to α, identically. In the underactuated case, on the other hand, the motion of $\dot p$ is constrained and only the steering control input r is available. In this case, we must rather control q to p, identically, and consequently $\upsilon_q(\psi,t)$ must adhere to the motion of $\dot p$.

B. Control design

For the task of stabilizing $\mathcal{B}$, let $\dot q=\upsilon_q(\psi,t)-\omega_q$ and $\dot\alpha=\upsilon_\alpha(q,\psi,\alpha,\theta,t)-\omega_\alpha$, where ω_q and ω_α will be used as design variables. Define the control Lyapunov function

$$\begin{aligned}W(p,\psi,q,\alpha,\theta)=&\frac{(1-\lambda)}{2}(p-q)^\top(p-q)\\&+\frac{(1-\lambda)}{2}(\psi-\alpha)^2+\frac{\lambda}{2}(\alpha-\alpha_{los}(q,\theta))^2\end{aligned} \tag{50}$$

where $\lambda\in(0,1)$ is a tuning weight. Differentiating W gives

$$\begin{aligned}\dot W=&(1-\lambda)(p-q)^\top(\dot p-\upsilon_q)+(1-\lambda)(\psi-\alpha)(r-\upsilon_\alpha)\\&-K_\alpha\lambda(\alpha-\alpha_{los}(q,\theta))^2\\&+\left[(1-\lambda)(p-q)^\top+\lambda(\alpha-\alpha_{los}(q,\theta))\alpha^q_{los}(q,\theta)\right]\omega_q\\&+\left[(1-\lambda)(\psi-\alpha)-\lambda(\alpha-\alpha_{los}(q,\theta))\right]\omega_\alpha.\end{aligned} \tag{51}$$

Since $\dot p=\upsilon_q(\psi,t)$, the first term vanishes. The second term is made negative definite by selecting a proper control law for r. The third term is already negative definite by the feedback term in υ_α. The fourth term is recognized as $-W^q(p,\psi,q,\alpha,\theta)\omega_q$ and the fifth term as $-W^\alpha(p,\psi,q,\alpha,\theta)\omega_\alpha$, and according to the maneuvering design methodology in Section II-C we select ω_q and ω_α to make these terms negative by gradient feedbacks. The result is summarized in the following theorem.

Theorem 7: For (26) with motion constrained to $u(t)\equiv U_d(t)$ and $v(t)\equiv 0$, and $p_d(\theta)$ satisfying Assumption 5, define the error coordinates $\tilde p:=p-q$, $\tilde\psi:=\psi-\alpha$, $\tilde\alpha:=\alpha-\alpha_{los}(q,\theta)$ and apply the global diffeomorphism $(p,\psi,q,\alpha,\theta)\mapsto(\tilde p,\tilde\psi,q,\tilde\alpha,\theta)$. Then the control law

$$\dot q=\upsilon_q(\psi,t)-\gamma_q W^q(p,\psi,q,\alpha,\theta)^\top,\qquad \gamma_q>0 \tag{52a}$$

$$\dot\alpha=\upsilon_\alpha(q,\psi,\alpha,\theta,t)-\gamma_\alpha W^\alpha(p,\psi,q,\alpha,\theta),\ \gamma_\alpha>0 \tag{52b}$$

$$r=-K_\psi(\psi-\alpha)+\upsilon_\alpha(q,\psi,\theta,t),\qquad K_\psi>0 \tag{53}$$

with υ_q and υ_α as defined in (48) and (49), renders the set

$$\mathcal{B}'=\left\{(\tilde p,\tilde\psi,q,\tilde\alpha,\theta):\ \tilde p=0,\ \tilde\psi=0,\ \tilde\alpha=0\right\} \tag{54}$$

UGES. In this set the dynamic tasks (46) and (47) are satisfied, and with (35) it follows from Proposition 6 that the original path-following objective (27) is solved. ◊

Proof: For the noncompact set (54) we have $\left|(\tilde p,\tilde\psi,q,\tilde\alpha,\theta)\right|_{\mathcal{B}'}=\left|(\tilde p,\tilde\psi,\tilde\alpha)\right|$. According to Lemma 8, let $x_1:=\mathrm{col}(\tilde p,\tilde\psi,\tilde\alpha)$, $x_2:=\mathrm{col}(q,\theta)$, and $(u_1,u_2)=0$. For $\left|(\tilde p,\tilde\psi,\tilde\alpha)\right|$ bounded, it follows from the fact that $\alpha^q_{los}(q,\theta)=-\frac{\Delta}{e(q,\theta)^2+\Delta^2}\mathrm{row}\left(-\sin(\psi_d(\theta)),\ \cos(\psi_d(\theta))\right)$ is bounded by $\frac{\sqrt2}{\Delta}$ and Assumption 5 that the right-hand sides of (52a) and (35a) satisfy the sector growth condition in Lemma 8 and the system is finite escape-time detectable through $\left|(\tilde p,\tilde\psi,q,\tilde\alpha,\theta)\right|_{\mathcal{B}'}$. For (50) we have the following bounds

$$d_1\left|(\tilde p,\tilde\psi,\tilde\alpha)\right|^2\le W(p,\psi,q,\alpha,\theta)\le d_2\left|(\tilde p,\tilde\psi,\tilde\alpha)\right|^2$$

where $d_1=\min\left\{\frac{\lambda}{2},\frac{1-\lambda}{2}\right\}$ and $d_2=\frac12$. With the control law (52) and (53) inserted in (51), we get

$$\begin{aligned}\dot W\le&-K_\psi(1-\lambda)\tilde\psi^2-K_\alpha\lambda\tilde\alpha^2\\&-\gamma_q\left|(1-\lambda)\tilde p^\top+\lambda\alpha^q_{los}(q,\theta)\tilde\alpha\right|^2\\=&-\gamma_q x_1^\top H(q,\theta)x_1<0\end{aligned} \tag{55}$$

where

$$H(q,\theta)=\begin{bmatrix}(1-\lambda)^2I & 0 & \lambda(1-\lambda)\left(\alpha^q_{los}\right)^\top\\ 0 & \frac{K_\psi}{\gamma_q}(1-\lambda) & 0\\ \lambda(1-\lambda)\alpha^q_{los} & 0 & \frac{K_\alpha}{\gamma_q}\lambda+\lambda^2\left(\alpha^q_{los}\right)\left(\alpha^q_{los}\right)^\top\end{bmatrix}$$

is a symmetric positive definite matrix. Since $\alpha^q_{los}(q,\theta)$ is bounded by $\frac{\sqrt2}{\Delta}$, it follows that there exists $d_3>0$,

$d_3 \leq \gamma_q \inf_{(\zeta,\pi)} \lambda_{\min}(H(\zeta,\pi))$ such that $\gamma_q x_1^\top H(q,\theta) x_1 \geq d_3 x_1^\top x_1$, $\forall (q,\theta)$, and hence

$$\dot{W} \leq -d_3 \left| (\tilde{p}, \tilde{\psi}, \tilde{\alpha}) \right|^2 . \tag{56}$$

By application of Theorem 9, UGES of (54) follows. ∎

C. Simulations

Investigation of the closed-loop equations shows that (35) and (52) form a dynamic guidance filter, taking the vessel states (p, ψ) as input and providing the signals $(\alpha, \upsilon_\alpha)$ as outputs to the feedback control law (53) that correspondingly steers the vessel towards and along the path. This filter possesses three gradient tuning functions. The first for $\dot{\theta}$ ensures for $\mu \gg 1$ that $\theta(t)$ rapidly minimizes (32) by driving the along-track error rapidly to zero with respect to the virtual position q. The second for $\dot{q}$ and third for $\dot{\alpha}$ ensures for $\gamma_q, \gamma_\alpha \gg 1$ a rapid minimization of (50) with respect to q and α, respectively. With λ chosen small, this typically ensures that $q(t)$ rapidly approaches $p(t)$ and $\alpha(t)$ rapidly approaches $\psi(t)$. The dynamics for $\dot{\theta}$ should be the fastest, since this needs to rapidly respond to the motion of $q(t)$. Then the convergence $q(t) \to p(t)$ ensures that the LOS angle becomes correctly calculated. Finally, we see that the minimization of $(\alpha(t) - \psi(t))^2$ quickly eliminates the feedback term in the control law (53) such that the control signal $r(t)$ is mainly governed by the feedforward signal $\upsilon_\alpha(t)$.

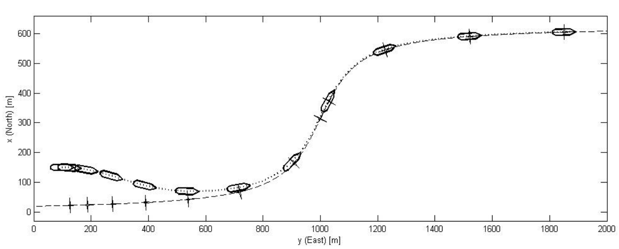

Fig. 2. Ship voyage, showing convergence to and along the path.

A simulation has been conducted to emulate a vessel with length $L = 80$ m, forward speed $U_d = 10$ m/s, and maximum speed of rotation $|r| \leq r_{\max} = 2$ deg/s. The path is parametrized as

$$x_d(\theta) = x_c + k \arctan\left(\left(\frac{\theta}{\kappa}\right), \quad y_d(\theta) = y_c + \theta, \tag{57}$$

where $\kappa = 100$, $k = 200$, $x_c = 100\pi$, $y_c = 1000$, and the LOS parameter $\Delta = 50$. The feedback gains are $K_\alpha = K_\psi = 0.025$, and the gradient gains are $\mu = 10$, $\gamma_q = \gamma_\alpha = 2$, and $\lambda = 0.05$. The vessel is initialized at $p_0 = (150, 100)$ and $\psi_0 = 90°$, while $(\theta_0, q_0, \alpha_0) = (0, 0, 0, 0)$.

Figures 2 and 3 show the overall response in position and heading, where the heading slowly converges to the LOS heading and the ship moves correspondingly to and along the track. In Figure 4 the transients in the dynamic guidance states are shown. In the fast time scale it is seen how the vessel position and heading remains fairly constant, while (θ, q, α) rapidly responds in accordance to their gradient tuning functions.

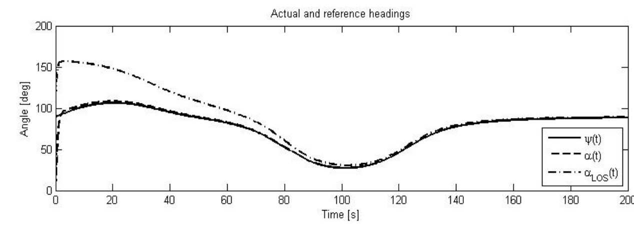

Fig. 3. The desired heading $\alpha(t)$ converges rapidly to $\psi(t)$ and then slowly to $\alpha_{LOS}(t)$.

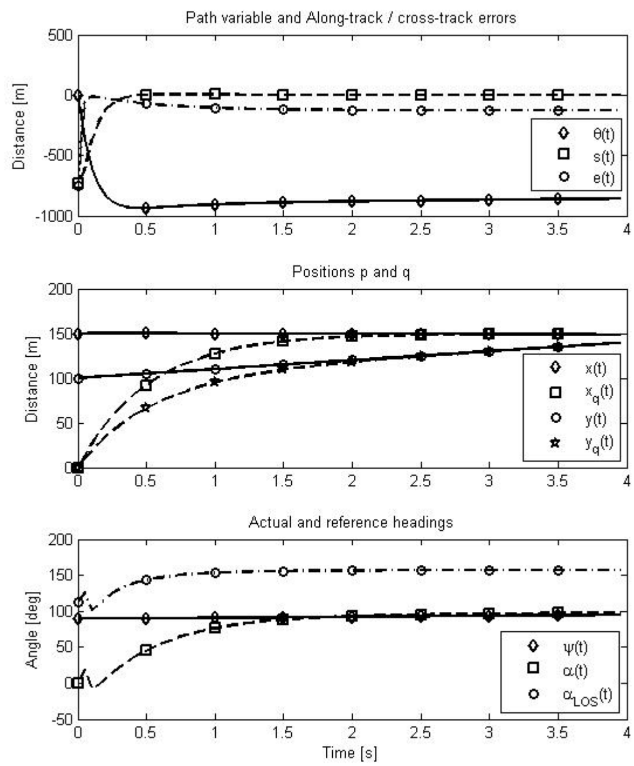

Fig. 4. Transients during first 4 sec. Upper plot shows the rapid transient of $\theta(t)$ resulting in $s(t) \to 0$. The middle plot and lower plot show the rapid transients of $q(t)$ and $\alpha(t)$ to $p(t)$ and $\psi(t)$, respectively.

IV. CONCLUSION

In this paper the maneuvering problem, as previously defined for stabilizing 1-dimensional manifolds, have been generalized by considering q-dimensional manifolds to be rendered UGAS in the geometric task of the control problem. The other task, termed the dynamic task, was to assign a desired dynamic behavior to the closed-loop system when constrained to the manifold. As in earlier works, parametrizing the desired manifold by a variable $\xi \in \mathbb{R}^q$ gives an extra degree of freedom for design that can be utilized to achieve other objectives in addition to the dynamic behavior on the manifold. In particular, the dynamics of $\dot{\xi}$ was allowed to take feedback from the system states in order to shape the transients. It was shown that the gradient update law, as earlier reported for the 1-dimensional case, could be utilized with success also in this case of higher order manifolds in order to rapidly minimize the distance of the state x to the manifold. However, also other tuning functions are allowed if properly designed according to Proposition 2.

A case study was conducted for a simplified vessel, where the objective was to control the vessel to and along a path by using only the steering input and the LOS method. The geometric task was then to control the vessel position and heading to the 3-dimensional manifold in the state space where the LOS method becomes activated. Once on this manifold, then the LOS method ensures as a dynamic task that the vessel converges to and follows the path as intended. Simulation have been done to illustrate the effectiveness and properties of the resulting dynamic control law.

V. Appendix

For the interconnected system

$$\begin{aligned} \dot{x}_1 &= f_1(x_1, x_2, u_1) \\ \dot{x}_2 &= f_2(x_1, x_2, u_2) \end{aligned} \tag{58}$$

where $x_1(t) \in \mathbb{R}^{n_1}$ and $x_2(t) \in \mathbb{R}^{n_2}$ are the states, $u_1(t) \in \mathcal{U}_1 \subset \mathbb{R}^{m_1}$ and $u_2(t) \in \mathcal{U}_2 \subset \mathbb{R}^{m_2}$ are inputs where $\mathcal{U}_1, \mathcal{U}_2$ are compact sets, and the vector fields f_1, f_2 are smooth, let the set of interest be

$$\mathcal{A} := \{(x_1, x_2) \in \mathbb{R}^{n_1} \times \mathbb{R}^{n_2} : |x_1|_{\mathcal{A}_1} = 0\}, \tag{59}$$

where $\mathcal{A}_1 \subset \mathbb{R}^{n_1}$ is compact. This gives $|(x_1, x_2)|_{\mathcal{A}} = |x_1|_{\mathcal{A}_1}$. The following lemma and theorem are given in [1, Appendix A.5]:

Lemma 8: If for each compact set $\mathcal{X} \subset \mathbb{R}^{n_1}$ there exist $L > 0$ and $c > 0$ such that:

$$|f_2(\xi, x_2, \upsilon)| \leq L\,|x_2| + c, \qquad \forall x_2 \in \mathbb{R}^{n_2}, \tag{60}$$

uniformly for all $(\xi, \upsilon) \in \mathcal{X} \times \mathcal{U}_2$, that is, f_2 satisfies a sector growth condition in x_2, then the system (58) is finite escape-time detectable through $|\cdot|_{\mathcal{A}}$.

The following theorem shows stability of (59) in the sense of Lyapunov with respect to (58):

Theorem 9: Assume that the sector bound (60) in Lemma 8 holds for (58). If, in addition, there exist a smooth function $V : \mathbb{R}^{n_1} \times \mathbb{R}^{n_2} \to \mathbb{R}_{\geq 0}$ and $\mathcal{K}_\infty$-functions α_i, $i = 1, \ldots, 4$, such that

$$\alpha_1\left(|x_1|_{\mathcal{A}_1}\right) \leq V(x_1, x_2) \leq \alpha_2\left(|x_1|_{\mathcal{A}_1}\right) \tag{61}$$

and

$$\begin{aligned} V^{x_1}(x_1, x_2) f_1\left(x_1, x_2, u_1\right) + V^{x_2}(x_1, x_2) f_2\left(x_1, x_2, u_2\right) \\ \leq -\alpha_3\left(|x_1|_{\mathcal{A}_1}\right) + \alpha_4\left(|u|\right) \end{aligned} \tag{62}$$

hold, where $u := \mathrm{col}(u_1, u_2) \in \mathcal{U}_1 \times \mathcal{U}_2$, then the system (58) is ISS with respect to the closed, 0-invariant set (59). In the case when $u_1 = 0$ and $u_2 = 0$ then the closed, forward invariant set (59) is UGAS with respect to (58), and if $\alpha_i(|x|_{\mathcal{A}_1}) = c_i\,|x|^r_{\mathcal{A}_1}$ for $i = 1, 2, 3$, where c_1, c_2, c_3, r are strictly positive reals with $r \geq 1$, then (59) is UGES with respect to (58).

See [1, Appendix A.5] or [13] for the proof.

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