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Optimal control of a class of linear nonautonomous parabolic PDE via two-parameter semigroup representation

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Abstract— **This paper considers the two-parameter semigroup representation of a class of parabolic partial differential equation (PDE) with time and spatially dependent coefficients. The properties of the PDE which are necessary for the initial and boundary value problem to be posed as a linear nonautonomous evolution equation on an appropriately defined infinite-dimensional function space are presented. Using these properties, the associated nonautonomous operator generates a two-parameter semigroup which yields the generalized solution of the initial and boundary value problem. The explicit expression of the two-parameter semigroup is provided and enables the application of optimal control theory for infinite dimensional systems.**

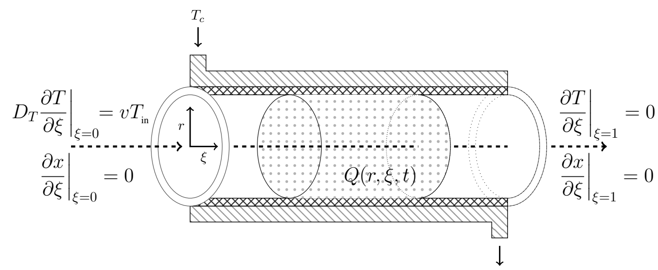

Fig. 1. Simple process diagram of an axisymmetric tubular reactor system with dynamic concentration $x(r,\xi,t)$, temperature $T(r,\xi,t)$ and consumption/generation term $Q(r,\xi,t)$.

I. INTRODUCTION

Many chemical and materials engineering transport-reaction processes exhibit time-varying and non-uniform characteristics which appear as time and spatially dependent coefficients in the diffusion-convection-reaction equations used to model the concentration or temperature dynamics. The dependence of the system parameters on space and time are often due to changes in the operating conditions and/or as a result of the process itself, each of which have significant impacts on the underlying dynamics of the transport phenomena. Examples of these processes include the time varying parameters which are due to the spatial domain evolution in crystal growth processes and non-homogeneous media in packed bed and catalytic tubular reactor systems, see [1]. In the latter example, catalyst fouling and coke deposition are common occurrences in industrial processes which lead to suboptimal reactant conversion, equipment failure and similar undesirable production outcomes, see [2], [3]. The reactor temperature dynamics can also affect the reaction kinetics in nonisothermal processes. Therein lies the need to incorporate these time and spatially dependent features into the underlying diffusion-convection-reaction model for both, the analysis of the process dynamics, and for the purpose of control synthesis. The standard approach to the regulator design utilizes infinite-dimensional systems theory whereby the PDE is converted into an abstract evolution equation on an appropriately defined function space. This approach relies on the expression of the evolution system operator as an analytic semigroup. In the case where the PDE parameters are time-dependent, the resulting system is nonautonomous and requires the satisfaction of additional properties in contrast to the autonomous case. The general theory for nonautonomous evolution equations has been extensively developed and includes many important results on the existence of generalized solutions, methods in the construction of the associated semigroups, and optimal control synthesis, see [4], [5], [6], [7], [8], [9], [10]. However, the actual analytic expression of the semigroups generated by nonautonomous operators are limited to a relatively few number of examples. The determination of the semigroup expression is a critical step in the implementation of the established control theory.

In this work, we consider a general nonhomogeneous diffusion-convection-reaction PDE with time and spatially dependent coefficients and present the properties which are necessary in order for the PDE to be represented as a linear nonautonomous parabolic evolution system. These specific properties, which have been established in [8], enable the construction of the associated semigroup and moreover yields an expression which forms the generalized solution of the PDE system. In particular, we provide the explicit form of the generalized solution of the nonautonomous parabolic evolution equation which is represented in terms of a *two-parameter semigroup*, $U(t,s)$. The evolutionary system is used in the synthesis of the linear quadratic regulator which requires the solution of the time-dependent differential Riccati equation.

This paper is organized as follows: Section II begins with an introduction to the conventional notation and function space setting which will be used throughout the paper. The properties of the PDE operator which are essential in the representation as an abstract evolution system are introduced. The Section III contains the main results of this work including the explicit expression and the properties of the two-parameter evolution operator. An illustrative example

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of a tubular reactor with time-dependent diffusion, axial dispersion, reaction kinetic coefficients and nonhomogeneous generation-consumption term, is presented. The Section IV presents the general optimal control framework in terms of the two-parameter evolution operator, and Section V provides a brief summary of results.

II. Preliminaries

A. Notation and function space description

The following notation will be used throughout this work: A general Banach space will be denoted as $\mathcal{X}$. If $\mathcal{Y}$ is a Banach space, $\mathcal{L}(\mathcal{X},\mathcal{Y})$ denotes the space of bounded linear operators $\mathcal{T}:\mathcal{X}\to\mathcal{Y}$ and $\mathcal{L}(\mathcal{X})=\mathcal{L}(\mathcal{X},\mathcal{X})$. The time index t is taken in the interval $[0,T]$ for notational convenience. The spatial domain is an open set of $\mathbb{R}^n$ and will be denoted as Ω with smooth boundary $\partial\Omega$. Spatial points are denoted by $\xi\in\mathbb{R}^n$. The space $C^k([0,T];\mathcal{X})$ consists of all functions which are k times continuously differentiable, with $k\in\mathbb{N}$, defined in the time interval $[0,T]$ and taking values in $\mathcal{X}$. The space $C^k(\Omega)$, $k\in\mathbb{N}$ consists of the functions having all derivatives up to order k continuous on Ω. The Hilbert spaces $L^2(\Omega)$ with inner product $\langle\cdot,\cdot\rangle$, $H_0^{m,p}(\Omega)$ and $H^{m,p}(\Omega)$ with norm $\|\cdot\|_{m,p}=\left(\sum_{\alpha\leq m}\int_\Omega|D^\alpha x|^p d\xi\right)^{1/p}$ where $\alpha=(\alpha_1,\dots,\alpha_n)$ is a multiindex, $|\alpha|=\alpha_1+\cdots\alpha_n$, with $D^\alpha x=\partial^{|\alpha|}x/\partial\xi_1^{\alpha_1}\cdots\partial\xi_n^{\alpha_n}$, $D^k x:=\{D^\alpha x,\,|\alpha|=k,\,k\in\mathbb{N}\}$, each follow with standard definitions and properties, see [11], [12], [13].

B. Nonautonomous parabolic system representation

In this paper we consider a class of parabolic partial differential equation defined on a bounded domain Ω in $\mathbb{R}^n$ with smooth boundary $\partial\Omega$, and expressed by the initial and boundary value problem as,

$$\begin{aligned}
&\frac{\partial x}{\partial t}+A(\xi,t,D)x=f(\xi,t) && \text{on} \quad \Omega\times(0,T]\\
&\frac{\partial x}{\partial \nu}=0 && \text{on} \quad \partial\Omega\times(0,T]\\
&x(\xi,0)=x_0(\xi) && \text{in} \quad \Omega
\end{aligned}\tag{1}$$

with operator,

$$A(\xi,t,D):=\sum_{|\alpha|\leq 2}a_\alpha(\xi,t)D^\alpha \tag{2}$$

where ν is the outward normal vector to $\xi\in\partial\Omega$. The Eq.1-2 are assumed to satisfy the following conditions:

C1. The coefficients $a_\alpha(\xi,t)$ are continuous in $\xi\in\bar{\Omega}$ and

$$|a_\alpha(\xi,t)-a_\alpha(\xi,s)|\leq L_0|t-s|^\beta$$

for $s,t\in[0,T]$, constants $L_0>0$ and $\beta\in(0,1]$;

C2. For every $\xi\in\bar{\Omega}$, there exists a constant $\epsilon>0$ and vector $\boldsymbol{\eta}\in\mathbb{R}^n$ where the principal part of $A(\xi,t,D)$ satisfies

$$-\sum_{|\alpha|=2}a_\alpha(\xi,t)\boldsymbol{\eta}^\alpha\geq\epsilon|\boldsymbol{\eta}|^2$$

such that the operator $A(\xi,t,D)$ is strongly elliptic in Ω for each $t\in[0,T]$;

C3. The function $f(\xi,t)\in L^2(\Omega)$ and

$$[f]_\mu:=\sup_{s,t\in[0,T]}\frac{|f(\xi,t)-f(\xi,s)|}{|t-s|^\mu}<\infty$$

such that $f(\xi,t)$ is Hölder continuous in $t\in[0,T]$.

Consider the family of linear operators $\{A(t)\}_{t\in[0,T]}$ associated with the elliptic operator $A(\xi,t,D)$ in the Eq.2.

Definition 1: Let the domain of the operator $A(t)$ be defined as

$$D(A(t)):=H^{1,2}(\Omega)\cap H^{2,2}(\Omega) \tag{3}$$

where

$$A(t)x:=A(\xi,t,D)x \tag{4}$$

for state $x\in D(A(t))$, such that the operator $A(t):D(A(t))\subset L^2(\Omega)\to L^2(\Omega)$.

The nonautonomous parabolic evolution system representation of the PDE system in the Eq.1 is given as the following initial value problem on the state space $\mathcal{X}=L^2(\Omega)$

$$\frac{dx(t)}{dt}+A(t)x(t)=f(t),\qquad x(0)=x_0 \tag{5}$$

for the initial state $x_0\in L^2(\Omega)$. The solution to the parabolic system in the Eq.5 is defined as a *generalized solution* of the initial and boundary value problem in the Eq.1. In order to determine this solution, consider the following result from [8, Lemma 6.1,Theorem 6.2, Chapter 7].

Proposition 1: Given that the operator $A(\xi,t,D)$ in the Eq.2 satisfies the conditions C1 and C2, then there exists a constant $c\geq 0$ such that $\{A(t)+cI\}_{t\in[0,T]}$ satisfies the following properties:

C4. The domain of the operator $D(A(t))$ is independent of $t\in[0,T]$;

C5. There exists a constant L such that

$$\|(\lambda I-A(t))^{-1}\|\leq\frac{L}{|\lambda|} \tag{6}$$

for $\lambda\in S_\delta$ where $\rho(A(t))\supset S_\delta:=\{\lambda\in\mathbb{C};|\arg\lambda|<\delta\,,\delta\in(\pi/2,\pi]\}\cup\{0\}$ for every $t\in[0,T]$, Re$\lambda\leq 0$;

C6. There exist constants L and $\alpha\in(0,1]$ such that for $s,t,\tau\in[0,T]$

$$\left\|(A(t)-A(s))A(\tau)^{-1}\right\|\leq L|t-s|^\alpha$$

Furthermore, if $f(\xi,t)$ satisfies C3, then for every $x_0\in L^2(\Omega)$, the nonautonomous parabolic evolution system in the Eq.5 has a unique generalized solution which is expressed in terms of a *two-parameter semigroup*, $U(t,s)$.

Remark 1: The boundary conditions imposed on the original PDE system in the Eq.1 were taken to be of homogeneous Neumann type. One can consider the case of homogeneous Dirichlet boundary conditions with the additional modification to the Definition 1 whereby $D(A(t))=H_0^{1,2}(\Omega)\cap H^{2,2}(\Omega)$ and proceed in the following section without loss of generality. However, in the most general case of mixed boundary conditions, the time dependent coefficients $a_\alpha(\xi,t)$ appear in the boundary data so that $D(A(t))$ is no longer independent of $t\in[0,T]$. Although this general restriction that $D(A(t))$ should be independent of t is lifted in [10] by modifications and additions to the conditions C1-C6, this situation requires some additional theory outside of the scope of the present work, see [9].

III. Main results

This section begins with a brief overview of some technical aspects of spectral theory which are necessary in the following development of the main Theorem 1 which includes the explicit representation of the two-parameter semigroup. In particular, we consider the homogeneous form of the Eq.1 with operator $A(t)$ defined in the Definition 1. The corresponding initial value problem on the state space $\mathcal{X}=L^2(\Omega)$ is given by:

$$\frac{dx(t)}{dt}+A(t)x(t)=0, \qquad x(s)=x_s \tag{7}$$

where $0\le s<t\le T$. From the Proposition 1, for every $0\le s<T$ and $x_s\in L^2(\Omega)$ there exists a unique solution $x(t)$ of the evolution system in the Eq.7 expressed as:

$$x(t)=U(t,s)x_s \tag{8}$$

where $U(t,s)$ is the two-parameter semigroup which is presented in the second part of this section.

A. Spectral characteristics

One can note that in the case when the coefficients of the operator $A(\xi,t,D)$ are time independent and satisfy the strong elliptic condition C2, then for the associated linear operators $A=A(\xi,D)$ with domain $D(A)=H^{1,2}(\Omega)\cap H^{2,2}(\Omega)$ it follows that $A:D(A)\to L^2(\Omega)$ is a closed operator in $L^2(\Omega)$ and $-A$ is the infinitesimal generator of an analytic C_0-semigroup on $L^2(\Omega)$ and is expressed as the one-parameter evolution equation $T(t)=e^{At}$, $t\ge 0$, with point spectrum $\sigma_p(A)\subset\sigma(T)$ consisting of isolated eigenvalues of A which satisfy $A\phi=\lambda\phi$ where $\phi(\xi)\in L^2(\Omega)$ are the associated eigenfunctions of $\lambda\in\sigma_p(A)$. However, for the nonautonomous operator $A(t)$, the time dependence of the PDE coefficients can appear in the expression of the eigenvalues and associated eigenfunctions. The construction of the two-parameter semigroup $U(t,s)$ in the following section requires the extension of this general framework to account for the possibility of time-dependent eigenvalues $\{\lambda(t)\}_{t\in[0,T]}$ and the associated family of time dependent eigenfunctions $\{\phi(\xi,t)\}_{t\in[0,T]}$ for the operator $A(t)$.

For each $t\in[0,T]$, the operator $A(t)$ is closed and densely defined and the resolvent $R_\mu(A(t),\mu)=(A(t)-\mu)^{-1}$ is compact, for $\mu\in\rho(A(t))$ where the resolvent set is defined as $\rho(A(t)):=\{\lambda(t)\in\mathbb{C}: \ (A(t)-\lambda(t))$ is one to one $(A(t)-\lambda(t))^{-1}:D\to D$ is bounded$\}$. Since $A(t)$ satisfies the strong elliptic property, for each $t\in[0,T]$ the spectrum of $A(t)$, denoted $\sigma(A(t))$ consists of isolated eigenvalues $\{\lambda_n(t)\}_{n=1}^\infty$ with finite multiplicity and no finite accumulation points, i.e. $\sigma(A(t))$ is discrete for each $t\in[0,T]$ and moreover, the eigenspace associated with a given eigenvalue is finite-dimensional, see [13]. Denote the projection on the n^{th} eigenfunction for each $t\in[0,T]$ as $E_n(\,\cdot\,)=\langle\,\cdot\,,\phi(t)\rangle\phi_n(t)$. The condition C6 is verified by the following where for all $t\in[0,T]$ and $x\in D(A(t))$:

$$\begin{aligned}R_\mu(A(t),\mu)x &= (A(t)-\mu)^{-1}x\\ &= \sum_n(\lambda_n(t)-\mu)^{-1}E_n(x)\\ &\le \max_{\substack{n\\ t\in[0,T]}}(\lambda_n(t)-\mu)^{-1}\sum_n E_n(x)\\ &= \max_{\substack{n\\ t\in[0,T]}}(\lambda_n(t)-\mu)^{-1}x\end{aligned}$$

Then there exist positive constants M and k such that,

$$\|(A(t)-\mu)^{-1}\|\le M(\mu-k)^{-1} \tag{9}$$

It follows that there exists a sector,

$$S_\omega=\{\lambda\in\mathbb{C}:|\text{arg}\lambda|<\frac{\pi}{2}+\omega\}\backslash\{0\}, \quad \omega\in(0,\pi/2]$$

contained in $\rho(A(t))$ and $\sigma(A(t))\subset\mathbb{C}\backslash S_\omega$, which means that $A(t)$ is a *sectorial operator* and is the infinitesimal generator of a family of analytic semigroups on $L^2(\Omega)$ which is expressed as the two-parameter semigroup $U(t,s)$. In the following section, we propose a particular form of this two-parameter system, see [8], [14].

B. Representation of the two-parameter semigroup

Consider the evolution operator $A(t)$ which generates a family of analytic semigroups for each $t\in[0,T]$ and is represented as the two-parameter semigroup $U(t,s)$ by the following Theorem.

Theorem 1: Let $\phi_n(\xi)\in C^\infty(\Omega)$ and $\psi_n(\xi)\in C^\infty(\Omega)$, $n\in\mathbb{N}$, be the set of biorthogonal eigenfunctions corresponding to $A(t)$ and the adjoint, $A^*(t)$, respectively. Allowing the possibility of dependence on time, denote $\phi_n(t):=\phi_n(\xi,t)\in C^1([0,T];\mathcal{X})$ and $\psi_m(t):=\psi_m(\xi,t)\in C^1([0,T];\mathcal{X})$, which at each $t\in[0,T]$ are biorthogonal, where $\mathcal{X}=L^2(\Omega)$. The corresponding set of eigenvalues (possibly time-dependent) are denoted $\lambda_n(t)\in C^1([0,T])$. Let $D(A(t)):=H^{1,2}(\Omega)\cap H^{2,2}(\Omega)$ and consider the operator $A(t):D(A(t))\to L^2(\Omega)$ defined as:

$$A(t):=\sum_{n=1}^\infty\Lambda_n(t)\langle\,\cdot\,,\psi_n\rangle\phi_n \tag{10}$$

with

$$\Lambda_n(t)=\left\{\left(t\frac{d}{dt}\lambda_n+\lambda_n\right)\phi_n+\frac{\partial}{\partial t}\phi_n\right\}\phi_n^{-1} \tag{11}$$

The operator $A(t)$ is the infinitesimal generator of the two-parameter semigroup $U(t,s)$, with $0\le s\le t\le T$, and defined as:

$$U(t,s)x(s):=\sum_{n=1}^\infty e^{\lambda_n(t)t}e^{-\lambda_n(s)s}\langle x(s),\psi_n(s)\rangle\phi_n(t) \tag{12}$$

for $x(s)\in L^2(\Omega)$.

The following Corollary immediately follows Theorem 1.

Corollary 1: The operator $U(t,s)$ satisfies the following properties:

P1. $U(t,t) = I$, $U(t,s) = U(t,r)U(r,s)$ for $0 \leq s \leq r \leq t \leq T$

P2. For $x(s) \in L^2(\Omega)$

$$A(t)U(t,s)x(s) = \sum_{n=1}^{\infty}\left[\left(t\frac{d}{dt}\lambda_n(t) + \lambda_n(t)\right)\phi_n(t) + \frac{\partial}{\partial t}\phi_n(t)\right]\cdot e^{\lambda_n(t)t-\lambda_n(s)s}\langle x(s), \psi_n(s)\rangle = \frac{\partial U(t,s)}{\partial t}x(s)$$

and similarly

$$-U(t,s)A(s) = \frac{\partial U(t,s)}{\partial s}$$

Moreover, provided that C1 and C2 hold, from [8, Theorem 6.1, Chapter 5] we also have the following bounds:

P3. $\|U(t,s)\| \leq L_1$, $\|A(t)U(t,s)\| \leq L_2(t-s)^{-1}$, and $\|A(t)U(t,s)A(s)^{-1}\| \leq L_3$ for constants L_i.

Then by using the Theorem 1, the solution of the homogeneous evolution equation in the Eq.7 is given by the Eq.8 where $U(t,s)$ is the two-parameter semigroup in the Eq.12. Moreover, provided that C3 holds, for every $x_s \in L^2(\Omega)$, $0 \leq s \leq t \leq T$, the solution of the nonhomogenous nonautonomous parabolic evolution system in the Eq.5 has a unique solution $x(t)$ which is given by,

$$x(t) = U(t,s)x_s + \int_s^t U(t,\tau)f(\tau)d\tau \tag{13}$$

Remark 2: The analysis and representation of solutions of the Eq.7 has also been studied in [6] where the nonautonomous linear operator $A(t)$ is considered in the form $A(t) = (A - B(t))$, and where the possibility of time-dependent eigenvalues is treated in a different way from the one presented in this work. We only comment here that the approach in [6] presumes that the nonautonomous operator can be separated into time invariant and time dependent parts and similarly for the eigenvalues $\lambda(t) = \lambda_p + \lambda_c(t)$.

C. Example: Tubular reactor model with time dependent coefficients

Consider the tubular reactor setup depicted in the Fig.1 in the axisymmetric region $\Omega \in \mathbb{R}^2$ with spatial points $\xi \in [0,1]$ and $r \in [0,1]$. The concentration dynamics of a single reactant $x(r,\xi,t)$ are described by the second order PDE

$$\begin{aligned}
&\frac{\partial x}{\partial t} = a(t)\Delta x - b(t)\frac{\partial x}{\partial \xi} - c(t)x + Q(r,\xi,t)\\
&\frac{\partial x}{\partial r}(0,\xi,t) = 0, \quad \frac{\partial x}{\partial r}(1,\xi,t) = 0\\
&\frac{\partial x}{\partial \xi}(r,1,t) = 0, \quad \frac{\partial x}{\partial \xi}(r,1,t) = 0
\end{aligned} \tag{14}$$

with the initial condition $x_0(r,\xi) \in L^2(\Omega)$, where Δ is the Laplacian operator and homogeneous Neumann boundary conditions are assumed at each of $r = 0$, $r = 1$, $\xi = 0$ and $\xi = 1$. The term $Q(r,\xi,t)$ corresponds to, for example, the consumption or generation due to the time-evolution of the reactor temperature, and it is assumed that $Q(r,\xi,t) \in C^1([0,T], L^2(\Omega))$. The time dependent parameters $a(t)$, $b(t)$ and $c(t)$ are assumed to be positive for all $t \in [0,T]$ and of class $C^1([0,T],\mathbb{R})$, satisfy the condition C1, and correspond to the coefficients of diffusion, fluid superficial velocity and reaction kinetics, respectively, see [15], [16]. The associated linear nonautonomous operator is defined as $A(t) = a(t)\Delta - b(t)(\partial/\partial\xi) - c(t)$ with domain $D(A(t)) = H^{1,2}(\Omega) \cap H^{2,2}(\Omega)$. The condition C2 is immediately satisfied since the coefficient $a_2(r,\xi,t) = -a(t)$. It is known that the operator $A(t)$ is the negative of a Sturm-Liouville operator for each $t \in [0,T]$, see [16], [17], and is the infinitesimal generator of a C_0-semigroup for each $t \in [0,T]$, with $\sup_{n\geq 1, t\in[0,T]} \lambda_n(t) < 0$, and $\{0\} \notin \sigma(A(t))$. This implies that the operator $A(t)$ has bounded inverse for each $t \in [0,T]$, i.e. $\|A(t)^{-1}\| \leq L_1$ for constant L_1. Denoting $\|\cdot\|_{m,p}$ the $H^{m,p}$ norm, the condition C6 can be verified by direct calculation:

$$\begin{aligned}
&\left\|(A(t) - A(s))A(\tau)^{-1}x\right\|_{0,2}^2\\
&\leq L_1\left[L_0|t-s|^\beta(\|x\|_{1,2}^2 + \|x\|_{0,2}^2)\right.\\
&+ \left.\|(b(s)-b(t))\nabla x\|_{0,2}^2 + \|(c(t)-c(s))x\|_{0,2}^2\right]\\
&\leq L_2|t-s|^\beta\left[2(\|x\|_{0,2}^2 + 2\|x\|_{1,2}^2\right]\\
&\leq L_3|t-s|^\beta\|x\|_{1,2}^2
\end{aligned}$$

At each $t \in [0,T]$ the eigenvalue problem $A(t)\phi(r,\xi,t) = \lambda(t)\phi(r,\xi,t)$ can be solved which yields the eigenvalues $\{\lambda(t)\}_{t\in[0,T]}$ expressed as

$$\lambda_{mn}(t) = -a(t)\left\{(m\pi)^2 + \beta_n^2\right\} - \frac{b(t)^2}{4a(t)} - c(t) \tag{15}$$

The associated family of time-dependent eigenfunctions $\phi_m^{(1)} \in C^1([0,T], L^2(0,1))$, $m \in \mathbb{N}$, are determined as:

$$\phi_m^{(1)}(\xi,t) = B_m(t)\exp\left(\frac{b(t)}{2a(t)}\xi\right)\cdot\left(\cos(m\pi\xi) - \frac{b(t)}{2a(t)m\pi}\sin(m\pi\xi)\right) \tag{16}$$

The coefficients,

$$B_m(t) = \sqrt{2}\left(1 + \left(\frac{b(t)}{2a(t)m\pi}\right)^2\right)^{-\frac{1}{2}} \tag{17}$$

orthonormalize $\phi_m^{(1)}(\xi,t)$ with respect to the adjoint eigenfunctions $\psi_m^{(1)} \in C^1([0,T], L^2(0,1))$ determined as,

$$\psi_m^{(1)}(\xi,t) = \exp\left(-\frac{b(t)}{a(t)}\xi\right)\phi^{(1)}(\xi,t) \tag{18}$$

which correspond to the adjoint operator $A^*(t) = a(t)\Delta + b(t)(\partial/\partial\xi) - c(t)$ and the eigenvalue problem $A^*(t)\psi = \lambda\psi$. The eigenfunctions in $r \in [0,1]$ for all $t \in [0,T]$ and $n \in \mathbb{N}$

are given in terms of Bessel functions J_p of the first kind and p^{th} order, and are determined as:

$$\phi_n^{(2)}(r) = \frac{\sqrt{2}}{J_0(\beta_n)} J_0(\beta_n r) \tag{19}$$

with adjoints $\psi_n^{(2)}(r) = r\phi_n^{(2)}(r)$. The coefficients $\beta_n = \{0, 3.83, 7.016, 10.173, 13.323, 16.471, \cdots\}$ are the n^{th} zeros of J_1 type Bessel functions which are readily available from appropriate tables for both of $\phi_n^{(2)} \in C^\infty(0,1)$ and the corresponding adjoints $\psi_n^{(2)} \in C^\infty(0,1)$. The eigenvalues in the Eq.15 are real and negative for all $t \in [0,T]$ so that the spectrum $\sigma(A(t))$, which is the same as $\sigma(A^*(t))$, is discrete and lies in the left half-plane of $\mathbb{C}$, $\{0\} \notin \sigma(A(t))$, with one dimensional eigenspaces, and growth bound $\omega_0 \in \mathbb{R}$,

$$\omega_0 = \sup_{\substack{m,n\geq 1 \\ t\in[0,T]}} \text{Re}(\lambda_{mn}(t)) < 0 \tag{20}$$

The operator $A(t)$ is expressed as in the Eq.10, is the infinitesimal generator of a family of stable C_0-semigroups, and expressed as the two-parameter evolution operator in the Eq.12 with $\lambda(t) = \lambda_{mn}(t)$, $\phi(t) = \phi_m^{(1)}(\xi,t)\phi_n^{(2)}(r)$ and $\psi(t) = \psi_m^{(1)}(\xi,t)\psi_n^{(2)}(r)$. Finally, for $Q(r,\xi,t) \in L^2(\Omega)$ and also satisfying the conditions C3, the solution to the PDE system in the Eq.14 is given as the generalized solution:

$$x(t) = U(t,s)x_s + \int_s^t U(t,\tau)Q(\tau)d\tau \tag{21}$$

for $0 \leq s \leq t \leq T$ and $x_s \in L^2(\Omega)$.

IV. Control problem

In this section we consider the control problem:

$$\frac{dx(t)}{dt} = A(t)x(t) + B(t)u(t), \qquad x(0) = x_0 \tag{22}$$

where $A(t) : D(A(t)) \to \mathcal{X}$ is the nonautonomous operator considered in the previous sections on the state space $\mathcal{X} = L^2(\Omega)$. The input term $B(t)u(t)$ corresponds to the function $f(\xi,t) = b(\xi,t)u(t)$ in the Eq.1 and it is assumed that $b(\xi,t) \in C([0,T],\mathcal{X})$ where $B(t) \in \mathcal{L}(\mathcal{U},\mathcal{X})$ is the linear input operator, $\mathcal{U}$ is the input space, and $B(t)u(t)$ is assumed to be continuous for all $u \in \mathcal{U}$.

A. Spectral assignability

The incorporation of the time dependent feature of the eigenvalues into the expression for the two-parameter system $U(t,s)$ has interesting implications in the study of the controllability and observability conditions of the operator $A(t)$ and the system in the Eq.22. In particular, one of the primary requisites in designing a feedback regulator is the ability of the controller to stabilize or augment the stability of the system. Among the controllability criteria is the implied arbitrary spectral assignability by state feedback. The spectral assignability of state feedback systems in the form of the Eq.22 has been considered in several works, see for example [18], [19], [20]. The results therein included the following conditions which have been modified accordingly to reflect the nonautonomous operator $A(t)$ considered in this present work. The conditions for stabilizability of the system in the Eq.22 include:

A1. $A(t)$ is an unbounded spectral operator with discrete spectrum $\sigma(A(t))$ and normalized eigenvectors ϕ_n. The eigenvalues are distinct and the eigenspaces are one-dimensional.

A2. $\inf_{\substack{i\neq j \\ t\in[0,T]}} |\lambda_i(t) - \lambda_j(t)| = \varepsilon > 0$

A3. $\sup_{\substack{j\leq n<\infty \\ t\in[0,T]}} \sum_{\substack{j=1 \\ j\neq n}}^{\infty} \frac{1}{|\lambda_j(t) - \lambda_n(t)|^2} < \infty$

A4. $0 \notin \sigma(A(t)), \quad \inf|\lambda_j(t)| > 0, \quad \sum_{j=1}^{\infty} \frac{1}{|\lambda_j(t)|^2} < \infty$

For unstable systems it is presumed that there is an index set J such that $\text{Re}\lambda_j \geq 0$ for $j \in J$ and $\text{Re}\lambda_j < 0$ for $j \notin J$ which means that there exists only a finite number of unstable modes. In [18] it is mentioned that even if A1-A3 holds and the system is controllable, the closed loop spectrum Γ is not arbitrarily assignable even for a finite number of unstable modes. For example, in the case where both the time dependence and potential instability of the system is only due to the zero order terms of the PDE, the set of unstable modes may be time dependent which appears in the expression of the eigenvalues. Then the spectrum $\sigma(A(t))$ consists of eigenvalues with stable and possible unstable parts $-\lambda_p + \lambda_u(t)$ which may be stable or unstable overall (finite or infinite) depending on the evolution of the generation term. With this in mind, we note that the example system presented in the Section III-C with $-c(t)x < 0$ and $Q(r,\xi,t) = 0$ (dissipative case) satisfies A1-A4 at each $t \in [0,T]$ which suggests the controllability criterion is satisfied. However, formal results on the general controllability and observability criteria in terms of the two-parameter semigroup $U(t,s)$ determined in this work remains an objective of future study.

B. Optimal control formulation

The operator $A(t)$ generates a family of C_0-semigroups in $L^2(\Omega)$ for all $t \in [0,T]$ and from Theorem 1 and Corollary 1, generates the two-parameter evolution operator $U(t,s)$ which satisfies the properties P1-P3. Then, the Eq.22 has a unique (mild) solution given by,

$$x(t) = U(t,0)x_0 + \int_0^t U(t,s)Bu(s)ds \tag{23}$$

for $x_0 \in L^2(\Omega)$ with $0 \leq s \leq t \leq T$, with $B \in C([0,T],\mathcal{L}(\mathcal{U},\mathcal{X}))$, see [21]. The standard optimal control formulation considers the quadratic minimization problem,

$$\begin{aligned} &\min_u J(u(t)) := \\ &\min_u \int_0^T \left(|Cx(s)|^2 + |Ru(s)|^2\right) d\tau + \langle Qx(T), x(T)\rangle \end{aligned} \tag{24}$$

over all inputs $u \in L^2([0,T],\mathcal{U})$ subject to the differential constraint of the Eq.22. The operator $Q \in \mathcal{L}(\mathcal{X})$ is self-adjoint and nonnegative and $R \in \mathcal{L}(\mathcal{U})$ is coercive, where

Q is the state weight operator, R is the input penalty operator, $C \in C([0,T], \mathcal{L}(\mathcal{X}, \mathcal{Y}))$ is the output measurement operator. Since $A(t)$ generates a C_0-semigroup on $L^2(\Omega)$ for all $t \in [0,T]$ which gives the state evolution in the Eq.23, the optimization problem in the Eq.24 has the unique minimizing solution $u_{\min}(t) \in C([0,T], \mathbb{R})$ given by the feedback formula,

$$u_{\min}(t) = -R^{-1}B^*\Pi(t)x_{\min}(t) \tag{25}$$

with the optimal cost related to the initial state as $J(u_{\min}) = \langle \Pi(0)x_0, x_0\rangle$. The operator $\Pi(t) \in \mathcal{L}(\mathcal{X})$ is the strongly continuous, self adjoint, nonegative solution of the differential Riccati equation,

$$\begin{aligned}\frac{d}{dt}\Pi(t) + A^*(t)\Pi(t) + \Pi(t)A(t)\\ - \Pi(t)BR^{-1}B^*\Pi(t) + C^*(t)C(t) = 0\end{aligned} \tag{26}$$

with $\Pi(T) = Q$. The mild form expression of the Eq.26 is given in terms of $U(t,s)$ and $x \in L^2(\Omega)$ as:

$$\begin{aligned}&\Pi(t)x = U^*(T,t)QU(T,t)x\\ &+ \int_t^T U^*(\tau,t)C^*(\tau)C(\tau)U(\tau,t)x d\tau\\ &- \int_t^T U^*(\tau,t)\Pi(\tau)B(\tau)R^{-1}B^*(\tau)\Pi(\tau)U(\tau,t)x d\tau\end{aligned}$$

The adjoint of $U(t,s)$ is denoted $U^*(t,s)$ which corresponds to the operator:

$$A^*(t) := \sum_{n=1}^{\infty} \Lambda_n^*(t)\langle \cdot, \phi_n\rangle\psi_n \tag{27}$$

with

$$\Lambda_n^*(t) = \left\{ \left(t\frac{d}{dt}\lambda_n + \lambda_n \right)\psi_n + \frac{\partial}{\partial t}\psi_n \right\}\psi_n^{-1} \tag{28}$$

such that $A^*(t)$ is the infinitesimal generator of the two-parameter semigroup $U^*(t,s)$, with $0 \le s \le t \le T$, and defined as:

$$U^*(t,s)x(s) := \sum_{n=1}^{\infty} e^{\lambda_n(t)t}e^{-\lambda_n(s)s}\langle x(s), \phi_n(s)\rangle\psi_n(t) \tag{29}$$

In this case that $\langle A(t)\phi, \psi\rangle = \langle \phi, A(t)\psi\rangle$, the operator $A(t)$ is self adjoint and $U(t,s) = U^*(t,s)$.

V. SUMMARY

In this paper, the properties which are necessary in representing a class of parabolic PDE with time and spatially dependent coefficients as a linear nonautonomous evolution equation were presented. The solution of this equation is given in terms of a two-parameter semigroup $U(t,s)$ and the explicit expression for this operator was presented along with some of the key properties.

An illustrative example of a nonhomogeneous tubular reactor system with time dependent diffusivity, superficial velocity and reaction coefficients was provided. The optimal control problem for the linear nonautonomous parabolic system was discussed and the linear quadratic regulator formulation which requires the solution of the time-dependent differential Riccati equation was expressed in terms of the two-parameter semigroup $U(t,s)$.

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