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Matrix Norm Approach for Control of Linear Time-Delay Systems*

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Abstract— **The aim of this paper is to present some results on the control synthesis of time-delay linear systems. Our objective is to find linear controllers able to increase the first stability window imposing that the delay-free system is stable. Our method treats time-delay systems control design with numeric routines based on Linear Matrix Inequalities (LMI) arisen from classical linear time invariant system theory coupled together with a unidimensional search. The proposed algorithm is simple, efficient and easy to be numerically implemented. Some examples illustrating state and output feedback design are solved and discussed in order to shed the light on the most relevant characteristic of the theoretical results. The paper ends with some discussion on further theoretical extensions of the proposed methodology.**

Index Terms— **Time-delay Systems, Linear Systems, State Feedback Design, Output Feedback Design.**

I. INTRODUCTION

Most dynamical systems present delays in their inner structure [1], due to phenomena as, for example, transport, propagation or communication, but, most of the time, for the sake of simplicity, they are ignored. On the other hand, those delays can be the cause of bad performance or even instability, and therefore, in order to proper analyze and design controllers for such systems, it is mandatory to take into account their effects. Another important source of delay is the feedback loop itself, with this delay induced by the sensors, actuators and, in more modern digital controllers, the time of calculation. Finally, among the recent applications, we cite networked control systems [2].

Starting from the studies of [3] and [4], many results regarding the analysis and control of such systems have been achieved, specially over the last decades, as it is discussed, among many others, in the books [5], [6], [7], the survey paper [8], and references therein.

When dealing with any dynamical systems, one of the basic questions we need to answer concerns the stability. For systems with delay, we can go even further, and be interested on this property as a function of the delay itself [9]. It is well known that the phenomenon of stability windows might happen, meaning that the system can lose and recover its stability when we start to increase the numerical value of the delay.

The numerical determination of the limits of stability of a linear time-delay system with respect to the delay is a well known research subject [10]. In the case of commensurate delays, all the stability windows can be directly and reliably obtained with a small computational effort [11], [12], [13], whereas in the non-commensurate case, although still feasible, the determination even of only the first window is much more involved [14], [15].

Our goal is to present explicit delay-dependent design procedures for state and output feedback control design. We end to provide a controller able to increase as much as possible $\tau^\star$ such that the closed-loop system remains stable for any $\tau \in [0, \tau^\star)$. To this purpose, we will rely on some properties that relates the norm of the matrices appearing in the state-space description, the position where the crossings of poles over the imaginary axis appear and the rate of displacement of those poles as a function of an increasing delay.

To the best of the authors' knowledge, such an approach is completely new and it allows to better exploit the intrinsic features of the roots of the corresponding characteristic equation. Furthermore, the numerical design procedure is entirely based on the solution of LMIs together with a line search.

The paper is organized as follows. Section II is devoted to the problem statement and the presentation of some existing results that will be used in sequel. Section III brings a numerical procedure for the state feedback, whereas in section IV we are concerned with the output feedback design problem. Each section presents some academic examples to put in evidence the more relevant aspects of the results. The paper ends with a conclusion in Section V where the main contributions are briefly discussed.

The notation used throughout the paper is standard. More precisely, capital letters denote matrices and small letters denote vectors. For scalars, small Greek letters are used. $\mathbb{R}(\mathbb{R}_+)$ and $\mathbb{C}$ are the set of real (positive real) and complex numbers, and $\jmath = \sqrt{-1}$ is the imaginary unit. For complex scalars, $\Re(\cdot)$ and $\Im(\cdot)$ denote real and imaginary parts, respectively. For complex vector $x \in \mathbb{C}^n$, $x^\sim$ indicates its conjugate transpose. For real matrices or vectors $(')$ indicates transposition. The spectrum of a complex matrix Q is denoted by $\sigma(Q)$ with each eigenvalue being denoted by $\sigma_i(Q)$, and its spectrum radius is given by $\rho(Q) = \max_i |\sigma_i(Q)|$. The identity matrix of any dimension is denoted simply by I. Finally, for partitioned Hermitian matrices, the symbol $(\bullet)$ denotes generically each of its symmetric blocks.

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II. Problem Statement

In this section, we consider linear time-delay systems of retarded type described by the following delay differential equation

$$\dot{x}(t) = \sum_{i=0}^{N} A_i x(t-\tau_i), \tag{1}$$

where $x(t) \in \mathbb{R}^n$ is the state variable, $A_i \in \mathbb{R}^{n\times n}$ for all $i \in \{0, 1, \ldots, N\}$ are real matrices and $0 = \tau_0 < \tau_1 < \tau_2 < \ldots < \tau_N$ are the delays. It is important to stress that $\tau_0 = 0$ making the time delay model of general form.

It is clear that system (1) is exponentially stable if and only if all roots of its characteristic equation:

$$\det \Delta(s) = 0, \tag{2}$$

where $\Delta : \mathbb{C} \to \mathbb{C}^{n\times n}$,

$$\Delta(s) = sI_n - \sum_{i=0}^{N} A_i e^{-s\tau_i}, \tag{3}$$

are in the open left half-plane [4]. The characteristic equation of system (1) given by (2) can be rewritten as

$$C(s, \overline{\tau}_1, \ldots, \overline{\tau}_H) = p(s) + \sum_{i=1}^{H} q_i(s) e^{-s\overline{\tau}_i} = 0; \tag{4}$$

where $p(s)$, $q_i(s)$ are polynomials with real coefficients, $H \leq n * N$ and $\overline{\tau}_i$ for all $i \in \{1, \ldots, H\}$ depend on A_i and τ_i for all $i \in \{0, \ldots, N\}$ in a complicated but well-known way.

If $(s, \overline{\tau}_1, \ldots, \overline{\tau}_H)$ is a simple root of (4), then a small perturbation on one of the elements of the delay, $\overline{\tau}_j \leftarrow \overline{\tau}_j + \epsilon$ keeping all the other delays the same, will provide a solution of (4) with the form $(s^\star, \overline{\tau}_1, \ldots, \overline{\tau}_j + \epsilon, \ldots, \overline{\tau}_H)$, where

$$s^\star = s + \sum_{k=1}^{\infty} \alpha_k \epsilon^k \tag{5}$$

and

$$\alpha_1 = s \frac{q_j(s) e^{-s\tau_j}}{p'(s) + \sum_{i=1}^{H} (q_i'(s) - \tau_i q_i(s)) e^{-s\tau_i}}. \tag{6}$$

Here, $p'(s)$ and $q_i'(s)$ denote the derivative of the polynomials $p(s)$ and $q_i(s)$ with respect to s, respectively.

Even if we consider a small change in an element τ_j of the original equation (1), the important point is that we still have the complex scalar variable "s" which factors out of the expression. This implies that, when crossing through the imaginary axis, the roots closer to the real axis will tend to move less when we increase some of the elements of the delay, even though certainly, in some cases, the other terms might change this characteristic.

The next proposition, stated in [16], provides an envelope curve around the roots of the characteristic equation (2) that will be essential to our purposes. As it will be seen afterwards, it is easy to calculate and provides precise results whenever the state variable $x(t) \in \mathbb{R}^n$ of the time-delay system has relatively small dimension.

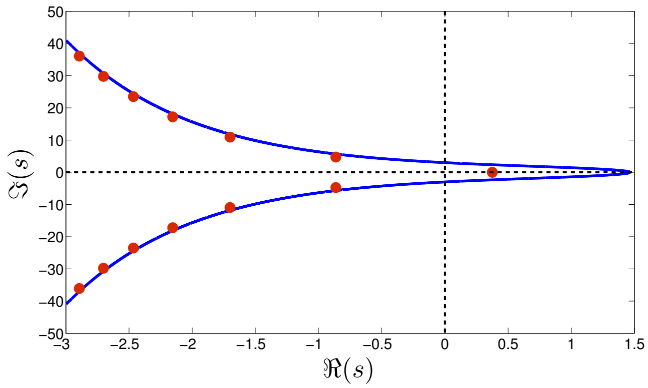

Fig. 1. Envelop and zeros of $\Delta(s) = s + 1 - 2e^{-s}$

Proposition 1 *If s_0 is a characteristic root of the system (1), then it satisfies*

$$|s_0| \leq \sum_{i=0}^{N} \|A_i\|_2 e^{-\Re(s_0)\tau_i}. \tag{7}$$

Proof: We follow the steps of [16]. The expression $\Delta(s_0) = 0$ is alternatively expressed as

$$s_0 \in \sigma\left(\sum_{i=0}^{N} A_i e^{-s_0\tau_i}\right). \tag{8}$$

But since the spectral radius of a matrix is the infimum of all its induced norms, it follows that

$$\begin{aligned} |s_0| &\leq \left\| \sum_{i=0}^{N} A_i e^{-s_0\tau_i} \right\|_2 \\ &\leq \sum_{i=0}^{N} \|A_i\|_2 |e^{-s_0\tau_i}|, \end{aligned} \tag{9}$$

from where (7) follows directly. ■

In the following example we will illustrate this key result of reference [16], which bounds every characteristic root inside the envelope given in (7).

Example 1 *Let us consider the scalar system*

$$\dot{x}(t) = -x(t) + 2x(t-1). \tag{10}$$

Its characteristic equation is given by $\Delta(s) = s + 1 - 2e^{-s}$ which yields $p(s) = s+1$, $q_1(s) = -2$ and $\overline{\tau}_1 = 1$. Applying (7) with equality, we obtain the envelop given by the solid line in Figure 1, whereas the dots show the exact location of the poles of system (1) calculated by the QPmR algorithm, see [15]. As it can be seen, the result is remarkably precise in this particular case.

Other envelops have been considered in [17] and [18], which could provide less conservative results. But, in our case, if we consider a root in the imaginary axis, that is

$s_0 = \jmath\omega_0$, equation (7) provides the upper bound

$$|\omega_0| \leq \sum_{i=0}^{N} \|A_i\|_2, \tag{11}$$

meaning that the interval over the imaginary axis where we can find a root of the characteristic equation is independent of the numerical value of the delay.

The next results will show how the 2-norm can be changed for any induced matrix Q-norm

$$\|A\|_Q^2 = \max_i \sigma_i(A'QA) \tag{12}$$

where $Q = Q' \in \mathbb{R}^{n\times n}$ is an arbitrary positive definite real matrix.

Proposition 2 *Let $Q = Q' > 0 \in \mathbb{R}^{n\times n}$ be given. If $\mu \in \mathbb{R}_+$ is such that*

$$\mu Q > A'QA \tag{13}$$

then $\rho(A) < \sqrt{\mu}$.

Proof: Multiplying the inequality (13) by any eigenvector $x_\ell \in \mathbb{C}^n$ associated to some eigenvalue $\sigma_\ell(A)$ through the right and its conjugate transpose $x_\ell^\sim$ through the left we see that the inequality

$$\mu x_\ell^\sim Q x_\ell > |\sigma_\ell(A)|^2 x_\ell^\sim Q x_\ell \tag{14}$$

holds for all $\ell = 1, \ldots, n$ proving thus the condition on the spectral radius $\rho(A)$. ■

Lemma 1 *Let $Q = Q' > 0 \in \mathbb{R}^{n\times n}$ be given. If $\mu \in \mathbb{R}_+$ is such that*

$$\mu Q > \sum_{i=0}^{N} A_i'QA_i \tag{15}$$

then any solution of the characteristic equation (2) of system (1) on the imaginary axis $s = \jmath\omega$ satisfies $|\omega| \leq \sqrt{\mu(N+1)}$.

Proof: We start by noting that for any $i \in \{0, \ldots, N\}$, the relation

$$\begin{bmatrix} A_i'QA_i & A_i'e^{\jmath\omega\tau_i} \\ \bullet & Q^{-1} \end{bmatrix} \geq 0 \tag{16}$$

is always verified by application of the Schur Complement with respect to the second column and row. We can sum up all the $N+1$ terms to achieve

$$\begin{bmatrix} \sum_{i=0}^{N} A_i'QA_i & \sum_{i=0}^{N} A_i'e^{\jmath\omega\tau_i} \\ \bullet & (N+1)Q^{-1} \end{bmatrix} \geq 0 \tag{17}$$

where we recall that $\tau_0 = 0$ in order to easy the notation. Applying again the Schur Complement with respect the second column and row, we obtain

$$\sum_{i=0}^{N} A_i'QA_i \geq \frac{1}{N+1}\left(\sum_{i=0}^{N} A_i'e^{\jmath\omega\tau_i}\right) Q \left(\sum_{i=0}^{N} A_i e^{-\jmath\omega\tau_i}\right) \tag{18}$$

and therefore, using (15)

$$(N+1)\mu Q > \left(\sum_{i=0}^{N} A_i'e^{\jmath\omega\tau_i}\right) Q \left(\sum_{i=0}^{N} A_i e^{-\jmath\omega\tau_i}\right). \tag{19}$$

Now we take into account that since $\jmath\omega$ is a characteristic root of the system, it is an eigenvalue of the complex matrix $\sum_{i=0}^{N} A_i e^{-\jmath\omega\tau_i}$, as is has been shown in (8). Therefore, considering the eigenvector x of that matrix related to the eigenvalue $\jmath\omega$, we have

$$\jmath\omega x = \left(\sum_{i=0}^{N} A_i e^{-\jmath\omega\tau_i}\right) x. \tag{20}$$

From this point we multiply both sides through the left by $Q^{1/2}$ and calculate the square 2-norm of both vectors, resulting in

$$|\omega|^2 x^\sim Q x = x^\sim \left(\sum_{i=0}^{N} A_i'e^{\jmath\omega\tau_i}\right) Q \left(\sum_{i=0}^{N} A_i e^{-\jmath\omega\tau_i}\right) x. \tag{21}$$

Using (19) this implies that

$$|\omega|^2 x^\sim Q x < (N+1)\mu x^\sim Q x \tag{22}$$

allowing us to complete the proof. ■

So, our strategy to increase the limit of stability is two-fold. At the same time that we will minimize the left side of (15) in order to limit the crossing frequencies, and therefore their velocities, we will try to place the poles for the delay-free system $\dot{x}(t) = (\sum_{i=0}^{N} A_i)x(t)$ further away from the imaginary axis. For that, we will do an iterative process, where we will increase the minimum distance from the imaginary axis of the poles of the delay-free system and minimize the corresponding induced Q-norm of the state-space matrices. For each point of the iterative process, we can compute explicitly the limit of stability.

III. State feedback design

We now move our attention to the time delay system

$$\dot{x}(t) = \sum_{i=0}^{N} A_i x(t-\tau_i) + Bu(t), \tag{23}$$

which is controlled by means of a state feedback control law $u(t) = \sum_{i=0}^{N} K_i x(t-\tau_i) \in \mathbb{R}^m$ to be designed. Connecting it to (23), we get a closed-loop system state space realization of the form

$$\dot{x}(t) = \sum_{i=0}^{N} (A_i + BK_i)x(t-\tau_i) \tag{24}$$

and we will try to maximize the first delay interval where the system remains stable. For the commensurate case, the maximum delay can be easily defined, whereas for the non-commensurate one, we will consider that the delays are homogeneously modified accordingly to $\tau_i = \gamma T_i$ where $T_i > 0$ for all $i \in \{0, \ldots, N\}$ and our goal is to maximize the positive scalar γ.

The first part of the procedure involves imposing some desired location of the poles for the closed-loop delay-free system

$$\dot{x}(t) = \left(\sum_{i=0}^{N} A_i + BK_i\right) x(t). \tag{25}$$

We recall that all eigenvalues $\sigma_i(A)$ of a matrix A are such that $\Re(\sigma_i(A)) < -\alpha$, $\alpha > 0$, if and only if there exists a matrix $Q = Q' > 0$ satisfying the inequality, see [19]

$$A'Q + QA < -2\alpha Q. \tag{26}$$

Hence, multiplying (26) by $P = Q^{-1}$ to the left and to the right, the condition for the closed-loop delay-free system (25) to have all its poles in the region $\Re(s) < -\alpha$ can be assured by the existence of matrices $P = P' > 0$ and Y_i for all $i \in \{0, \dots, N\}$ such that

$$\mathcal{A}P + P\mathcal{A}' + B\mathcal{Y} + \mathcal{Y}'B' < -2\alpha P \tag{27}$$

where $\mathcal{A} = \sum_{i=0}^{N} A_i$ and $\mathcal{Y} = \sum_{i=0}^{N} Y_i$. In the affirmative case, the state-feedback gains can be obtained from the simple formulas

$$K_i = Y_i P^{-1} \tag{28}$$

for all $i = \{0, \dots, N\}$. We notice that whenever $\alpha > 0$ is fixed, (27) is a linear matrix inequality with respect to all involved matrix variables, that is $P > 0$ and $\mathcal{Y}$.

Now we move to the second part of the algorithm. From this point, we will try to minimize the right hand-side of the relation

$$|\omega| \leq \sqrt{(N+1)\mu} \tag{29}$$

To this end, applying the Schur complement in (15) with A_i replaced by the closed-loop matrices $A_i + BK_i$ for all $i \in \{0, \dots, N\}$ and multiplying each row and column by $P = Q^{-1}$ we get

$$\begin{bmatrix} \mu P & PA_0' + Y_0'B' & \dots & PA_N' + Y_N'B' \\ \bullet & P & \dots & 0 \\ \bullet & \bullet & \dots & 0 \\ \bullet & \bullet & \bullet & P \end{bmatrix} > 0. \tag{30}$$

Therefore, even though we cannot minimize directly the upper bound (11) using LMIs, this is possible for the upper bound indicated in (29) a fact that puts in evidence its intrinsic theoretical importance. Notice that the minimization with respect to the positive scalar $\mu > 0$ is done by unidimensional search.

In the end of the process, if it was successful to find any solution, we achieve that the closed-loop delay-free system is stable and that any crossing of poles through the imaginary axis will happen at $|\omega| \leq \sqrt{(N+1)\mu}$, and therefore, possibly with a low speed. Repeating this strategy for different values of $\alpha > 0$, and for each one of those calculating the maximum value of the delay which guarantees the stability of the system, completes the strategy.

Hence, the procedure can be summarized by the determination of the function

$$\mu(\alpha) = \inf_{\mu, P, Y_0, \dots, Y_N} \{\mu \ : \ (27), (30)\} \tag{31}$$

for increasing $\alpha > 0$ and calculating the associated value of the maximum first interval delay preserving closed-loop stability $\tau(\alpha)$. The existence of $\tau(\alpha) > 0$ for some $\alpha > 0$ requires that the delay-free system be stabilizable.

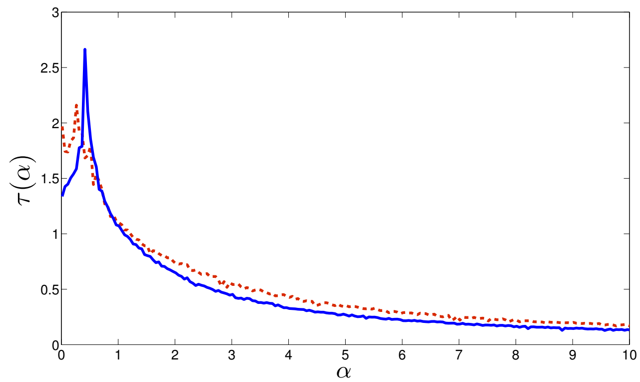

Fig. 2. Maximum stability margin for closed-loop system (32).

Remark 1 *Notice that some restrictions on the controller can be easily applied. For example, if we are searching for a finite dimension linear controller of the form $u(t) = K_0 x(t)$, initially we have only to impose $Y_i = 0$ for all $i \in \{1, \dots, N\}$ and apply the same procedure.*

Example 2 *To illustrate the results obtained until now, let us consider a second-order example borrowed from [20], where the matrices corresponding to the state space realization (23) are as follows:*

$$\left[A_0 \mid A_1 \mid B \right] = \left[\begin{array}{cc|cc|c} 0 & 0 & -1 & -1 & 0 \\ 0 & 1 & 0 & -0.9 & 1 \end{array} \right]. \tag{32}$$

We notice that although the pair (A_0, B) is not controllable, the pair $(A_0 + A_1, B)$ is, which is our requirement for the applicability of the procedure.

Figure 2 shows in solid line the case $u(t) = K_0 x(t) + K_1 x(t-\tau)$ and in dashed line the memoryless case, for which we have imposed $K_1 = 0$. For the first case, the maximum interval such that the system is asymptotic stable we were able to get was $\tau \in [0, 2.6644)$, whereas in the second case it was $\tau \in [0, 2.1605)$. The corresponding gains, for the first case, were

$$\begin{aligned} K_0 &= \begin{bmatrix} -0.2540 & -2.0267 \end{bmatrix}, \\ K_1 &= \begin{bmatrix} -0.2540 & -0.1267 \end{bmatrix}. \end{aligned}$$

and, for the second case (memoryless controller), the procedure provided the gain

$$K_0 = \begin{bmatrix} -0.3148 & -1.7284 \end{bmatrix}.$$

It is interesting to mention that, in [20], considering only asymptotic stability which is possible to be obtained by setting the $\mathcal{H}_\infty$-level large enough, the maximum value assuring the existence of a state feedback stabilizing controller is $\tau_{max} = 1.28$. Also, as expected, in Figure 2, the maximum value of the delay provided by the solid curve is clearly greater than the one provided by the dashed one.

IV. Output Feedback Design

We now turn our attention to the design of a dynamic full-order output feedback controller to time-delay systems whose state space realization is given by

$$\dot{x}(t) = \sum_{i=0}^{N} A_i x(t-\tau_i) + Bu(t) \quad (33)$$

$$y(t) = \sum_{i=0}^{N} C_{yi} x(t-\tau_i) \quad (34)$$

where, in addition to the assumptions and the variables defined in the previous section, $y(t) \in \mathbb{R}^p$ is the measured signal. Hence, in this section we design a full-order dynamic output feedback controller with the following structure

$$\dot{\hat{x}}(t) = \sum_{i=0}^{N} \hat{A}_i \hat{x}(t-\tau_i) + \hat{B}y(t) \quad (35)$$

$$u(t) = \sum_{i=0}^{N} \hat{C}_i \hat{x}(t-\tau_i) + \hat{D}y(t) \quad (36)$$

where $\hat{x}(t) \in \mathbb{R}^n$. When connected to (33)-(34) the closed-loop system rewrites as:

$$\dot{\xi}(t) = \sum_{i=0}^{N} F_i \xi(t-\tau_i) \quad (37)$$

where $\xi(t) = [x(t)' \ \hat{x}(t)']' \in \mathbb{R}^{2n}$ is the state variable and the indicated matrices stand for

$$F_i = \begin{bmatrix} A_i + B\hat{D}_c C_{yi} & B\hat{C}_i \\ \hat{B}_i C_{yi} & \hat{A}_i \end{bmatrix} \quad (38)$$

for all $i \in \{0, \ldots, N\}$.

As in the state feedback case, we first consider the delay-free problem. For that, we need to find a matrix $P = P' > 0$, with P and its inverse partitioned as [21]

$$P = \begin{bmatrix} X & U \\ \bullet & \hat{X} \end{bmatrix}, \qquad P^{-1} = \begin{bmatrix} Y & V \\ \bullet & \hat{Y} \end{bmatrix} \quad (39)$$

such that $\mathcal{F}P + P\mathcal{F}' < -2\alpha P$, with $\mathcal{F} = \sum_{i=0}^{N} F_i$. In order to linearize this constraint, we introduce the following nonsingular matrix

$$T = \begin{bmatrix} I & Y \\ 0 & V' \end{bmatrix}, \quad (40)$$

and multiply the mentioned constraint by T to the right and T' to the left, arriving at $T'\mathcal{F}PT + T'P\mathcal{F}'T < -2\alpha T'PT$, and we notice that

$$T'PT = \begin{bmatrix} X & I \\ \bullet & Y \end{bmatrix}. \quad (41)$$

Also, considering the traditional one-to-one change of variables, see [21] for details

$$\begin{bmatrix} M_i & F \\ L_i & R \end{bmatrix} = \begin{bmatrix} V & YB \\ 0 & I \end{bmatrix} \hat{K}_{ci} \begin{bmatrix} U' & 0 \\ C_{yi}X & I \end{bmatrix} + \begin{bmatrix} Y \\ 0 \end{bmatrix} A_i \begin{bmatrix} X & 0 \end{bmatrix} \quad (42)$$

where

$$\hat{K}_{ci} = \begin{bmatrix} \hat{A}_i & \hat{B} \\ \hat{C}_i & \hat{D} \end{bmatrix} \quad (43)$$

for all $i \in \{0, \ldots, N\}$, we get

$$T'\mathcal{F}PT = \begin{bmatrix} \mathcal{A}X + B\mathcal{L} & \mathcal{A} + BR\mathcal{C}_y \\ \mathcal{M} & Y\mathcal{A} + F\mathcal{C}_y \end{bmatrix} \quad (44)$$

where $\mathcal{M} = \sum_{i=0}^{N} M_i$, $\mathcal{L} = \sum_{i=0}^{N} L_i$ and $\mathcal{C}_y = \sum_{i=0}^{N} C_{yi}$. Continuing as we did in the second part of the state feedback control design, we see that, applying the Schur complement and multiplying each row and column of (15) by $T'P$ and PT respectively, we achieve

$$\begin{bmatrix} \mu T'PT & T'PF_0'T & \ldots & T'PF_N'T \\ \bullet & T'PT & \ldots & 0 \\ \bullet & \bullet & \ldots & 0 \\ \bullet & \bullet & \bullet & T'PT \end{bmatrix} > 0. \quad (45)$$

where the terms $T'PF_i'T$ for all $i \in \{0, \ldots, N\}$ have the same form as $T'P\mathcal{F}'T$ given in (44), and to get the exact expression we only need to substitute $\mathcal{A}$, $\mathcal{M}$, $\mathcal{L}$ and $\mathcal{C}_y$ by A_i, M_i, L_i and C_{yi}, respectively. As before, we have to handle two coupled LMIs in order to minimize μ for a given $\alpha > 0$.

One might notice that no choice about U or V was made. Indeed, in order to recover the state-space realization matrices of the controller (35)-(36), one of those matrices must be imposed and the other calculated from the relation $XY + UV' = I$. But if we rewrite equation (42) as

$$\begin{bmatrix} \hat{A}_i & \hat{B} \\ \hat{C}_i & \hat{D} \end{bmatrix} = \begin{bmatrix} V^{-1} & 0 \\ 0 & I \end{bmatrix} S_i \begin{bmatrix} (U')^{-1} & 0 \\ 0 & I \end{bmatrix} \quad (46)$$

where the matrix S_i

$$S_i = \begin{bmatrix} S_{1i} & S_2 \\ S_{3i} & S_4 \end{bmatrix} = \begin{bmatrix} S_{1i} & F - YBR \\ L_i - C_{yi}X & R \end{bmatrix} \quad (47)$$

with $S_{1i} = M_i - YA_iX - S_2C_{yi}X - YBL_i$, we can immediately see that S_i does not depends on V and U. On the other hand, if we consider the controller transfer function given by

$$\begin{aligned} C(s) &= \sum_{i=0}^{N} \hat{C}_{yi} e^{-s\tau_i} \left(sI - \sum_{i=0}^{N} \hat{A}_i e^{-s\tau_i} \right)^{-1} \hat{B} + \hat{D} \\ &= \sum_{i=0}^{N} S_{3i} e^{-s\tau_i} \left(sVU' - \sum_{i=0}^{N} S_{1i} e^{-s\tau_i} \right)^{-1} S_2 + S_4 \\ &= \sum_{i=0}^{N} S_{3i} e^{-s\tau_i} \left(s(I - YX) - \sum_{i=0}^{N} S_{1i} e^{-s\tau_i} \right)^{-1} S_2 + S_4 \end{aligned}$$

which does not depend on V and U and not even on $\hat{X}$ or $\hat{Y}$. This implies that, as long as the choice of V or U be an invertible matrix, with the other being calculated by the relation $XY + UV' = I$, it can be made arbitrary. In our numerical example, we have chosen $U = -X$. Notice that the minimization of $\mu > 0$ subject to all involved constraints can be done by any LMI solver coupled together with a line

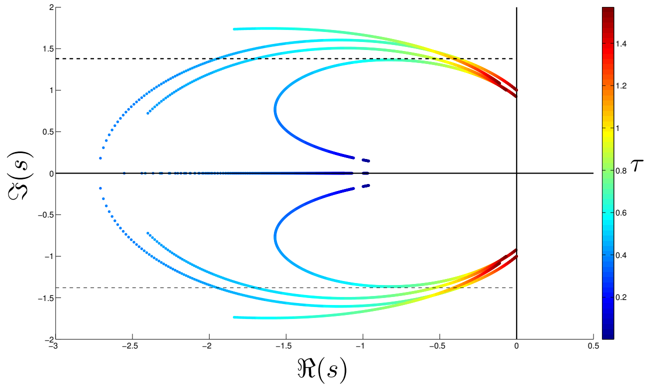

Fig. 3. Root-loci of the closed-loop system.

search.

Example 3 *Let us now consider the same second-order example from [20] in order to illustrate the output feedback control design. The time-delay system (33)-(34) matrices are the same as in the previous example and*

$$\left[\, C_{y0} \mid C_{y1} \,\right] = \left[\, 0 \quad 1 \mid 0 \quad 0 \,\right].$$

For this system we have applied the proposed algorithm to generate a sequence of stabilizing controllers imposing that the delay-free system had poles further away of the imaginary axis and minimizing t in (45) in order to limit the frequency of crossings. The best result was obtained for $\alpha = 0.9620$*, where the associated controller matrices were given by*

$$\hat{A}_0 = \begin{bmatrix} 0 & 1.0805 \\ -0.0097 & -0.6673 \end{bmatrix},\ \hat{A}_1 = \begin{bmatrix} -1 & -1.0004 \\ 0.0327 & -0.8061 \end{bmatrix}$$

$$\hat{C}_0 = \begin{bmatrix} 0.0097 & 0.5997 \end{bmatrix},\ \hat{C}_1 = \begin{bmatrix} -0.0327 & -0.0939 \end{bmatrix}$$

$$\hat{B} = \begin{bmatrix} 1.0805 \\ -0.5494 \end{bmatrix},\ \hat{D} = -0.5182$$

and stability is guaranteed for all $\tau \in [0, 1.5708)$*. In [20], the maximum delay obtained reported was* $\tau_{max} = 1.28$*, which implies that our procedure provides to an improvement of more than* 20%*.*

Figure 3 shows the root-loci of the closed-loop system with respect to $\tau \in [0, 1.5708)$*. It is interesting to observe that the upper bound in (29) given by* $|\omega| \leq 1.3802$ *is represented in the same figure by the two dashed lines. It is apparent that the* ω*-crossing is approximately* $\omega \approx 1$ *indicating a good precision of the proposed method.*

V. Conclusions

In this paper, we have proposed a new design procedure for time-delay state and output feedback design. It is based on the norms of the matrices of the state-space realization, since it was shown how these norms can affect the rate of displacement of the poles around the imaginary axis as we increase the numerical value of the delay.

One important feature of this method is that it makes possible the control design by manipulating classical LMIs coupled together with a line search. As a consequence, it opens the possibility to handle time-delay systems with high number of state variables and number of delays. Some simple examples illustrate the theoretical results. There are problems still open that deserve future research efforts, as for instance the use of some less conservative envelops and the generalization of the procedure to neutral systems.

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