

An Input-Output approach to the robust synchronization of dynamical systems with an application to the Hindmarsh-Rose neuronal model

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Abstract—Motivated by a deeper understanding of the mechanisms involved in neuronal synchronization, we extend an input/output approach recently proposed to analyze networks of nonlinear dynamical operators defined in the extended L_2 space. This extension allows to cover a wider class of systems, by tolerating some heterogeneity among the operators involved. We apply this result to a network of heterogeneous Hindmarsh-Rose neurons and provide an analytical justification of rather counter-intuitive synchronization phenomena observed in simulation.

I. INTRODUCTION

Synchronization in a network of agents can be interpreted as the appearance of a correlated behavior among its constituting dynamical systems. It finds applications in many physical, engineering, medical, and biological fields. The problem of finding sufficient conditions under which synchronization can be guaranteed is particularly challenging when the components of the network are heterogeneous. Nonetheless, such a heterogeneity is common in many biological applications, in particular in the study of neuronal synchronization. Recently, a promising method has been developed [17] to provide explicit conditions on the agents' dynamics and on their interconnection topology for a network of identical systems to synchronize. This approach relies on the input/output properties of the agents involved, and thus requires little knowledge on the individual dynamics. This feature is of particular interest for neuronal synchronization, in which parameter identification is often hard to achieve in a precise manner. However, this result imposes for the time-being that all agents composing the network have the same dynamics, which constitutes a restrictive constraint in view of the typical heterogeneities between neuronal cells. The aim of this paper is therefore to extend this method to make it cope some heterogeneity between the agents and to apply it to a population of heterogeneous Hindmarsh-Rose neuronal models [8].

In networks of neuronal cells, signaling occurs both internally, through the interaction of the different ionic currents, and externally, through synaptic coupling. Following the framework

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introduced in [17], the model we rely on explicitly takes into account internal and external interconnections by viewing each component of the network (referred to as a *compartment*) as an interconnection of subsystems (referred to as *species*) represented by operators in the extended L_2 space. The input to each operator includes both the influence of the other species within the same compartment, and a diffusive coupling term between the same species in different compartments, as well as exogenous disturbances. Similarly to [17], [2], [18], [1], the dynamical properties of the isolated subsystems as well as the algebraic properties of the interconnection are summarized in the so-called *dissipativity* matrix, whose diagonal stability implies the robust synchronization of the network. This approach is similar to classical works on large-scale systems such as [22], [11]. The robustness property is quantified through L_2 gain conditions that can be explicitly computed for particular interconnection structures.

Other recent related works have used different approaches to study synchronization in networks of nonlinear systems. In [20], [6], [19] the authors exploits the incremental passivity of the underlying dynamics. In [14], [13] the authors use a convergent dynamics approach. All these works heavily use a state-space formalism, which requires a detailed knowledge of the underlying dynamics, as opposed to the purely input-output approach used in this note.

This paper generalizes the results in [17] in the following ways: i) the elements belonging to the same species are not required to be identical, ii) the obtained synchronization conditions are weaker, and iii) the bound on the synchronization error is explicitly computed, thus paving the way to the study of interconnected systems forced by external inputs (*e.g.* control signals).

The paper is organized as follows. In Section II, we recall the formalism of [17] and adapt it to heterogenous compartments. In Section III, the needed input/output properties are defined and illustrated through some academic examples. The main result is provided in Section IV, and its application to a network of Hindmarsh-Rose neuronal models is presented in Section V. All the proofs are available in the online available extended version [4].

II. PRELIMINARIES AND PROBLEM STATEMENT

Let L_{2e}^m denotes the extended L_2 space [21] of signals $w : \mathbb{R}_{\geq 0} \rightarrow \mathbb{R}^m$, such that the truncation $w_T := w|_{[0,T]}$ is in $L_2^m([0,T])$, for all $T \geq 0$. In other words, L_{2e}^m is made of all signals that are square-integrable on any finite interval. Given any $T \geq 0$, for all $w, v \in L_{2e}^m$, the scalar product of w_T and v_T is denoted by $\langle w, v \rangle_T$. We write $\|w\|_T$ for the L_2^m norm of

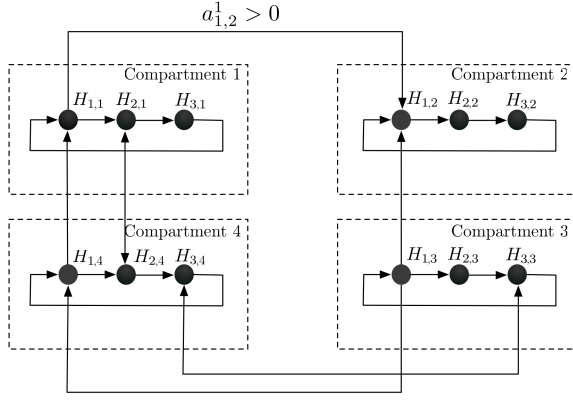


Fig. 1. An illustration of the interconnection structure

$w|_T$. Given any square matrix A , $\text{spect}(A)$ denotes the set of its eigenvalues.

The system under analysis is given by the diffusive interconnection of n compartments, each composed of N subsystems that we refer to as species [17]. The compartments are structurally identical, in the sense that they contain the same number of species, and that the internal interconnection is common to all compartments. The heterogeneity comes into play at the level of the species, *i.e.* the members of one species in different compartments are allowed to be different. The class of heterogeneities that we can take into account with the present approach will be detailed in the following sections.

Each species $k \in \{1, \dots, N\}$ in the compartment $j \in \{1, \dots, n\}$ is described through a nonlinear operator $H_{k,j} : L_{2e}^m \rightarrow L_{2e}^m$, and its input-output behavior is given by

$$y_{k,j} = H_{k,j} v_{k,j}, \quad v_{k,j} \in L_{2e}^m. \quad (1)$$

The inputs are given by

$$v_{k,j} = w_{k,j} + \sum_{i=1}^N \sigma_{k,i} y_{i,j} + \sum_{z=1}^n a_{j,z}^k (y_{k,z} - y_{k,j}), \quad (2)$$

where $w_{k,j}$ models exogenous disturbances, $\sum_{i=1}^N \sigma_{k,i} y_{i,j}$ models the input-output coupling among different species in the same compartment j , and $\sum_{z=1}^n a_{j,z}^k (y_{k,z} - y_{k,j})$ represents the diffusive coupling between the same species k in different compartments (see Figure 1). The coefficients $a_{i,j}^k$, $k = 1, \dots, N$, $i, j = 1, \dots, n$ are non-negative. They represent the interconnection structure among different compartments. We assume no self-loops, that is $a_{j,j}^k = 0$, for all $k = 1, \dots, N$ and all $j = 1, \dots, n$. As highlighted by the species superscript k , the coefficient $a_{i,j}^k$ may vary from species to species, allowing for different interconnection topologies between different species.

The internal interconnection structure is quantified by the $N \times N$ matrix

$$\Sigma := [\sigma_{k,i}]_{k,i=1,\dots,N}. \quad (3)$$

Moreover, for all $\eta \in \mathbb{R}^N$, we let

$$E_\eta := \Sigma - \text{diag}(\eta_1, \dots, \eta_N). \quad (4)$$

We respectively denote by $Y_k := \text{col}(y_{k,1}, \dots, y_{k,n})$, $V_k := \text{col}(v_{k,1}, \dots, v_{k,n})$, $W_k := \text{col}(w_{k,1}, \dots, w_{k,n})$, the vectors of outputs, inputs, and exogenous disturbances of the same species k . Given a set of vectors Z_k , $k = 1, \dots, N$, we

indicate the stacked vector by $Z := \text{col}(Z_1, \dots, Z_N)$, for example we indicate the stacked vector of outputs by $Y := \text{col}(Y_1, \dots, Y_N) \in L_{2e}^{nN}$.

The closed-loop input (2) can then be condensed as

$$V_k = W_k + \sum_{i=1}^N \sigma_{k,i} Y_i - L_k Y_k, \quad \forall k = 1, \dots, N, \quad (5)$$

where $L_k := [l_{i,j}^k]_{i,j=1,\dots,n} \in \mathbb{R}^{n \times n}$ is the Laplacian matrix associated to the k -th diffusive interconnection, defined as

$$l_{i,j}^k := \begin{cases} \sum_{z=1}^n a_{i,z}^k, & i = j \\ -a_{i,j}^k, & i \neq j. \end{cases} \quad (6)$$

The connectivity properties of the diffusive interconnection can be associated to the algebraic properties of L_k [5]. In particular, the algebraic connectivity λ_k can be extended to the case of directed graphs [23] as

$$\lambda_k := \min_{z \in \bar{1}_n^+, |z|=1} z^T L_k z. \quad (7)$$

To analyze the synchronization of the interconnected system (1)-(2), we compare the outputs of the same species in different compartments. The mean output $\bar{Y}_k \in L_{2e}^m$ of a species k is defined as

$$\bar{Y}_k := \frac{1}{n} \sum_{j=1}^n y_{k,j}, \quad k = 1, \dots, N. \quad (8)$$

By defining the vector of synchronization error

$$Y_k^\Delta := \text{col}(y_{k,1} - \bar{Y}_k, \dots, y_{k,n} - \bar{Y}_k), \quad (9)$$

we have that $Y_k^\Delta = 0$ if and only if the outputs are synchronized, meaning that $y_{k,1} = y_{k,2} = \dots = y_{k,n}$. A natural quantity to characterize the degree of synchronization of the species k over the time window $[0, T]$, $T \geq 0$, is thus given by $\|Y_k^\Delta\|_T$. In the sequel, the same notation is used to define the vectors V_k^Δ and W_k^Δ .

III. DEFINITIONS AND FIRST EXAMPLES

In order to study the synchronization of (1)-(2) we introduce some operator properties that will be extensively used in the paper.

The next definition characterizes an *incremental* input-output property that relates output differences to input differences for pairs of operators. It thus constitutes a natural instrument to study synchronization of input-output operators. This definition is the natural generalization of [17, Definition 1] to the case of heterogeneous populations.

Definition 1 (Mutual relaxed co-coercivity) Given $\gamma \in \mathbb{R}$ and $\mathcal{I} \subset \mathbb{N}$, a family \mathcal{H} of input-output operators $H_i : L_{2e}^m \rightarrow L_{2e}^m$, $i \in \mathcal{I}$, is said to be γ -mutual relaxed co-coercive if, for all $i, j \in \mathcal{I}$, there exists $\beta_{i,j} \in \mathbb{R}$, such that, for all $u_i, u_j \in L_{2e}^m$, and all $T \geq 0$,

$$\gamma \|H_i u_i - H_j u_j\|_T^2 \leq \langle (H_i u_i - H_j u_j), (u_i - u_j) \rangle_T + \beta_{i,j}. \quad (10)$$

The constants $\beta_{i,j}$ are called the *biases*.

Let us illustrate Definition 1 through some examples that will be helpful for the development of Section V.

Example 1 (Scalar systems affine in the inputs) With the same computation as in (cf. [17, Section V.A]), a scalar

dynamical system $\dot{x} = -f(x) + u$ with arbitrary initial conditions and output $y = x$ can be shown to define mutually relaxed co-coercive operators, provided its right-hand side satisfies a one sided Lipschitz condition as the one studied in [13]. In particular, a scalar linear dynamics $\dot{x} = -ax + bu$, with arbitrary initial conditions defines mutually co-coercive operators with co-coercivity constant $\gamma = \frac{a}{b}$.

For state space models Definition 1 is satisfied by incrementally output feedback passive systems [20], [12] with small heterogeneities as shown in the following proposition.

Proposition 1 (Incremental passivity with small heterogeneity) Consider a family of input-output dynamics

$$\begin{cases} \dot{x}_j &= f(x_j, u_j) + \delta f_j(x_j, v_j) \\ x_j(0) &= x_j^0 \\ y_j &= h_k(x_j) \end{cases} \quad (11)$$

where $u_j, y_j \in \mathbb{R}^m$, $x_j \in \mathbb{R}^p$, and $v_j \in \mathbb{R}^q$, for all $j = 1, \dots, n$. Note that $f : \mathbb{R}^p \times \mathbb{R}^m \rightarrow \mathbb{R}^p$ is common to all x_j , while the heterogeneity comes from the term $\delta f_j : \mathbb{R}^p \times \mathbb{R}^q \rightarrow \mathbb{R}^p$. Suppose that the input-output operators associated to (11) are well defined. Suppose that there exists a smooth function $V : \mathbb{R}^p \rightarrow \mathbb{R}_{\geq 0}$ and a constant $\gamma \in \mathbb{R}$ such that along the trajectories of (11), for all $i, j = 1, \dots, n$,

$$\begin{aligned} \dot{V}(x_i - x_j) &\leq -\gamma(y_i - y_j)^2 + (u_i - u_j)^T(y_i - y_j) + \\ &\left[\frac{\partial V}{\partial x}(x_i - x_j) \right]^T (\delta f_i(x_i, v_i) - \delta f_j(x_j, v_j)). \end{aligned} \quad (12)$$

Given two functions $v_i, v_j : \mathbb{R}_{\geq 0} \rightarrow \mathbb{R}^q$, suppose that, for all initial conditions x_i^0, x_j^0 , there exists a constant $C \geq 0$ such that, for all input functions $u_i, u_j \in L_{2e}^m$, along the trajectories of (11),

$$\left\| \left[\frac{\partial V}{\partial x}(x_i - x_j) \right]^T (\delta f_i(x_i, v_i) - \delta f_j(x_j, v_j)) \right\|_T \leq C, \quad (13)$$

for all $T \geq 0$. Then the input-output operators associated to (11) are mutually relaxed co-coercive with co-coercivity constant γ and biases $\beta_{i,j} = V(x_i^0 - x_j^0) + C$.

The heterogeneity $\delta f_j(x_j, v_j)$ depends both on the state of the system, and on the (possibly time varying) parametric uncertainties v_j . The right hand side of (13) represents the energy added to incremental storage function V by the presence of heterogeneities. Relation (13) requires this energy to be finite and independent from the systems inputs. When $\delta f_j = 0$, for all $j = 1, \dots, n$, the only heterogeneities are due to different initial conditions. In this case Proposition 1 says that identical incrementally passive systems define a family of mutually relaxed co-coercive operators. The following example illustrates a particular situation in which condition (13) can be checked

Example 2 (Passive linear system with heterogeneous non-controllable part) Consider a family of non-controllable passive system with (possibly nonlinear) heterogeneities in the non-controllable part of the form

$$\begin{cases} \dot{x}_j^c &= A_c x_j^c + B u_j + A_{uc} x_u, \\ \dot{x}_j^u &= A_u x_j^u + \delta f_j(x_j^u) \\ x_j(0) &= x_j^0 \\ y_j &= C x_j^c \end{cases} \quad (14)$$

where $y_j, u_j \in \mathbb{R}^m$, $x_j^c \in \mathbb{R}^{p_c}$, $x_j^u \in \mathbb{R}^{p_u}$, and $[A_c, B]$ is controllable, for all $j = 1, \dots, n$. Suppose that the non-controllable part exponentially synchronizes, that is there exists $\alpha_{i,j}, b_{i,j} > 0$ such that, for all initial condition $x_j^u(0), x_i^u(0)$,

$$|x_j^u(t) - x_i^u(t)| \leq b_{i,j} |x_j^u(0) - x_i^u(0)| e^{-\alpha_{i,j} t}, \quad \forall t \geq 0, \quad (15)$$

for all $i, j = 1, \dots, n$. By picking a suitable quadratic incremental Lyapunov function, it is possible to show that (12) and (13) are satisfied, and thus (14) defines a family of mutually co-coercive operators. Intuitively, this comes from the fact that the controllable part is linear and output feedback passive, and thus output feedback incrementally passive, while the derivative along the incremental trajectories of the non-controllable part has finite integral due to the exponential synchronization property. Detailed computations are omitted due to space constraint, but can be found in the on-line available preprint [4].

IV. ROBUST SYNCHRONIZATION RESULTS

A. Statement

The following theorem is an extension of [17, Theorem 1] to the case of heterogeneous dynamics. Its proof follows the steps of [17, Theorem 1] and is omitted due to space constraints.

Theorem 1 Consider the network (1)-(2). Suppose that the following assumptions are satisfied:

- 1) For each $k = 1, \dots, N$, the family of operators $\mathcal{H}_k := \{H_{k,j}\}_{j=1, \dots, n}$ is γ_k -mutually relaxed co-coercive, $\gamma_k \in \mathbb{R}$.
- 2) For each $k = 1, \dots, N$, $\tilde{\gamma}_k := \gamma_k + \lambda_k > 0$, where λ_k is the algebraic connectivity (7) of the interconnection graph associated to the species k .
- 3) The dissipation matrix $E_{\tilde{\gamma}}$, as defined in (4), is diagonally stable.

Then, there exist $\rho, \beta > 0$, such that

$$\|Y^\Delta\|_T \leq \rho \|W^\Delta\|_T + \beta, \quad \forall T \geq 0. \quad (16)$$

In particular, letting $d_i > 0$, $i = 1, \dots, N$, be such that $DE_{\tilde{\gamma}} + E_{\tilde{\gamma}}^T D < 0$, where $D = \text{diag}(d_1, \dots, d_N)$, the L_2 -gain ρ in (16) is given by

$$\rho := \frac{\max_{i=1, \dots, N} \{d_i\}}{\min \left\{ \text{spect}(-DE_{\tilde{\gamma}} - E_{\tilde{\gamma}}^T D) \right\}}. \quad (17)$$

Theorem 1 ensures that the synchronization error Y^Δ of the system is small (in the L_2 norm) provided that the input dispersion W^Δ is small. In particular the closed-loop system has finite incremental L_2 -gain from the incremental input W^Δ to the incremental output Y^Δ . With respect to [17], apart from the less conservative assumptions, Theorem 1 provides an explicit expression for the L_2 -gain. Notice that the gain can be made arbitrarily small, by reducing the eigenvalues of the matrix $DE_{\tilde{\gamma}} + E_{\tilde{\gamma}}^T D$.

B. L_2 -gain for particular interconnection topologies

In the following lemmas, we give the explicit computation of the incremental L_2 -gain ρ , appearing in Theorem 1 for two particular compartmental interconnection topologies.

Lemma 1 (L_2 gain for cyclic feedback compartmental interconnections [2]) Suppose that the compartmental coupling

is given by a cyclic feedback. If $\tilde{\gamma}_i > 0$, for all $i = 1, \dots, N$, and

$$1 - r \cos\left(\frac{\pi}{N}\right) > 0, \quad (18)$$

where

$$r := \sqrt[N]{\prod_{i=1}^N \frac{1}{\tilde{\gamma}_i}}, \quad (19)$$

then the dissipativity matrix $E_{\tilde{\gamma}}$ is diagonally stable, and the incremental L_2 -gain (17) of the closed-loop system is given by

$$\rho = \frac{1}{(1 - r \cos(\frac{\pi}{N})) \min_{i=1, \dots, N} \tilde{\gamma}_i} \tilde{\delta} \quad (20)$$

where

$$\tilde{\delta} = \frac{\max \left\{ 1, (r\tilde{\gamma}_2)^2, (r^2\tilde{\gamma}_2\tilde{\gamma}_3)^2, \dots, (r^{N-1}\tilde{\gamma}_2 \dots \tilde{\gamma}_N)^2 \right\}}{\min \left\{ 1, (r\tilde{\gamma}_2)^2, (r^2\tilde{\gamma}_2\tilde{\gamma}_3)^2, \dots, (r^{N-1}\tilde{\gamma}_2 \dots \tilde{\gamma}_N)^2 \right\}}.$$

The form of the incremental L_2 -gain for a cyclic interconnection (20) can be readily used for synthesis purposes. It suggests in particular that the algebraic connectivity of the interconnection topologies associated to different species must be chosen in such a way that the secant condition (18) is satisfied with a large margin, that is $r \cos(\pi/N) \ll 1$, and that the minimum closed-loop co-coercivity constant $\min_{i=1, \dots, N} \tilde{\gamma}_i$ should be large. Noticing that r is given by the inverse of the geometrical mean $\bar{\gamma}$ of the algebraic connectivities, that is

$$\bar{\gamma} := \sqrt[N]{\prod_{i=1}^N \tilde{\gamma}_i}, \quad (21)$$

the term $\tilde{\delta}$ imposes that the set $\{\tilde{\gamma}_i\}_{i=1, \dots, N}$ should be as homogeneous as possible, meaning that $\frac{\tilde{\gamma}_i}{\bar{\gamma}} \sim 1$, for all $i = 1, \dots, N$. The last condition can be interpreted as avoiding ‘‘bottle-necks’’ effects in the feedback cycle given by species that synchronize with a slower rate than the others. This kind of homogeneity condition is often encountered in the study of the *synchronizability* of a given interconnection topology [10], [7].

In the following lemma we specialize the computation of the incremental L_2 gain to the case of antisymmetric input-output interconnections.

Lemma 2 (L_2 gain for antisymmetric compartmental interconnections) *Suppose that the dissipativity matrix $E_{\tilde{\gamma}}$ has the form*

$$E_{\tilde{\gamma}} = A_N - \text{diag}(\tilde{\gamma}_1, \dots, \tilde{\gamma}_N),$$

where A_N denotes an antisymmetric $N \times N$ matrix and $\tilde{\gamma}_i > 0$ for all $i = 1, \dots, N$. Then $E_{\tilde{\gamma}}$ is diagonally stable and the incremental L_2 -gain (17) is given by

$$\rho := \frac{1}{\min_{i=1, \dots, N} \{\tilde{\gamma}_i\}}.$$

The L_2 gain obtained for antisymmetric input-output interconnections is independent of the size of the system, and takes into account the minimum co-coercivity constant only. This fact reflects the observation that any antisymmetric input-output interconnection can be decomposed into a family of two-dimensional negative feedbacks. Notice that both Lemma 1 and Lemma 2 provide the same L_2 -gain for a two-dimensional negative feedback.

C. Application to differential equations

The results of Theorem 1 can be used to analyze synchronization in systems described with a state space formalism

$$\begin{cases} \dot{x}_{k,j} &= f_{k,j}(x_{k,j}, v_{k,j}), \\ y_{k,j} &= h_k(x_{k,j}) \end{cases} \quad (22)$$

where $y_{k,j}, v_{k,j} \in \mathbb{R}^m$, $x_{k,j} \in \mathbb{R}^{p_k}$, for all $k = 1, \dots, N$, and all $j = 1, \dots, n$. Its proof is a straightforward application of Theorem 1 and is omitted.

Corollary 1 *Assume that the nonlinear operators $H_{k,j}$, $k = 1, \dots, N$, $j = 1, \dots, n$, associated to (22) with some initial conditions $x_{k,j}^0 \in \mathbb{R}^{p_k}$ are well defined. Consider the closed-loop system defined by (22), with inputs as in (2), and suppose that the conditions in Theorem 1 are satisfied. Then, there exists $\rho, \beta > 0$, such that*

$$\|Y^\Delta\|_T \leq \rho \|W^\Delta\|_T + \beta, \quad \forall T \geq 0, \quad (23)$$

where ρ is given as in the statement of Theorem 1. If in addition $W^\Delta \in L_2$, then the output asymptotically synchronizes.

As opposed to [17, Corollary 1], the vector field and initial conditions do not need to be identical among different compartments. In particular, the requirement of zero-state reachability, assumed in [17, Corollary 1] is not required.

V. ROBUST SYNCHRONIZATION IN NETWORKS OF HINDMARSH-ROSE NEURONS

A. The Hindmarsh-Rose model and its input-output representation

The Hindmarsh-Rose (HR) model, first introduced in [8], is a qualitative model of neuronal bursting dynamics. That is, its trajectories mimic the behavior of bursting neurons. The HR dynamics is defined by the three following coupled differential equations

$$\begin{aligned} \dot{x} &= -ax^3 + bx^2 + I + y - z_{x,i} + u_x \\ \dot{y} &= c - dx^2 - y \\ \dot{z} &= r(s(x + \bar{z} + u_z) - z_i). \end{aligned} \quad (24)$$

The first variable x models the membrane voltage, the second y models fast Na^+ and K^+ currents through the membrane, and the third z models slow Ca^{2+} currents. I is a parameter that models external currents through the membrane. u_x models other exogenous electrical inputs (heterogeneities, coupling with other cells, noise, etc.), while u_z models the diffusion of Ca^{2+} ions in the cell. $a, b, c, d, r, s, \bar{z}$ are free parameters that change the qualitative behavior of the system by inducing bifurcations in the underlying dynamics. The (x, y) -subsystem accounts for the excitable spike generation mechanism. The variable z provides a slow¹ adaptation mechanism that let the (x, y) -subsystem switch between resting and spiking, which corresponds to a bursting behavior (see [8] for more details). It is then natural to consider the (x, y) -subsystem as single bi-dimensional biological species $X := (x, y)^T$, that interacts in an input-output fashion with the (slow) z species. Letting $y_X := x$ be the output of the X species, and $y_z := z$ be the output of the z species, the HR neurons, with arbitrary initial conditions

¹ $r \ll 1$ in the Equation (24)

(x_0, y_0, z_0) , can equivalently be modeled as the interconnection of the input/output operators H_X and H_z , modeling the X and z species, respectively, defined by $H_X : v_X \mapsto y_X$ and $H_z : v_z \mapsto y_z$, where

$$\begin{cases} \dot{X} &= F(X) + Bv_X \\ y_X &= x \\ X(0) &= X_0 := (x_0, y_0)^T \end{cases} \quad (25)$$

$$\begin{cases} \dot{z} &= r(s(\bar{z} + v_z) - z) \\ y_z &= z \\ z(0) &= z_0 \end{cases} \quad (26)$$

with $F(X) := \begin{pmatrix} -ax^3 + bx^2 + y \\ c - dx^2 - y \end{pmatrix}$, $B := \begin{pmatrix} 1 \\ 0 \end{pmatrix}$, $v_X := I + w_X - z$ and $v_z := w_z + x$ are the inputs to the X and z species, and w_X and w_z are the external current and the external calcium diffusion, respectively. This configuration corresponds to a compartmental input-output interconnection matrix, as defined in (3),

$$\Sigma_{HR} := \begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix}. \quad (27)$$

In order to apply Corollary 1 to the synchronization of the family of operators (25)-(26) according to the methodology developed in the previous sections, we have to check that the operators are well defined and study their mutual co-coercivity.

Since the operator associated to the slow z species is defined by the one-dimensional linear system $\dot{z} = r(s(\bar{z} + v_z) - z)$ with input v_z and output z , it is well defined [21]. Moreover, it follows directly from Example 1 that an ensemble of input-output operators (26) with arbitrary initial conditions defines a family of mutually co-coercive operators with

$$\gamma_z := 1/s. \quad (28)$$

The following propositions, whose proofs are provided in [4], establish that the operators associated to the X species are well defined, and form a family of mutually co-coercive operators.

Proposition 2 *For all initial conditions $X_0 \in \mathbb{R}^2$, the operator H_X defined in (25) is well defined.*

Proposition 3 *For all $X_{10}, X_{20} \in \mathbb{R}^2$, the input-output operators associated to (25) with initial conditions X_{10} and X_{20} are mutually relaxed co-coercive with co-coercivity constant $\gamma_X := -\frac{d^2}{2} - b^2$.*

B. Network of Hindmarsh-Rose neurons

In the following we consider the diffusive interconnection of $n \in \mathbb{N}_{\geq 2}$ HR neuronal compartment (25)-(26). The initial conditions specifying the input-output behavior of each compartment are assumed to be arbitrary. We let λ_X and λ_z be the algebraic connectivity associated to the X and z species, respectively. The dissipativity matrix $E_{\tilde{\gamma}}$, as defined in (4), is then given by

$$E_{\tilde{\gamma}} = \begin{bmatrix} \tilde{\gamma}_X & -1 \\ 1 & \tilde{\gamma}_z \end{bmatrix}, \quad (29)$$

where $\tilde{\gamma}_X = \gamma_X + \lambda_X$ and $\tilde{\gamma}_z = \gamma_z + \lambda_z$. From Lemma 1 or Lemma 2, $E_{\tilde{\gamma}}$ is diagonally stable provided $\lambda_X > -\gamma_X$. Moreover, in the case of zero inputs, the boundedness of the

trajectories of each subsystem follows directly from [12, Proposition 1]. Hence, from (28), Proposition 2, and Proposition 3, all the conditions of Corollary 1 are satisfied, provided that

$$\lambda_X > \frac{d^2}{2} + b^2. \quad (30)$$

At the light of these considerations, we are able to provide analytical results on the robust synchronization of a network of HR neurons, as stated in the following proposition, whose proof can be found in [4].

Proposition 4 *Consider a network of $n \in \mathbb{N}_{\geq 2}$ HR neurons (25)-(26). Let W_X^Δ and W_z^Δ be the incremental input of the X and z species, respectively, and W^Δ the resulting incremental input of the network, as defined in Section II. Let Y_X^Δ and Y_z^Δ be the synchronization errors (9) associated to the X and z species respectively, and Y^Δ the resulting network synchronization error. Suppose that (30) is satisfied. Then there exists $\beta \geq 0$, such that, for all $T \geq 0$,*

$$\|Y^\Delta\|_T \leq \rho_{HR} \|W^\Delta\|_T + \beta, \quad (31)$$

where

$$\rho_{HR} := \frac{1}{\min(\tilde{\gamma}_X, \tilde{\gamma}_z)}. \quad (32)$$

We point out that, as opposed to [17], our approach does not require to check the zero-state reachability of the HR model.

C. Numerical simulation

To illustrate the results, we run numerical simulations on a biologically meaningful interconnection topology. The parameters used in the simulation are as follows: $a = 0.01, b = 0.3, c = 0.1, d = 0.5, r = 0.001, s = 4, \bar{z} = 7$. We have considered the interaction of two distinct neuronal populations. Each population is composed by qualitatively identical neurons. The first population contains the *active* neurons. By *active* we mean neurons with an *endogenous* rhythmic activity. The second population is composed of silent neurons, that is neurons that are endogenously at rest. The two different behaviors are obtained by fixing different values of w_z , which models the extracellular calcium concentration. A low value ($w_z = 2$) corresponds to the active population, while silent neurons are characterized by a higher value ($w_z = 4$). Moreover, each neuron is affected by an heterogeneous electrical input w_X .

We suppose that the electrical coupling between neurons belonging to the same population is absent. On the contrary, each active neuron is connected to all the silent neurons, and vice-versa, with the same coupling strength K_X . This kind of interconnection represents a simplification of the interaction between different brain neuronal subpopulations. Indeed, for neurons in the sub-thalamic zone, most of the synapses of a neuron belonging to some specific area project outside the interested area [16]. On the contrary, neurons that are located inside the same area, share the same physical medium. In this way, the natural diffusion in the cellular surrounding of the ions generating the currents in the neuron membrane constitutes a further type of diffusive coupling [15]. In the HR neuron, this can be modeled by a diffusive coupling in the z species, describing the dynamics of Ca^{2+} ions concentration in the cell. We thus suppose that each neuron belonging to a given population is coupled to all the other neurons of this population through diffusive coupling in the z species with homogeneous

coupling strength K_z . If the two species have the same number $M \in \mathbb{N}_{>0}$ of neurons, the Laplacian matrices associated to the interconnection topologies among the X and z species are given by

$$L_X = - \begin{bmatrix} 0_{M \times M} & 1_{M \times M} \\ 1_{M \times M} & 0_{M \times M} \end{bmatrix} + MI_N, \quad (33)$$

and

$$L_z = - \begin{bmatrix} 1_{M \times M} & 0_{M \times M} \\ 0_{M \times M} & 1_{M \times M} \end{bmatrix} + MI_N. \quad (34)$$

The interconnection between the X species corresponds to a complete bipartite graph. If K_X is the coupling strength, the algebraic connectivity is given by $\lambda_X = MK_X$ [3, page 5]. Since the graph associated to the z species is not connected its algebraic connectivity is zero [3]. By picking $K_X > \left(\frac{d^2}{2} + b^2\right)/M$ and $K_z \geq 0$, all the assumption of Proposition 4 are satisfied. In the simulation we have picked $M = 10$, $K_X = \left(\frac{d^2}{2} + b^2 + 0.5\right)/M$ and $K_z = 0.25/M$. With this choice we get, from Lemma 1 or Lemma 2, $\rho_{HR} = 4$.

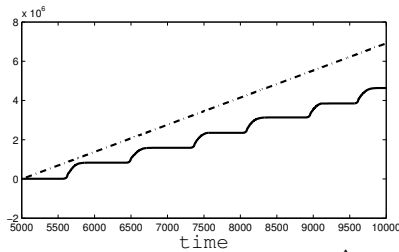


Fig. 2. Evolution of the synchronization error $\|Y^\Delta\|_T$ (solid line) and of the predicted bound $\rho\|W^\Delta\|_T$ (dashed line) after the diffusive coupling between the active and silent populations is activated. The bias has been removed for clarity.

Figure 2 shows the evolution of the synchronization error. When the coupling is switched on at time $t = 5000$, The theoretical integral bound predicted by Theorem 1 is satisfied. The prediction of a robust synchronous behavior is associated to important counterintuitive phenomena. Neurons belonging to the active population are not synchronized before the coupling with the silent population is not present, due to heterogeneities (Figure 3, left). Even though they are not directly coupled, they become synchronized once they start to interact with the silent neuronal population. At the same time, the silent neuronal population starts to show global oscillations at the same frequency (Figure 3, right). This kind of mutually induced oscillations is typical of the interaction between excitatory and inhibitory neuronal populations [9].

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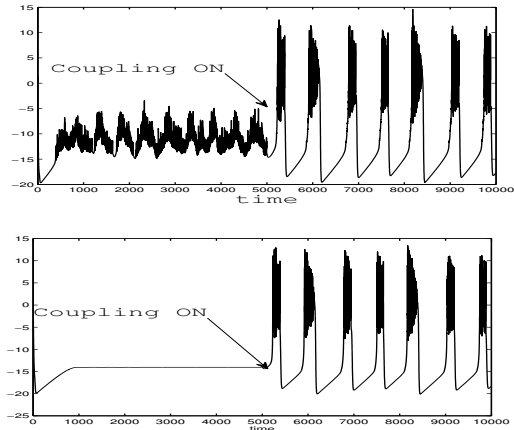


Fig. 3. Mean membrane voltage of the active (top) and silent (bottom) population before and after the coupling between the two populations is switched on. After the coupling is activated, the system shows a global bursting oscillation even though no interconnections are present inside each neuronal population.

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