# Set-point Tracking in Mode-Observable Switching Linear Systems

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*Abstract*— This paper addresses the problem of set-point tracking for a continuous-time process modeled by a multi-input multi-output (MIMO) linear system that may switch, in unknown and unpredictable fashion, among different modes taken from a finite set. The proposed methodology relies on an high level controller which, from time to time, can switch on in feedback with the process a set-point controller, from a family of candidate controllers, based on a real-time estimation of the current process mode. It is shown that, under certain conditions, global exponential stability can be achieved for any slow-on-the-average process mode switching sequence.

## I. INTRODUCTION

In recent years, switching systems have attracted significant research efforts both in theory and applications, as they allow one to describe the behavior of a large class of plants resulting from the interactions of continuous dynamics, discrete dynamics, and logic decisions [1]. These contributions have been basically of a two-fold nature: on one side [2], [3], [4], [5], [6], [7], [8], [9] several studies have focused on mode observability and mode estimation; on the other side, the main interest has been devoted to stability and stabilization problems. Within this latter source of contribution, however, the major emphasis has been on basic issues, namely the characterization of the control laws which can ensure stability to the switching system under the assumption that an exact knowledge of the current process mode is available in real-time or with delay [10], [11], [12].

As a matter of fact, the departure from the assumption that an exact knowledge of the process mode sequence is available poses major challenges. Indeed, such a departure (akin to the step from gain-scheduling to adaptive control) must invariably be undertaken by adopting specific mechanisms for estimating the current process mode on the grounds of the available data. To the best of the authors' knowledge, there are only few contributions which address the case where the knowledge of the plant configuration is not available, neither in real time nor with delay [13], [14], [15], [16]. In addition, fundamental issues such as how to deal with persistent disturbances or how to satisfy control objectives other than stability are as yet largely unexplored.

A recent paper [17] has considered the zero regulation problem for continuous-time processes modeled by a multi-input multi-output (MIMO) linear system which are subject to persistent disturbances and may switch, in unknown and unpredictable fashion, among different modes taken from a finite set. The solution there proposed enjoys the following features: i) It is realized via an high level control scheme whereby a controller, selected from a finite family of candidate controllers, is at any time switched-on in feedback to the plant; ii) The control scheme relies on an adaptive control strategy where the controller selection is made in accordance with the current process mode estimate.

The present paper aims at extending the approach of [17] to the case of set-point tracking as well as to discuss how this additional control objective can affect the process mode estimate. Intuitively, the possibility to properly infer the current process mode, and hence, to properly reconfigure the control action is closely connected to plant mode observability considerations [2], [3]. As will be seen, in contrast with the zero regulation problem, the mode observability conditions can fail to hold in the set-point tracking case, because such an additional control objective forces the system along nonzero terminal trajectories. Nonetheless, it is shown that by adopting sensible modifications to the mode estimator it is possible to preserve exponential stability for any slow-on-the-average process mode switching sequence and further ensure the offset-free tracking property whenever the switched system makes this objective conceptually achievable. For the sake of brevity, all the proofs are omitted.

*Notations.* Given a matrix $M$, $M^\top$ is its transpose and $\|M\| = \left[\lambda_{\max}(M^\top M)\right]^{1/2}$ its norm, where $\lambda_{\max}$ denotes the maximum eigenvalue. Given a measurable time function $v : \mathbb{R}^+ \in \mathbb{R}^n$ and a time interval $\underline{\mathcal{I}} \subseteq \mathbb{R}^+$, we denote its $\mathcal{L}_2$ norm on $\mathcal{I}$ as $\|v\|_{2,\mathcal{I}} = \sqrt{\int_{\mathcal{I}} |v(t)|^2 dt}$. Finally, $\mathcal{L}_2(\mathcal{I})$ denotes the sets of square integrable time functions on $\mathcal{I}$.

## II. PROBLEM FORMULATION AND BACKGROUND

Consider a plant $\mathcal{P}_{\sigma(t)}$ described by a continuous-time switching linear system of the form

$$\left.\begin{aligned} \dot{x}(t) &= A_{\sigma(t)}\, x(t) + B_{\sigma(t)}\, u(t) \\ y(t) &= C_{\sigma(t)}\, x(t) \end{aligned}\right\} \qquad (1)$$

where $t \in \mathbb{R}_+$; $x \in \mathbb{R}^n$ is the plant state, $u \in \mathbb{R}^p$ is the control input, $y \in \mathbb{R}^p$ is the plant output and $\sigma \in \mathcal{N}$, $\mathcal{N} := \{1, 2, \ldots, N\}$ is the plant mode. $A_i$, $B_i$, and $C_i$, $i \in \mathcal{N}$, are constant matrices of appropriate dimensions. It is supposed that the unknown and unobserved switching signal $\sigma : \mathbb{R}^+ \to \mathcal{N}$ belongs to the class $\mathfrak{G}$ of all the functions that are piecewise constant, right continuous, and admit no Zeno

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behavior (*i.e.* the number of switching instants is finite on every finite interval).

Further, let $y_r \in \mathbb{R}^p$ be a constant set-point to be tracked by the output and $e(t) = y(t) - y_r$ denote the corresponding tracking error. The aim is to find possibly nonlinear feedback controls which globally asymptotically stabilize (1) and possibly yields offset-free asymptotic tracking of constant set-points $y_r$. To this end, we consider a one degree-of-freedom linear switching controller $\mathcal{C}_{\hat{\sigma}(t)}$ of the form

$$\left.\begin{array}{rcl} \dot{q}(t) &=& F_{\hat{\sigma}(t)}\, q(t) + G_{\hat{\sigma}(t)}\, e(t) \\ u(t) &=& H_{\hat{\sigma}(t)}\, q(t) \end{array}\right\} \tag{2}$$

where $q \in \mathbb{R}^m$ is the controller state vector and $\hat{\sigma} \in \mathcal{N}$ is the controller mode; $F_i$, $G_i$, $H_i$, and $K_i$, $i \in \mathcal{N}$, are constant matrices of appropriate dimensions. The switching signal $\hat{\sigma} : \mathbb{R}^+ \to \mathcal{N}$ is supposed to be known and belonging to $\mathfrak{S}$. Hereafter, for the sake of simplicity, both the plant $\mathcal{P}_i$ and the controller $\mathcal{C}_j$ will be understood to be controllable and observable for any fixed indices $i$ and $j$, respectively. Then, given (2), the adopted approach consists of suitably selecting a discrete family $\{(H_i, F_i, G_i)\}_{i=1}^{N}$ and a switching signal $\hat{\sigma}$ generating mechanism in such a way that the regulated plant have the stated stability properties.

In connection with the design of the switching controller, some preliminary observations are in order, which pertains to the frozen-time analysis. According to the internal model principle [18], [19], a classic approach adopted to asymptotically reject constant disturbances and achieve offset-free tracking of constant set-points is to enforce an integral action from $e$ to $u$. It is well-known that, given a time-invariant plant $\mathcal{P}_i \sim (C_i, A_i, B_i)$ of the form (1), a stabilizing linear error-feedback law $\mathcal{C}_j \sim (H_j, F_j, G_j)$ exists which ensures an offset-free steady-state tracking only if

$$\det \begin{bmatrix} A_i - \lambda_i I & B_i \\ C_i & 0 \end{bmatrix} \neq 0, \qquad i = 1, 2, \ldots, p \tag{3}$$

The latter is indeed a necessary and sufficient condition for the existence of a stabilizing linear error-feedback law with integral action for system (1) under controllability of $(A_i, B_i)$ and observability of $(C_i, A_i)$ [18].

A property closely related to (3) is the following. Assume that (3) holds for all $i \in \mathcal{N}$, and consider a left and a right polynomial matrix fraction description (MFD) of the plant

$$H_{\sigma(t)}(s) = P_{\sigma(t)}^{-1}(s)\, Q_{\sigma(t)}(s) = \overline{Q}_{\sigma(t)}(s)\, \overline{P}_{\sigma(t)}^{-1}(s) \tag{4}$$

where, for each $i \in \mathcal{N}$, $P_i$ and $Q_i$ ($\overline{P}_i$ and $\overline{Q}_i$) are left (right) coprime polynomial matrices of appropriate dimensions with $\det(sI - A_i) = \det P_i(s) = k \det \overline{P}_i(s)$, $k \in \mathbb{R}\backslash\{0\}$. Here, equation (4) is intended as a shorthand notation to mean that over each time interval where $\sigma(t) = i$, $y$ is the output of the LTI system with transfer matrix $H_i(s) = P_i^{-1}(s)\, Q_i(s) = \overline{Q}_i(s)\, \overline{P}_i^{-1}(s)$ and state at the beginning of this interval being initialized according to (1). Then, as a standard result of linear system theory (*e.g.* see [20]), (3) is equivalent to the fact that $s = 0$ is not a transmission zero of $H_i(s)$, and,

hence, $Q_i(0)$ is full-rank $\forall\, i \in \mathcal{N}$. Likewise, consider a left and a right polynomial MDF of the controller

$$K_{\sigma(t)}(s) = R_{\sigma(t)}^{-1}(s)\, S_{\sigma(t)}(s) = \overline{S}_{\sigma(t)}(s)\, \overline{R}_{\sigma(t)}^{-1}(s)\,. \tag{5}$$

As beforehand, for each $j \in \mathcal{N}$, $R_j$ and $S_j$ ($\overline{R}_j$ and $\overline{S}_j$) are left (right) coprime polynomial matrices of appropriate dimensions with $\det(sI - F_j) = \det R_j(s) = h \det \overline{R}_j(s)$, $h \in \mathbb{R}\backslash\{0\}$. Assuming that $\mathcal{C}_j$ has integral action, then $\mathcal{C}_j$ has the right MFD $\overline{S}_j(s)\, \overline{R}_j^{-1}(s) = \overline{S}_j(s)\, \Delta^{-1}(s)\, \check{R}_j^{-1}(s)$ with $\Delta(s) = s\, I$. If such a property is enforced for all candidate controllers, it follows that under (3) the characteristic polynomial $\varphi_{i/j}(s)$ of the the closed-loop $(\mathcal{P}_i/\mathcal{C}_j)$,

$$\begin{aligned} \varphi_{i/j}(s) &= \det \begin{bmatrix} P_i(s) & -Q_i(s) \\ -S_j(s) & R_j(s) \end{bmatrix} \\ &= h\, \det\left(P_i(s)\, \overline{R}_j(s) - Q_i(s)\, \overline{S}_j(s)\right) \end{aligned}$$

is such that $\varphi_{i/j}(0) \neq 0$, $\forall\, i, j \in \mathcal{N}$. Under the just mentioned conditions, the latter property is indeed a direct consequence of the fact that $R_j(0) = 0$, $\forall\, j \in \mathcal{N}$, $Q_i(0)$ is full-rank $\forall\, i \in \mathcal{N}$, and the fact that $\overline{R}_j$ and $\overline{S}_j$ are coprime.

## III. MODE-OBSERVABILITY OF FEEDBACK LINEAR SWITCHING SYSTEMS

In this section, based on the foregoing observations, attention will be devoted to the problem of inferring the plant mode $\sigma$ from the measured data. Toward this end, consider first the following state space realization for the closed-loop system $(\mathcal{P}_{\sigma(t)}/\mathcal{C}_{\hat{\sigma}(t)})$ resulting from the feedback interconnection of (1) with (2)

$$\left.\begin{array}{rcl} \dot{w}(t) &=& A^{cl}_{\sigma(t)/\hat{\sigma}(t)}\, w(t) + B^{cl}_{\hat{\sigma}(t)}\, y_r \\ z(t) &=& C^{cl}_{\sigma(t)/\hat{\sigma}(t)}\, w(t) + D^{cl}\, y_r \end{array}\right\} \tag{6}$$

where $w := [\, x^T \; q^T \,]^T \in \mathbb{R}^{n+m}$ and $z := [\, u^T \; e^T \,]^T \in \mathbb{R}^{2p}$ denote the closed-loop state and output response, respectively. Furthermore,

$$\begin{aligned} A^{cl}_{i/j} &:= \begin{bmatrix} A_i & B_i\, H_j \\ G_j\, C_i & F_j \end{bmatrix}, \quad B^{cl}_j := \begin{bmatrix} 0 \\ -G_j \end{bmatrix}, \\ C^{cl}_{i/j} &:= \begin{bmatrix} 0 & H_j \\ C_i & 0 \end{bmatrix}, \quad D^{cl} := \begin{bmatrix} 0 \\ -I \end{bmatrix}, \quad i, j \in \mathcal{N}. \end{aligned}$$

Let $z_{i/j}(t, t_0, w_0, y_r)$ denote the output response of (6) at time $t > t_0$ when the initial state at time $t_0$ is $w_0$, the reference is $y_r$, the controller switching signal is $\hat{\sigma}(\tau) = j$ for any $\tau \in [t_0, t]$, and the plant switching signal is $\sigma(\tau) = i$ for any $\tau \in [t_0, t]$.

The following notion of *distinguishability* between two plant modes can be introduced.

*Definition 1:* For system (6), two plant modes $i, i' \in \mathcal{N}$ with $i \neq i'$ are said to be *distinguishable* if

$$z_{i/j}(\cdot, t_0, w_0, y_r) \neq z_{i'/j}(\cdot, t_0, w_0', y_r) \quad \text{a.e. on } [t_0, t]$$

for any $t_0, t$ with $t > t_0$, $j \in \mathcal{N}$, $y_r \in \mathbb{R}^p$, and $w_0, w_0' \in \mathbb{R}^{n+m}$ with $w_0 \neq 0$ or $w_0' \neq 0$. $\qquad\square$

In words, two plant modes are distinguishable when, over any finite interval, they always lead to different data $z$ provided that the initial state is different from zero. As discussed next, in contrast with the analysis of autonomous switched systems (where $y_r$ is absent), the distinguishability of two plant modes depends in a crucial way on their zero-frequency gains. To see this, it is first convenient to recall the following result, which provides conditions for the state-space solution of the error-feedback regulation problem.

*Proposition 1:* [21] Consider a plant $\mathcal{P}_i \sim (C_i, A_i, B_i)$ with $(A_i, B_i)$ stabilizable and $(C_i, A_i)$ detectable. Let the controller $\mathcal{C}_j \sim (H_j, F_j, G_j)$ be stabilizing. Then, $\mathcal{C}_j$ provides asymptotic offset-free tracking if and only if there exist matrices $\Pi_i$, $\Xi_i$ and $\Sigma_{i/j}$ such that

$$A_i \Pi_i + B_i \Xi_i = \Pi_i E, \quad C_i \Pi_i = I,$$
$$F_j \Sigma_{i/j} = \Sigma_{i/j} E, \quad H_j \Sigma_{i/j} = \Xi_i \qquad \square$$

In connection with Proposition 1, it is an easy matter to see that, for any feedback interconnection $(\mathcal{P}_i/\mathcal{C}_j)$, $\Pi_i y_r$, $\Sigma_{i/j} y_r$ and $\Xi_i y_r$ represent the steady-state solutions associated to $x$, $q$ and $u$, respectively. It is a standard exercise to verify that a controller satisfying the requirements of Proposition 1 can be expressed in the observer-based form

$$\left.\begin{aligned} \dot{q}(t) &= \left( \begin{bmatrix} A_i & 0 \\ 0 & 0 \end{bmatrix} - G_j \begin{bmatrix} C_i & -I \end{bmatrix} \right) q(t) \\ &\quad + \begin{bmatrix} B_i \\ 0 \end{bmatrix} u(t) + G_j e(t) \\ u(t) &= \begin{bmatrix} K_j & (\Xi_i - K_j \Pi_i) \end{bmatrix} q(t) \end{aligned}\right\} \qquad (7)$$

where $K_j$, $G_j$ are such that

$$A_i + B_i K_j \quad \text{and} \quad \left( \begin{bmatrix} A_i & 0 \\ 0 & 0 \end{bmatrix} - G_j \begin{bmatrix} C_i & -I \end{bmatrix} \right)$$

are stability matrices. Further, in this case, $\Sigma_{i/j} = [\Pi_i^T \ I]^T$.

The introduction of Proposition 1 is motivated by the fact that it has a direct connection to the issue of plant modes distinguishability. Consider in fact that the output of (6)

$$z_{i/j}(t, t_0, w_0, y_r) = C_{i/j}^{cl} e^{A_{i/j}^{cl}(t - t_0)} w_0$$
$$+ C_{i/j}^{cl} \int_{t_0}^{t} e^{A_{i/j}^{cl}(t - \tau)} B_j^{cl} y_r \, d\tau + D^{cl} y_r$$

can also be rewritten as

$$z_{i/j}(t, t_0, w_0, y_r) = C_{i/j}^{cl} e^{A_{i/j}^{cl}(t - t_0)} \left( w_0 + (A_{i/j}^{cl})^{-1} B_{i/j}^{cl} y_r \right)$$
$$+ \left( D^{cl} - C_{i/j}^{cl} (A_{i/j}^{cl})^{-1} B_{i/j}^{cl} \right) y_r \qquad (8)$$

In equation (8), invertibility of $A_{i/j}^{cl}$, $i, j \in \mathcal{N}$, follows from the fact that $\varphi_{i/j}(s) = \det(sI - A_{i/j}^{cl})$ and that $\varphi_{i/j}(0) \neq 0$, $\forall \, i, j \in \mathcal{N}$ as pointed out at the end of Section II. Let

$$\mathcal{G}_{i/j} := D^{cl} - C_{i/j}^{cl} (A_{i/j}^{cl})^{-1} B_{i/j}^{cl}, \quad i, j \in \mathcal{N}$$

From equation (8), it is simple to conclude that it is impossible to distinguish between two plant modes $i$ and $i'$ in case

$\mathcal{G}_{i/j} = \mathcal{G}_{i'/j}$. In such a case, it is indeed sufficient to let $w_0 = -(A_{i/j}^{cl})^{-1} B_{i/j}^{cl} y_r$ and $w_0' = -(A_{i'/j}^{cl})^{-1} B_{i'/j}^{cl} y_r$ in order to obtain $z_{i/j}(\cdot, t_0, w_0, y_r) \equiv z_{i'/j}(\cdot, t_0, w_0', y_r)$.

*Proposition 2:* Two plant modes $i, i' \in \mathcal{N}$ with $i \neq i'$ can be distinguishable only if the matrix $\mathcal{G}_{i/j} - \mathcal{G}_{i'/j}$, is non-singular $\forall \, j \in \mathcal{N}$. $\qquad \square$

From Proposition 1, one sees that the indistinguishability condition corresponds exactly to the situation where the switched system is in steady-state and there exist two plant modes $i$ and $i'$ such that

$$0 = A_{i/j}^{cl} w_0 + B_j^{cl} y_r = A_{i/j}^{cl} \begin{bmatrix} \Pi_i \\ \Sigma_{i/j} \end{bmatrix} y_r + B_j^{cl} y_r$$
$$= A_{i'/j}^{cl} w_0' + B_j^{cl} y_r = A_{i'/j}^{cl} \begin{bmatrix} \Pi_{i'} \\ \Sigma_{i'/j} \end{bmatrix} y_r + B_j^{cl} y_r$$
$$z(t) = C_{i/j}^{cl} w_0 + D^{cl} y_r = \begin{bmatrix} \Xi_i \\ 0 \end{bmatrix} y_r = \mathcal{G}_{i/j} y_r$$
$$= C_{i'/j}^{cl} w_0' + D^{cl} y_r = \begin{bmatrix} \Xi_{i'} \\ 0 \end{bmatrix} y_r = \mathcal{G}_{i'/j} y_r$$

Despite this, it should be clear that, from the point of view of the control objectives, it is not necessary to require distinguishability when the switched system is in steady-state but only that the switched system be mode observable along all possible non steady-state trajectories. These concepts will be better formalized in the following.

### A. Non steady-state Mode-observability

The above analysis motivates the introduction of a different notion of distinguishability between two plant modes. To this end, consider first the following.

*Definition 2:* Let $z_{i/j}(\cdot, t_0, w_0, y_r)$ be the output of the unswitched feedback loop $(\mathcal{P}_i/\mathcal{C}_j)$. Then, $z_{i/j}(\cdot, t_0, w_0, y_r)$ is said to be a *steady-state* output trajectory on $[t_0, t]$ if $z_{i/j}(\cdot, t_0, w_0, y_r) \equiv \mathcal{G}_{i/j} y_r$ on $[t_0, t]$. $\qquad \square$

It is to be pointed out that each feedback loop $(\mathcal{P}_i/\mathcal{C}_j)$ (stable or not) admits a unique steady-state solution being $A_{i/j}^{cl}$ non singular. The following definition can be given.

*Definition 3:* For system (6), two plant modes $i, i' \in \mathcal{N}$ with $i \neq i'$ are said to be *non steady-state distinguishable* (NSS distinguishable, for short) if they are distinguishable along their non steady-state output trajectories.

Further, (6) is said to be *non steady-state mode-observable* (NSS mode-observable, for short) if any two different plant modes $i, i' \in \mathcal{N}$ are NSS distinguishable. $\qquad \square$

In practice, NSS mode-observability corresponds to the invertibility of the mapping from the any non steady-state output trajectory $z(\cdot)$ to the plant switching signal $\sigma(\cdot)$. As will be seen, NSS mode-observability allows one to reconstruct the unknown switching signal $\sigma(\cdot)$ from observation of $z(\cdot)$, provided that the switched system is not in steady-state. In this respect, necessary and sufficient conditions for NSS mode-observability of (6) are given in the following.

Let now $O_{i/j}^{(k)}$ denote the observability matrix of order $k$ of the feedback system $(\mathcal{P}_i/\mathcal{C}_j)$

$$O_{i/j}^{(k)} := \begin{bmatrix} C_{i/j}^{cl} \\ C_{i/j}^{cl} A_{i/j}^{cl} \\ \vdots \\ C_{i/j}^{cl} \left( A_{i/j}^{cl} \right)^{k-1} \end{bmatrix}.$$

It is worth pointing out that, since the pairs $(C_i, A_i)$ and $(F_j, H_j)$ are observable by hypothesis, then also $(C_{i/j}^{cl}, A_{i/j}^{cl})$ turns out to be observable, *i.e.* the observability matrix $O_{i/j}^{(k)}$ is full-rank for any $k \geq n+m$. The following lemma unveils that the *joint observability matrix*

$$\begin{bmatrix} \mathcal{O}_{i/j}^{(2n+2m)} & \mathcal{O}_{i'/j}^{(2n+2m)} \end{bmatrix}$$

plays a key role in determining NSS distinguishability of two plant modes $i$ and $i'$.

*Lemma 1:* Two plant modes $i, i' \in \mathcal{N}$ with $i \neq i'$ are NSS distinguishable if and only if their joint observability matrix is full-rank, i.e.,

$$\mathrm{rank} \begin{bmatrix} \mathcal{O}_{i/j}^{(2n+2m)} & \mathcal{O}_{i'/j}^{(2n+2m)} \end{bmatrix} = 2n+2m, \quad \forall j \in \mathcal{N}. \ (9)$$

As a consequence, the feedback system (6) is NSS mode-observable if and only if condition (9) holds for any pair of different plant modes $i, i' \in \mathcal{N}$. $\square$

As can be seen, the conclusions of Lemma 1 are closely connected to those derived in [2] and [3] for autonomous switching linear systems. In the next two sections, we describe how Lemma 1 can be used so as to extend the results of [17] to the case of set-point regulation.

## IV. SWITCHING CONTROLLER AND SUPERVISED SYSTEM

In this section, we discuss how stability of the feedback system (6) can be achieved by means of a suitable choice of the controller switching signal $\hat{\sigma}$. To this end, it is supposed that (1) and (2) satisfy the following basic requirements.

A1. Condition (3) holds for every $i \in \mathcal{N}$.
A2. For each plant mode $i \in \mathcal{N}$, the corresponding controller $\mathcal{C}_i$ is stabilizing, and $R_i(0) = 0$.
A3. The feedback system (6) is NSS mode-observable.

While A1 and A2 can be regarded as feasibility conditions in order to achieve stability and, possibly, the offset-free tracking property, A3 requires some additional arguments. In this respect, a sufficient condition to ensure NSS mode-observability is provided in the following.

As it emerges from the proof of Lemma 1, (9) is a necessary and sufficient condition for mode-observability (in the sense of Definition 1) of the autonomous system given by (6) when $y_r = 0$. Indeed, the joint observability matrix
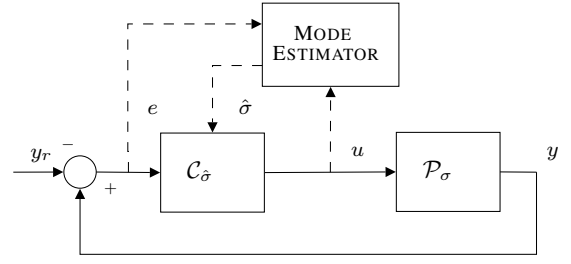


Fig. 1. Block diagram of the feedback control system with mode estimator.

equals the observability matrix of the autonomous system

$$\left. \begin{aligned} \dot{\chi}(t) &= \begin{bmatrix} A_{i/j}^{cl} & 0 \\ 0 & A_{i'/j}^{cl} \end{bmatrix} \chi(t) \\ \zeta(t) &= \begin{bmatrix} C_{i/j}^{cl} & C_{i'/j}^{cl} \end{bmatrix} \chi(t) \end{aligned} \right\} \quad (10)$$

obtained from the parallel connection of $(\mathcal{P}_i/\mathcal{C}_j)$ with $(\mathcal{P}_{i'}/\mathcal{C}_j)$. This implies that (9) holds under the same conditions which ensure mode-observability (in the sense of Definition 1) of the autonomous system given by (6) when $y_r = 0$. Then, next result follows as a simple variant of its counterpart in [17].

*Proposition 3:* Two plant modes $i, i' \in \mathcal{N}$ with $i \neq i'$ are NSS distinguishable if, for any $j \in \mathcal{N}$, the closed-loop characteristic polynomials $\varphi_{i/j}(s)$ and $\varphi_{i'/j}(s)$ are coprime. $\square$

According to Proposition 3, A3 holds provided that for any pair $i, i' \in \mathcal{N}$ with $i \neq i'$ and any $j \in \mathcal{N}$ the closed loop polynomials $\varphi_{i/j}(s)$ and $\varphi_{i'/j}(s)$ have no common roots. As discussed in [17], for single-input single-output systems, the conditions of Proposition 3 are also necessary.

### A. Dwell-time switching and mode estimator

The choice of the control action to use, among all the candidate controllers $\mathcal{C}_i, i \in \mathcal{N}$, is carried out in real-time by an high-level unit called *mode estimator*. At each time $t \in \mathbb{R}_+$, the mode estimator generates an estimate $\hat{\sigma}(t, z(\cdot)) \in \mathcal{N}$ of the current plant mode based on the measured data $z(\cdot)$ up to the current time $t$. As depicted in Figure 1, the estimate is then used as the controller switching signal, *i.e.* $\hat{\sigma}(t) = \hat{\sigma}(t, z(\cdot))$. The plant mode estimate is generated according to a *dwell-time switching logic* (DTSL, for short), whose functioning can be explained as follows. The mode estimator updates its estimate $\hat{\sigma}(t, z(\cdot))$ of the plant mode $\sigma(t)$ at discrete-time instants of the type $kT$ where $k \in \mathbb{Z}_+$ and $T$, a positive real, is the so called *dwell time*. This amounts to assuming the controller switching signal constant over each time interval $\mathcal{I}_k := [kT, (k+1)T)$, *i.e.* $\hat{\sigma}(t) = \hat{\sigma}_k, \quad \forall t \in \mathcal{I}_k$.

In order to generate $\hat{\sigma}_{k+1}$, the mode estimator employs a finite family of $N$ performance signals $\delta_{i/j}(z(\cdot), \mathcal{I}_k), i \in \mathcal{N}$, which provide a measure of the distance between the plant and the $i$-th model when the switched-on controller is $\mathcal{C}_j$.

Accordingly, one selects a value $i_* \in \mathcal{N}$ among those which achieve the minimum, *i.e.*

$$i_* \in \arg\min_{i \in \mathcal{N}} \delta_{i/\hat{\sigma}_k}(z(\cdot), \mathcal{I}_k).$$

If $\delta_{i_*/\hat{\sigma}_k}(z(\cdot), \mathcal{I}_k)$ is smaller than $\delta_{\hat{\sigma}_k/\hat{\sigma}_k}(z(\cdot), \mathcal{I}_k)$, then $\hat{\sigma}_{k+1}$ is set equal to $i_*$, otherwise the controller mode is left unchanged. It is to be noted that, as in [22], one could collect the data only in a subinterval of $\mathcal{I}_k$ of the type $[kT, kT+\tau]$ with $\tau \in (0, T)$, keeping the remaining subinterval $[kT + \tau, (k+1)T)$ for all the necessary computations. It is not difficult to verify that all next developments hold true also in this more general setting.

Thanks to the adoption of the DTSL, a simple criterion to generate the estimate $\hat{\sigma}_k$ can be devised. To this end, notice first that, whenever also the plant mode takes on a constant value, say $i$, over $\mathcal{I}_k$, the evolution of the plant input/output data on $\mathcal{I}_k$ can be written as

$$z(t) = z_{i/\hat{\sigma}_k}(t, kT, w(kT), y_r), \quad t \in \mathcal{I}_k.$$

Thus, the set $\mathcal{S}_{i/\hat{\sigma}_k}(\mathcal{I}_k)$ of all possible measured data on $\mathcal{I}_k$ associated with a plant mode $i$ and a controller mode $\hat{\sigma}_k$ corresponds to the affine subspace

$$\mathcal{S}_{i/\hat{\sigma}_k}(\mathcal{I}_k) := \Big\{ \hat{z} \in \mathcal{L}_2(\mathcal{I}_k) : \hat{z}(\cdot) = z_{i/\hat{\sigma}_k}(\cdot, kT, \hat{w}, y_r)$$
$$\text{on } \mathcal{I}_k, \text{ for some } \hat{w} \in \mathbb{R}^{n+m} \Big\} \quad (11)$$

where $y_r$ is a preassigned and fixed output reference. Therefore, next proposition directly descends from the definition of NSS mode-observability.

*Proposition 4:* Under assumption A3, for any two different plant modes $i, i' \in \mathcal{N}$ and any controller mode $\hat{\sigma}_k \in \mathcal{N}$,

$$\mathcal{S}_{i/\hat{\sigma}_k}(\mathcal{I}_k) \cap \mathcal{S}_{i'/\hat{\sigma}_k}(\mathcal{I}_k) =$$
$$\begin{cases} \{\mathcal{G}_{i/\hat{\sigma}_k} y_r\}, & \text{if } y_r \in \ker(\mathcal{G}_{i/\hat{\sigma}_k} - \mathcal{G}_{i'/\hat{\sigma}_k}) \\ \emptyset, & \text{otherwise} \end{cases}$$

where $\ker(\cdot)$ denotes the kernel. $\qquad\square$

In view of the above considerations, a convenient approach for estimating the plant mode $\sigma(\cdot)$ on $\mathcal{I}_k$ consists in choosing the index $i$ for which the distance between the observed data $z(\cdot)$ on $\mathcal{I}_k$ and the affine subspace $\mathcal{S}_{i/\hat{\sigma}_k}(\mathcal{I}_k)$ is minimal. More precisely, let $W_{i/j}(t, t_0)$ denote the *observability Gramian* of $(\mathcal{P}_i/\mathcal{C}_j)$ on $[t_0, t)$, *i.e.*

$$W_{i/j}(t - t_0) := \int_{t_0}^t \Psi_{i/j}^T(\xi, t_0)\, \Psi_{i/j}(\xi, t_0) d\xi, \quad (12)$$

$$\Psi_{i/j}(t, t_0) := C_{i/j}\, e^{A_{i/j}(t - t_0)}.$$

Then, at the generic time $(k+1)T$ the estimate $\hat{\sigma}_{k+1}$ can be obtained according to the minimum-distance criterion

$$\delta_{i/j}(z(\cdot), \mathcal{I}_k) := \min_{\hat{w} \in \mathbb{R}^{n+m}} \big\| z(\cdot) - z_{i/j}(\cdot, kT, \hat{w}, y_r) \big\|_{2, \mathcal{I}_k} \quad (13)$$
$$= \min_{\hat{w} \in \mathbb{R}^{n+m}} \big\| \zeta_{i/j}(\cdot, kT) - \Psi_{i/j}(\cdot, kT)\, \hat{w} \big\|_{2, \mathcal{I}_k}$$

where

$$\zeta_{i/j}(t, kT) :=$$
$$z(t) - C_{i/j}^{cl} \int_{kT}^t e^{A_{i/j}^{cl}(t-\tau)}\, B_j^{cl}\, y_r\, d\tau - D^{cl}\, y_r$$

Notice that, being the pair $(C_{i/j}^{cl}, A_{i/j}^{cl})$ completely observable by hypothesis, the observability Gramian $W_{i/j}(t - t_0)$ is positive definite for any $t > t_0$. Accordingly, the minimization in (13) yields

$$\delta_{i/j}(z(\cdot), \mathcal{I}_k)$$
$$= \Bigg( \int_{\mathcal{I}_k} \Bigg| \zeta_{i/j}(t, kT) - \Psi_{i/j}(t, kT) \left( W_{i/j}(kT) \right)^{-1}$$
$$\times \int_{\mathcal{I}_k} \left( \Psi_{i/j}(\xi, kT) \right)^\top \zeta_{i/j}(\xi, kT)\, d\xi \Bigg|^2 dt \Bigg)^{1/2}.$$

The relevant property of the criterion (13) is that when $\sigma(t)$ is constant in the interval $\mathcal{I}_k$, one has

$$\zeta_{\sigma_k/\hat{\sigma}_k}(t, kT) = \Psi_{\sigma_k/\hat{\sigma}_k}(t, kT)\, w(kT) \quad t \in \mathcal{I}_k.$$

and, hence, $\delta_{\sigma_k/\hat{\sigma}_k}(z(\cdot), \mathcal{I}_k) = 0$.

## V. STABILITY ANALYSIS

In this section we show how the previous results can be used to analyze the stability properties of the switched system (6). To this end, some preliminary observations are in order. Consider first that assumption A1 amounts to the existence of two positive reals $\mu$ and $\lambda$ such that

$$\| e^{A_{i/i}^{cl} t} \| \le \mu e^{-\lambda t}, \quad \forall t \in \mathbb{R}_+, \forall i \in \mathcal{N}. \quad (14)$$

Further, since the set $\mathcal{N}$ is finite, one has

$$\| e^{A_{i/j}^{cl} t} \| \le \theta e^{\rho t}, \quad \forall t \in \mathbb{R}_+, \forall i, j \in \mathcal{N} \quad (15)$$

for some positive reals $\theta$ and $\rho$.

The following result can be proven which is instrumental for next developments.

*Theorem 1:* Assume that A1-A3 hold and that the plant mode is constant on the time interval $[t_h, t_H)$. Then, if the criterion (13) is used in the DTSL, there exist finite positive real $\beta$ and $\gamma$ such that

$$|w(t)| \le \beta\, e^{-\lambda(t - t_h)} |w(t_h)| + \gamma |y_r|, \quad \forall t \in [t_h, t_H)$$

$$\square$$

Theorem 1 shows that, under the stated assumptions, the proposed minimum-distance criterion ensures exponential stability of the switched system whenever no plant variation occurs. As discussed next, similar conclusions hold true provided that the plant switching signal is sufficiently slow on the average. In this respect, let $N_\sigma(t, t_0)$ be the number of discontinuities of $\sigma$ in the interval $(t_0, t)$, then the following assumption is needed [23].

A4. There exist a positive real $\tau_D$, called *average dwell-time*, and a positive integer $N_0$, called *chatter bound*, such that

$$N_\sigma(t, t_0) \leq N_0 + \frac{t - t_0}{\tau_D}$$

for any $t, t_0 \in \mathbb{R}_+$ with $t > t_0$.

The main stability result of this section can be stated.

*Theorem 2:* Assume that assumptions A1-A4 hold and let the criterion (13) be used in the DTSL. Then,
i) If the average dwell-time $\tau_D$ is such that

$$\tau_D > [\log \mu + 2 \log \theta + (\lambda + \rho) 2T] / \lambda, \qquad (16)$$

there exist finite positive reals $\alpha_*$, $\beta_*$ and $\gamma_*$ such that

$$|w(t)| \leq \beta_* \, e^{-\alpha_*(t - t_0)} |w(t_0)| + \gamma_* |y_r|, \ \forall \, t \geq t_0 \quad (17)$$

ii) If the plant switching signal $\sigma(\cdot)$ is finitely convergent, we also have $\lim_{t \to \infty} e(t) = 0$. $\qquad \square$

It should be clear from the above analysis that, in contrast with the case where no plant variations occurs, it may be impossible to ensure the offset-free tracking property in the presence of plant variations. However, it is not difficult to verify that conditions do exist under which the offset-free tracking property can be recovered.

Let each candidate controller has the observer-based form (7). In case $\Pi_i = \Pi$ and $\Xi_i = \Xi$, $\forall \, i \in \mathcal{N}$, it is immediate to verify that the switched system (6) admits the coordinate transformation $\tilde{x} := x - \Pi y_r$, $\tilde{q} := q - \Sigma y_r$, where $\Sigma := [\Pi^T \ I]^T$. Under such circumstances the set-point tracking problem can be converted to an equivalent pure zero-regulation problem for the switched system

$$\left. \begin{array}{rcl} \dot{\tilde{w}}(t) & = & A^{cl}_{\sigma(t)/\hat{\sigma}(t)} \, \tilde{w}(t) \\ \tilde{z}(t) & = & C^{cl}_{\sigma(t)/\hat{\sigma}(t)} \, \tilde{w}(t) \end{array} \right\} \qquad (18)$$

where $\tilde{z} := [\tilde{u}^T \ e^T]^T$, $\tilde{u} := u - \Xi y_r$. Then, asymptotic stabilization of (18) implies asymptotic stabilization of (6) along with the offset-free tracking property. For instance, it is not difficult to show that, when $A_i$ is invertible $\forall \, i \in \mathcal{N}$, $\Sigma$ and $\Xi$ do exist provided that all the candidate plants have the same open-loop input-to-state and input-to-output gains.

## VI. CONCLUSIONS

Consideration has been given to the set-point tracking problem for MIMO switching linear systems thay may switch in unknown and unpredictable fashion, among different modes taken from a finite set. It is shown that suitable adaptive control schemes do exist which ensure exponential stability for any slow-on-the-average process mode switching sequence and further ensure the offset-free tracking property whenever the switched system makes this objective conceptually achievable. The positive assessment of robustness against the presence of persistent disturbances in the loop as reported in [17], indicates that similar desiderable properties are likely to be enjoyed by the present scheme: a topic currently under consideration by the authors of this paper.

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