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Actuator and sensor placement in linear advection PDE

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Abstract— **We study the problem of actuator and sensor placement in a linear advection partial differential equation (PDE). The problem is motivated by its application to actuator and sensor placement in building systems for the control and detection of a scalar quantity such as temperature and contaminants. We propose a gramian based approach to the problem of actuator and sensor placement. The special structure of the advection PDE is exploited to provide an explicit formula for the controllability and observability gramian in the form of a multiplication operator. The explicit formula for the gramian, as a function of actuator and sensor location, is used to provide test criteria for the suitability of a given sensor and actuator location. Furthermore, the solution obtained using gramian based criteria is interpreted in terms of the flow of the advective vector field. In particular, the almost everywhere uniform stability property and ergodic properties of the advective vector field are shown to play a crucial role in deciding the location of actuators and sensors. Simulation results are performed to support the main results of this paper.**

I. Introduction

In this paper, we study the problem of actuator and sensor placement in a linear advection partial differential equation. The problem is motivated by its application to actuator and sensor location in building systems for the purpose of control of temperature and detection of contaminants. Building systems in US account for 39 percent of total energy consumption [1]. Design of efficient building systems not only has a significant economic benefit but also social and environmental benefits.

The governing equations for building system fluid flows and scalar density are coupled nonlinear partial differential equations subjected to disturbances, various sources of uncertainties, and complicated geometry. Analysis of the building system with the full scale complexity leads to a finite element based computational approach to the actuator and sensor placement problem [2]. Such a purely computational based approach provides little insight into the obtained solution. An alternate system theoretic and dynamical systems based approach under some simplifying assumptions and physics can also be pursued [3], [4]. Such an approach provides useful insight and guidelines to the complex control problems involved in building system applications. In this paper, we pursue a similar approach for the location of actuator and sensor placement problem in building systems.

Under some simplifying assumptions and physics [3], [4], the system equations are modeled as a linear advection partial differential equation with inputs and outputs. We propose a gramian based approach to the actuator and sensor location problem. Because of the simplifying assumption made in the modeling of system equations, further research is needed for the application of the results to building systems. However, the results are an important first step towards its application to building systems. The main contribution of this paper is in providing an explicit formula for the controllability and observability gramians as a function of actuator and sensor locations and the advection velocity field. Technical conditions for the existence of infinite time gramians are also provided.

For complex vector fields that arise in the context of fluid flow problems, we use ergodic properties of the vector field to provide guidelines for the minimum number of actuators and sensors. Furthermore, local growth rate of the vector field, as captured by finite time Lyapunov exponents, is used to determine the areas of phase space from where the system trajectories expand the most. We provide simulation results using a two dimensional fluid flow vector field for the computation of finite time controllability and observability gramians.

The organization of the paper is as follows. In section II, we describe the problem and some preliminaries from the theory of partial differential equations. In section III, we present the main results of the paper. Simulation results are presented in section V followed by conclusion in section VI.

II. Preliminaries

We study the problem of optimal location of actuators in a linear advection partial differential equation. The motivation for this problem comes from the optimal location of actuators for the control of a scalar quantity, such as temperature, in a room denoted by $\rho(x,t)$. The evolution of $\rho(x,t)$, is governed by the velocity $v(x,t)$ field of the fluid flow. This velocity field is obtained as a solution of the following Navier Stokes equation:

$$\begin{array}{c}\frac{\partial v(x,t)}{\partial t}+v(x,t)\cdot\nabla v(x,t)=-\nabla p(x,t)+\frac{1}{Re}\triangle v(x,t)\\ \nabla\cdot v(x,t)=0,\end{array}\qquad(1)$$

where $x\in X\subset\mathbb{R}^N$ (with $N=3$ or 2) is the domain of the room, $v(x,t)$ is the velocity field, $p(x,t)$ is the pressure, and Re is the Reynolds number. The evolution of the scalar quantity $\rho(x,t)$ is governed by following linear controlled partial differential equation

$$\begin{aligned}\frac{\partial\rho}{\partial t}+v(x,t)\cdot\nabla\rho(x,t) &= \frac{1}{PrRe}\triangle\rho(x,t)+\sum_{k=1}^{N}\chi_{B_k}(x)u_k(t)\\ y_k(x,t) &= \chi_{A_k}(x)\rho(x,t),\quad k=1,\ldots,M\end{aligned}\qquad(2)$$

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where Pr is the Prandtl number, $\chi_{A_k}(x)$ is the indicator function on set $A_k \subset \Omega$, and $u_k(t) \in \mathbb{R}$ is the control input for $k = 1,\ldots,N$. The form of control input $\chi_B(x)u(t)$ and output measurement $\chi_A(x)\rho(x,t)$ is motivated by the fact that the actuation and sensing can be exercised only on a small region B and A of the physical space X respectively. The objective is to determine the optimal location of actuators and sensors and hence the determination of indicator functions $\chi_{B_k}(x)$ and $\chi_{A_k}(x)$. The terms $v(x,t)\cdot\nabla\rho(x,t)$ and $\triangle\rho(x,t)$ in (2) correspond to advection and diffusion respectively, with $D = \frac{1}{RePr}$ being the diffusion constant. Note that the advection diffusion equation (2) is decoupled from the Navier Stokes equation (1). In the case where the scalar density is temperature, this decoupling corresponds to the assumption that buoyancy forces have negligible or no effect. Furthermore, for simplicity of presentation of the main results of this paper, we now make the following assumptions.

Assumption 1: We replace the time varying velocity field $v(x,t)$ responsible for the advection of scalar density with the mean velocity field $f(x)$ i.e.,

$$f(x) := \frac{1}{T}\int_0^T v(x,t)dt.$$

Remark 2: Typically the velocity field information $v(x,t)$ is available over a finite time interval $[0,T]$ either from a simulation or from an experiment. Assumption 1 corresponds to linearizing the linear advection PDE along the mean flow field $f(x)$. It follows that if $v(x,t)$ is volume preserving i.e., $\nabla\cdot v(x,t) = 0$, then $\nabla\cdot f(x) = 0$ as well.

Assumption 3: We assume that the diffusion constant D in the advection diffusion equation (2) is zero.

Remark 4: As we will see in the simulation section, the assumption of zero diffusion constant is justified.

We next discuss a few preliminaries on semigroup theory of partial differential equations. Consider the following ordinary differential equation (ODE):

$$\dot{x} = f(x), \quad x(0) = x_0, \tag{3}$$

where $x \in X \subset \mathbb{R}^N$ a compact set. We denote by $\phi_t(x)$ the solution of ODE (3) starting from the initial condition x. ODE (3) is used to define two linear infinitesimal operators, $\mathscr{A}_K : L^2(X) \to L^2(X)$ and $\mathscr{A}_{PF} : L^2(X) \to L^2(X)$ defined as follows:

$$\mathscr{A}_K\rho = f\cdot\nabla\rho, \quad \mathscr{A}_{PF}\rho = -\nabla\cdot(f\rho).$$

The domains of the above operators are given as follows:

$$D(\mathscr{A}_K\rho) = \{\rho \in H^1(X) : \rho|_{\Gamma_o} = 0\},$$

$$D(\mathscr{A}_{PF}\rho) = \{\rho \in H^1(X) : \rho|_{\Gamma_i} = 0\},$$

where Γ_o and Γ_i are the outflow and inflow portions of the boundary ∂X defined as follows:

$$\Gamma_o = \{x \in \partial X : f\cdot\eta > 0\}, \quad \Gamma_i = \{x \in \partial X : f\cdot\eta < 0\},$$

where η is the outward normal to the boundary ∂X. The semigroups corresponding to the $\mathscr{A}_K$ and $\mathscr{A}_{PF}$ are called as Koopman ($\mathbb{U}_t$) and Perron-Frobenius ($\mathbb{P}_t$) operators respectively. These operators are defined as follows:

$$\mathbb{U}_t : L^2(X) \to L^2(X), \ (\mathbb{U}_t\rho)(x) = \rho(\phi_t(x))$$

$$\mathbb{P}_t : L^2(X) \to L^2(X), \ (\mathbb{P}_t\rho)(x) = \rho(\phi_{-t}(x))\left|\frac{\partial\phi_t(x)}{\partial x}\right|^{-1}$$

where $|\cdot|$ denotes the determinant.

III. Main results

The gramian based approach is one of the systematic approaches available for optimal placement of actuators and sensors. Using the gramian based approach, actuators and sensors are placed at a location where the degree of controllability and observability of the least controllable and observable state is maximized [5], [6].

A. Controllability gramian

For the construction of the controllability gramian, the advection-diffusion partial differential equation (2) using assumptions (1) and (3) for a single input case can be written as follows:

$$\frac{\partial\rho}{\partial t} + \nabla\cdot(f(x)\rho) = \chi_B(x)u(x,t); \tag{4}$$
$$\rho|_{\Gamma_i} = 0; \ \rho(x,0) = \rho_0(x).$$

In equation (4), we are assuming that the control input u is both a function of spatial variable x and time t. This assumption will typically not be satisfied in the building system application, however, making this assumption allows us to use existing results from linear PDE theory in the development of controllability gramian [5]. Since $m(X) >> m(B)$, where m is the Lebesgue measure, we expect the main conclusions of this paper to hold even when u is assumed to be only a function of time. The set B is the region of control in the state space X, and $u(x,t) \in L^2([0,\tau] : L^2(B))$ i.e., we have a control input that is square integrable in time and space, acting on the set B. The solution to (4) is given by the following:

$$\rho(x,t) \quad = \mathbb{P}_t\rho_0(x) + \int_0^t \mathbb{P}_{t-s}(\chi_B(x)u(x,s))ds.$$

We define the controllability operator $\mathscr{B}^\tau : L^2([0,\tau] : L^2(B)) \to L^2(X)$ as follows:

$$\mathscr{B}^\tau u \quad := \int_0^t \mathbb{P}_{t-s}(\chi_B(x)u(x,s))ds. \tag{5}$$

The adjoint of the controllability operator $\mathscr{B}^{\tau*} : L^2(X) \to L^2([0,\tau] : L^2(B))$ can be calculated and is given as follows:

$$(\mathscr{B}^{\tau*}z)(s) = \chi_B(x)\mathbb{U}_{(\tau-s)}z(x). \tag{6}$$

We have the following theorem on the controllability property of the PDE (4).

Theorem 5: Let $\mathscr{R}^\tau = \cup_{t=0}^\tau \phi_t(B)$. The PDE (4) is exactly controllable in a given time $\tau > 0$ for all initial and terminal states in the space $L^2(\mathscr{R}^\tau)$ i.e. given initial and terminal states $\rho_0(x)$ and $\rho_\tau(x)$ in $\mathscr{R}^\tau$, there exists a control $u(x,t) \in L^2([0,\tau] : L^2(B))$ such that $\rho(x,0) = \rho_0(x)$, and $\rho(x,\tau) = \rho_\tau(x)$, where $\rho(x,t)$ is the solution of (4).

We omit the proof for space constraint. Detailed proof could be obtained in [7].

Definition 6: The finite time controllability gramian $\mathscr{C}_B^\tau : L^2(X) \to L^2(X)$ for the PDE (4) is given by the following:

$$\mathscr{C}_B^\tau z = \mathscr{B}^\tau \mathscr{B}^{\tau *} z = \int_0^\tau \mathbb{P}_{(\tau-s)}(\chi_B(x)\mathbb{U}_{(\tau-s)} z(x))ds. \tag{7}$$

Furthermore, we have the following definition for the induced two norm of the operator $\mathscr{C}_B^\tau$:

$$||\mathscr{C}_B^\tau||_2^2 = \max_{z\in L^2(X), s.t. \|z\|_{L^2(X)}=1} \langle \mathscr{C}_B^\tau z, z\rangle_{L^2(X)} .$$

Theorem 7: The controllability gramian $\mathscr{C}_B^\tau : L^2(X) \to L^2(X)$ can be written as a multiplication operator as follows:

$$(\mathscr{C}_B^\tau z)(x) = \left(\int_0^\tau \mathbb{P}_t \chi_B(x) dt\right) z(x) \tag{8}$$

Proof:

$$\mathscr{C}_B^\tau z = \int_0^\tau \mathbb{P}_{(\tau-s)}(\chi_B(x)\mathbb{U}_{(\tau-s)} z(x))ds$$
$$= \int_0^\tau \mathbb{P}_s(\chi_B(x)\mathbb{U}_s z(x))ds = \int_0^\tau \mathbb{P}_s(\chi_B(x) z(\phi_s(x)))ds$$
$$= \int_0^\tau \chi_B(\phi_{-s}(x) z(x) \left|\frac{\partial \phi_s(x)}{\partial x}\right|^{-1} ds = \left[\int_0^\tau (\mathbb{P}_s \chi_B(x))ds\right] z(x).$$

■

The explicit formula for the controllability gramian from equation (8) in terms of multiplication operator can be used to provide an analytical expression for the minimum energy control input.

Claim 8: $\rho_B^\tau(x) := \int_0^\tau \mathbb{P}_t \chi_B(x)dt$ is strictly positive on $\mathscr{R}^\tau = \cup_{t=0}^\tau \phi_t(B)$ and hence $\mathscr{C}_B^\tau$ is invertible on $\mathscr{R}^\tau$ with the inverse given by

$$(\mathscr{C}_B^\tau)^{-1} z = \frac{z}{\rho_B^\tau(x)}, \quad \forall z \in L^2(\mathscr{R}^\tau). \tag{9}$$

Proof: Since $m(B) > 0$, and B evolves into $\phi_\tau(B)$ in time τ, for every $x \in \mathscr{R}^\tau$, there exist times $0 \le t_1(x) < t_2(x) \le \tau$ such that $x \in \phi_t(B)$ $\forall t \in [t_1(x), t_2(x)]$. Hence, by the positivity of $\mathbb{P}_t$ we have that $\mathbb{P}_t(\chi_B(x)) > 0$ $\forall t \in [t_1(x), t_2(x)] \subseteq [0, \tau]$. Hence we have the following:

$$\rho_B^\tau(x) = \int_0^\tau \mathbb{P}_t \chi_B(x)dt \ge \int_{t_1(x)}^{t_2(x)} \mathbb{P}_t \chi_B(x)dt > 0 \ \forall x \in \mathscr{R}^\tau.$$

This proves the claim. ■

Theorem 9: Let $\rho_\tau(x)$ and $\rho_0(x)$ be the elements of $L^2(\mathscr{R}^\tau)$, then the minimum energy control input that is required to steer the system from initial state $\rho_0(x)$ to final state $\rho_\tau(x)$ is given by following formula

$$u_{opt}(x,s) = \mathscr{B}^{\tau *}(\mathscr{C}_B^\tau)^{-1}(\rho_\tau(x) - \mathbb{P}_\tau \rho_0(x))$$
$$= \chi_B(x)\mathbb{U}_{\tau-s}\left(\frac{\rho_\tau(x) - \mathbb{P}_\tau \rho_0(x)}{\rho_B^\tau(x)}\right). \tag{10}$$

The minimum energy required is given by

$$||u_{opt}||^2 \tag{11}$$
$$= < (\rho_\tau(x) - \mathbb{P}_\tau \rho_0(x)), (\mathscr{C}_B^\tau)^{-1}(\rho_\tau(x) - \mathbb{P}_\tau \rho_0(x)) >_{L^2(\mathscr{R}^\tau)}$$
$$= \frac{||(\rho_\tau(x) - \mathbb{P}_\tau \rho_0(x))||^2_{L^2(\mathscr{R}^\tau)}}{\rho_B^\tau(x)}.$$

Proof: First, we note that controlling the initial state $\rho_0(x)$ to $\rho_\tau(x)$ is equivalent to reaching the final state $(\rho_\tau(x) - \mathbb{P}_\tau \rho_0(x))$ from the zero initial state i.e. $\rho_0(x) \equiv 0$. Hence, equivalently, we prove that $\hat{u}_{opt}(x,s) = \mathscr{B}^{\tau *}(\mathscr{C}_B^\tau)^{-1}(\rho_\tau(x))$ is the control input with minimum norm that reaches $\rho_\tau(x)$ in time τ. This, along with an explicit calculation of $\mathscr{B}^{\tau *}(\mathscr{C}_B^\tau)^{-1}(\rho_\tau(x))$ will prove the Theorem. Next, we consider the following set of admissible control inputs:

$$\mathscr{U} = \{u(x,t) \in L^2([0,\tau] : L^2(B)) : \mathscr{B}^\tau u = \rho_\tau\}.$$

We have the following:

$$\mathscr{B}^\tau \hat{u}_{opt} = \mathscr{B}^\tau \mathscr{B}^{\tau *}(\mathscr{C}_B^\tau)^{-1}\rho_\tau = \mathscr{B}^\tau \mathscr{B}^{\tau *}(\mathscr{B}^\tau \mathscr{B}^{\tau *})^{-1}\rho_\tau = \rho_\tau.$$

Hence, we have that $\hat{u}_{opt}(x,s) = \mathscr{B}^{\tau *}(\mathscr{C}_B^\tau)^{-1}(\rho_\tau(x)) \in \mathscr{U}$. Next, we define the following operator on $L^2([0,\tau] : L^2(B))$ $P^\tau = \mathscr{B}^{\tau *}(\mathscr{C}_B^\tau)^{-1}\mathscr{B}^\tau$. We observe the following:

$$\begin{aligned}(P^\tau)^2 &= \mathscr{B}^{\tau *}(\mathscr{C}_B^\tau)^{-1}\mathscr{B}^\tau \mathscr{B}^{\tau *}(\mathscr{C}_B^\tau)^{-1}\mathscr{B}^\tau = \mathscr{B}^{\tau *}(\mathscr{C}_B^\tau)^{-1}\mathscr{B}^\tau \\ &= P^\tau, (P^\tau)^* = (\mathscr{B}^{\tau *}(\mathscr{C}_B^\tau)^{-1}\mathscr{B}^\tau)^* = P^\tau. \end{aligned} \tag{12}$$

Hence, the operator P^τ is a projection operator on the space $L^2([0,\tau] : L^2(B))$. Then, we have the following from Bessel's inequality:

$$||u||^2 = ||(P^\tau)u||^2 + ||(I - P^\tau)u||^2 \ge ||(P^\tau)u||^2,$$

where the norm is on the space $L^2([0,\tau] : L^2(B))$. Now, let $u \in \mathscr{U}$ be arbitrary. This means $\mathscr{B}^\tau u = \rho_\tau$. Applying $\mathscr{B}^{\tau *}(\mathscr{C}_B^\tau)^{-1}$ on both sides, we get the following:

$$P^\tau u = \mathscr{B}^{\tau *}(\mathscr{C}_B^\tau)^{-1}\mathscr{B}^\tau u = \mathscr{B}^{\tau *}(\mathscr{C}_B^\tau)^{-1}\rho_\tau = \hat{u}_{opt}.$$

Hence, Bessel's inequality above gives $||u||^2 \ge ||\hat{u}_{opt}||^2$. Next, (10) and (11) can be easily shown using by an explicit calculation using (6) and (9). ■

Based on the formula for the controllability gramian, we propose the following criteria for the selection of optimal sensor location and hence the set B^*.

Actuator placement criteria

1) Maximizing the support of the controllability gramian operator i.e.,

$$B^* = \arg\max_{B\subset X} \ supp\left(\int_0^\tau \mathbb{P}_t \chi_B(x)dt\right) \tag{13}$$

2) If the support of controllability gramian is maximized or if more than one choice of set B leads to the same support then the decision can be made based on maximizing the 2-norm of the support i.e.,

$$B^* = \arg\max_{B\subset X} \|\int_0^\tau \mathbb{P}_t \chi_B(x)dt\|_{L^2(X)} .$$

Using the result of Theorem (5), it follows that criterion (1) maximizes the controllability in the state space, so that the control action in a small region $B \subset X$ will have an impact over larger portion of the state space. Furthermore, it follows from the explicit formula for the minimum energy control (10) from Theorem (9) that if the the actuator selection is made based on criteria (2) then the amount of control effort is minimized.

B. Observability gramian

For the construction of observability gramian, we consider the advection partial differential equation with output measurement as follows:

$$\begin{aligned}\frac{\partial \rho}{\partial t} &= \nabla\cdot(f\rho), \quad \rho|_{\Gamma_i}=0, \quad \rho(x,0)=\rho_0(x)\\ y(x,t) &= \chi_A(x)\rho(x,t) \end{aligned} \tag{14}$$

The observability operator $\mathscr{A}^\tau : L^2(X)\to L^2([0,\tau],L^2(A))$ for (14) is defined as follows:

$$(\mathscr{A}^\tau z)(x,\tau)=\chi_A(x)(\mathbb{P}_\tau z)(x).$$

The adjoint to the observability operator $\mathscr{A}^{\tau *}$: $L^2([0,\tau],L^2(A))\to L^2(X)$ can be written as follows:

$$\mathscr{A}^{\tau *}w(x,\tau))(x)=\int_0^\tau(\mathbb{U}_s\chi_A(x)w(x,s))ds$$

Definition 10 (Observability gramian): The finite time observability gramian $\mathscr{O}_A^\tau : L^2(X)\to L^2(X)$ for the PDE (14) is given by the following formula

$$(\mathscr{O}_A^\tau z)(x)=(\mathscr{A}^{\tau *}\mathscr{A}^\tau z)(x)=\int_0^\tau(\mathbb{U}_s\chi_A(x)\mathbb{P}_s z(x))ds. \tag{15}$$

The counterpart of Theorems (5) and (9) can be proved for the observability of system (14) using a duality argument.

Theorem 11: The observability gramian for (14) can be written as a multiplication operator as follows:

$$(\mathscr{O}_A^\tau z)(x)=\left[\int_0^\tau(\mathbb{U}_s\chi_A(x))ds\right]z(x). \tag{16}$$

Proof: We omit the proof due to space constraint but the proof follows along the lines of proof of theorem 7. ■

Following criteria can be used for the optimal location of sensor.

Sensor placement criteria

The finite time observability gramian can be used to decide the criteria for the optimal location of the sensor.

1) Maximizing the support of observability gramian operator

$$A^*=\arg\max_{A\subset X} supp\left(\int_0^\tau \mathbb{U}_t\chi_A(x)dt\right).$$

2) If the support of observability gramian is maximized or if more than one choice of set B leads to the same support then the decision can be made based on maximizing the 2-norm of the support i.e.,

$$A^*=\arg\max_{A\subset X}\parallel\int_0^\tau \mathbb{U}_t\chi_A(x)dt\parallel_{L^2(X)}$$

IV. Advective vector field and gramian

In this section, we provide an interpretation for the optimal actuator and sensor location problem in terms of the flow of the advection vector field. In particular, we observe that the (almost everywhere) stability and ergodic properties of the vector field play an important role in deciding the location of actuators and sensors. In this section, we omit proof of every individual theorem for space constraint. We refer the readers to [7] for the proofs.

A. Infinite time Controllability gramian

The notion of almost everywhere stability is extensively studied in [8] [9]. Furthermore, a PDE based approach is also provided for the verification of almost everywhere stability in [10]. We have the following theorem regarding the infinite time controllability gramian for vector fields that are almost everywhere uniformly stable:

Theorem 12: For vector fields that are stable in the almost everywhere uniform sense, we have

$$(\mathscr{C}_B^\infty z)(x)=\int_0^\infty \mathbb{P}_t\chi_B(x)dtz(x)=\rho_B(x)z(x). \tag{17}$$

where $\rho_B(x)$ is the positive solution of the following PDE

$$\nabla\cdot(f(x)\rho_B(x))=\chi_B(x);\rho|_{\Gamma_i}=0. \tag{18}$$

The integral $\int_X \mathscr{C}_B^\infty z(x)dx$ for the special case where $z(x)=\chi_A(x)$, the indicator function for the set A, has the interesting interpretation of residence time, which is defined as follows:

Definition 13: For an almost everywhere uniform stable vector field, consider any two measurable subsets A and B of $X\setminus B_\delta$, then the residence time of set B in set A is defined as the amount of time system trajectories starting from set B will spend in set A before entering the δ neighborhood of the equilibrium point $x=0$. We denote this time by T_B^A.

Theorem 14: The residence time T_B^A for an almost everywhere stable vector field $f(x)$ is given by following formula

$$T_B^A=\int_X \mathscr{C}_B^\infty\chi_A(x)dx.$$

B. Infinite time observability gramian

The infinite time observability gramian is defined under the assumption that the vector field $f(x)$ is globally asymptotically stable. First, we have the following Theorem that characterizes global asymptotic stability:

Theorem 15: Let B_δ be a δ neigborhood of $x=0$. Let $v(x)\in C^1(X\setminus\bar{B}_\delta)$ denote the solution of the following steady state transport equation:

$$\mathscr{A}_K v=f\cdot\bigtriangledown v=-v_0(x);v|_{\partial\bar{B}_\delta}=0, \tag{19}$$

where $v_0(x)$ satisfies

$$v_0(x)=0\ \forall x\in\bar{B}_\delta. \tag{20}$$

Then $x=0$ is globally asymptotically stable for (3) if and only if there exists a positive solution $v(x)\in C^1(X/\bar{B}_\delta)$ for (19) for all $v_0(x)>0\in C^1(X/\bar{B}_\delta)$ satisfying (20).

If $x=0$ is globally asymptotically stable, then $\Gamma_o\supseteq\partial\bar{B}_\delta$. Hence, by using a standard density argument of $C^1(X\setminus\bar{B}_\delta)$ in $L^2(X\setminus B_\delta)$, and using trace operator theory [11] for point values of H^1 functions, we can show the following Theorem:

Theorem 16: Let $v(x)\in\mathscr{D}(\mathscr{A}_K)\cap L^2(X\setminus\bar{B}_\delta)$ denote the solution of the following steady state transport equation:

$$\mathscr{A}_K v=f\cdot\bigtriangledown v=-v_0(x);v|_{\Gamma_o}=0, \tag{21}$$

Then $x=0$ is globally asymptotically stable for (3) if and only if there exists a positive solution $v(x)\in\mathscr{D}(\mathscr{A}_K)\cap L^2(X/\bar{B}_\delta)$ for (19) and for all $v_0(x)\in\mathscr{D}(\mathscr{A}_K)\cap L^2(X/\bar{B}_\delta)$.

Theorem 17: Let $x=0$ be a globally stable equilibrium point for $\dot{x}=f(x)$, then the infinite time observability gramian is well defined and we have

$$(\mathscr{O}_A^{\infty} z)(x) = \left[\int_0^{\infty} (\mathbb{U}_t \chi_A(x))dt\right] z(x) = V(x)z(x) \qquad (22)$$

where $V(x)$ is the positive solution of following steady state partial differential equation. $f(x)\cdot\nabla V(x) = -\chi_A(x)$.

C. Fluid flow vector field

Typically, the vector fields that arise in the context of fluid flow will not be globally stable. Although the infinite time gramian as defined in (17) cannot be defined for a typical vector field that arises in the context of fluid flows, the finite time gramians and the actuator and sensor placement criteria as defined in section III can be used for fluid flow vector fields. Asymptotic dynamics of a fluid flow vector field can be defined using the following notion of physical invariant measure from ergodic theory of dynamical systems.

Definition 18 (Physical invariant measure): Let $\mathscr{M}(X)$ be the space of probability measures on X. The vector field $f(x)$ is said to have a physical invariant measure μ, if μ is invariant under the flow generated by f i.e., $\mu(A) = \mu(\phi_{-t}(A))$ for all sets $A \in \mathscr{B}(X)$, the Borel sigma-algebra on X, and

$$\lim_{t\to\infty} \frac{1}{t}\int_0^t g(\phi_\tau(x))d\tau = \int_X g(x)d\mu(x)$$

for all continuous functions $g: X \to \mathbb{R}$ and positive Lebesgue measure initial conditions $x \in X$.

We have following theorem using it.

Theorem 19: Let μ_i for $i=1,2$ be two distinct physical invariant measures for $\dot{x}=f(x)$ and $X_i \subset X$ for $i=1,2$ be such that $m(X_i)>0$, $X_i \cap X_j = \emptyset$ for $i \neq j$, and for all $x \in X_{i(j)}$, we have

$$\lim_{\tau\to\infty} \frac{1}{\tau}\int_0^\tau g(\phi_t(x))dt = \int_X g(x)d\mu_{i(j)}(x) \qquad (23)$$

for all continuous functions $g: X \to \mathbb{R}$. Then for any two sets $B_i \subset X_i$ for $i=1,2$, we have

$$\phi_t(B_i)\cap B_j = \emptyset, \quad i \neq j. \qquad (24)$$

Remark 20: A trivial consequence of the above theorem is that the minimum number of actuators required for controllability is equal to the number of invariant measures.

In order to achieve effective control over shorter duration of time, it is essential that the actuator be placed at location where system dynamics expands and spreads over a larger region of the phase space. Finite time Lyapunov exponents can be used to measure the rate of expansion from different sets in the phase space [12]. Let us consider the ODE $\dot{x}(t) = f(x(t))$ and let $\phi_t(x)$ be the trajectory generated by it.

The finite-time Lyapunov exponent $\sigma_\tau(x)$ is defined as,

$$\sigma_\tau(x) = \frac{1}{\tau} ln\left(\sqrt{\lambda_{max}\left(\frac{d\phi_\tau^T}{dx}\frac{d\phi_\tau}{dx}\right)}\right), \qquad (25)$$

where λ_{max} is the maximum eigenvalue. This is the maximum finite time expansion rate for a given initial condition. FTLE can be defined for a set B as,

$$\Lambda_\tau(B) = \max_{x\in B} \ \sigma_\tau(x).$$

Remark 21: Naturally, in order to have a larger diameter for the set $\mathscr{R}^\tau$, the actuator set should be selected where the FTLE field has high value.

The ergodic average used in the definition of physical invariant measure motivates us to consider the following modification of observability gramian for a complex and possibly unstable vector field.

Theorem 22: Let the vector field $\dot{x}=f(x)$ have a unique physical invariant measure μ, then

$$\lim_{\tau\to\infty} \frac{1}{\tau}(\mathscr{O}_A^\tau z)(x) \qquad (26)$$

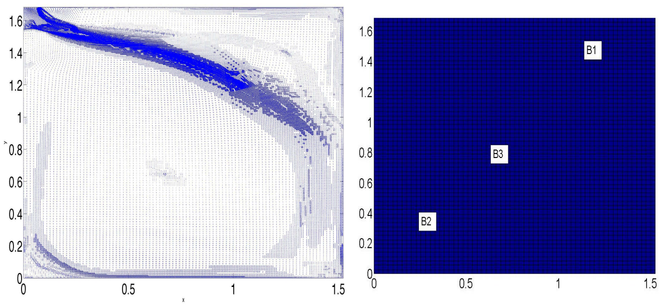
is well defined for Lebesgue almost all initial condition $x \in X$ and equals $\mu(A)z(x)$.

If we want to maximize the average gramian (26), then among all sets with equal Lebesgue measure, the sensor must be placed on the set A^* where the system dynamics spends most of the time. Furthermore, because of the ergodic property of the attractor set (time average equal to space average) the set A^* will also have maximum value of physical measure among all sets with equal Lebesgue measure.

V. Simulation

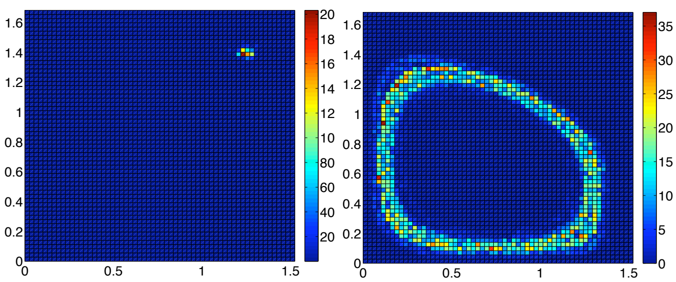
For the purpose of simulation, we only employ a two dimensional slice of a three dimensional velocity field as shown in Fig. 1a. The dimensions of the room are as follows: $0 \leq x \leq 1.52m$ and $0 \leq y \leq 1.68m$. The order of magnitude for the velocity field is $O(1)$. The Reynolds number of the flow is $Re = 76725$ and the Prandtl number $Pr = 0.729$. This makes $\frac{1}{PrRe} \approx O(10^{-5})$, and hence the zero diffusion constant assumption 3 made in this paper is justified. The Reynolds number for the flow rate is in turbulent range. The $k-\varepsilon$ model, which is Reynolds Average Navier-Stokes (RANS) model [13] is used to obtain the velocity field as shown in Fig. 1. A commercial CFD software Fluent was used to solve the coupled set of governing equations for pressure, temperature, turbulent kinetic energy, turbulent dissipation and velocity. No slip boundary condition was applied at all the walls. For the purposes of computation, we employ set oriented numerical methods for the approximation of P-F semigroup $\mathbb{P}_t$ [14]. The computational results for this section are obtained with actuators and sensors located at three different sets B_1, B_2, and B_3. The locations of these three sets are shown in Fig. 1b.

In Fig. 2b and Fig. 3a, we show the plots for the support of the controllability gramian after 10000 time steps corresponding to two different locations of actuator sets B_2 and B_3 respectively. The support of the controllability gramian corresponding to B_2 and B_3 locations of actuator sets is approximately the same and equals 1.6. However the 2-norm of the gramian corresponding to actuator location on set B_2 is equal to 38, while for B_3 it is equal to 35. Comparing figures 2a, 2b, and 3a, we see that the support of the gramian for actuator location at set B_1 is considerably smaller but it has considerably larger 2-norm compared to



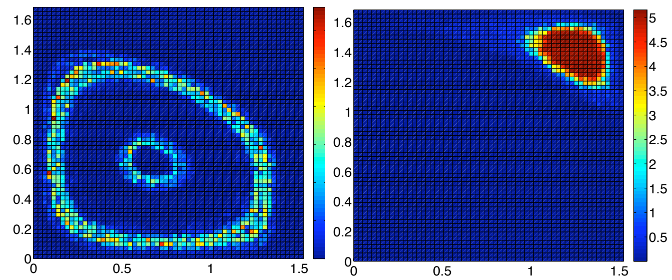
(a) (b)

Fig. 1. (a) Velocity field; b) Actuator locations on sets B_1, B_2 and B_3.



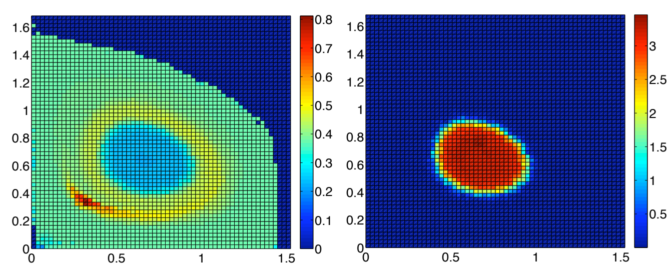
(a) (b)

Fig. 2. Controllability gramian after 10000 time iterations for actuator located at set a) B_1; b) B_2.



(a) (b)

Fig. 3. a) Controllability gramian after 10000 time iterations for actuator located at set B_3; b) Observability gramian after 1000 iterations with sensor location at B_1



(a) (b)

Fig. 4. Observability gramian after 1000 iterations with sensor location at a) B_2; b) B_3.

actuators locations at B_2 and B_3. The large value of gramian with small support in Fig. 1b can be very effective if one desires to perform localized control action. Comparing the support and the 2-norm of the gramian function, one can conclude that the actuator location corresponding to B_2 is optimal among B_1, B_2, and B_3.

VI. Conclusion

In this paper, controllability and observability gramian based test criteria is used to decide the suitability of given actuators and sensor location. As compared to currently existing purely computational based methods , our proposed approach provides a systematic and insightful method for deciding the sensors and actuators location in building system. In particular the ergodic properties of the vector field are shown to play an important role in actuator and sensor placement. In our future research work, the explicit formula for the gramians will be exploited to provide systematic algorithm for determining the optimal location of sensors and actuators. Furthermore some of the assumptions made in the derivation of control equations will be removed in future.

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