2011 50th IEEE Conference on Decision and Control and
European Control Conference (CDC-ECC)
Orlando, FL, USA, December 12-15, 2011

Deterministic Team Problems with Signaling

Ather Gattami

***Abstract*— This paper considers linear quadratic team decision problems where the players in the team affect each others information structure through their decisions. It shows that linear decisions are optimal and can be found by solving a linear matrix inequality.**

***Index Terms*— Team Decision Theory, Game Theory, Convex Optimization.**

I. INTRODUCTION

The team problem is an optimization problem, where a number of decision makers (or players) make up a team, optimizing a common cost function with respect to some uncertainty representing *nature*. Each member of the team has limited information about the global state of nature. Furthermore, the team members could have different pieces of information, which makes the problem different from the one considered in classical optimization, where there is only one decision function that has access to the entire information available about the state of nature.

Team problems seemed to possess certain properties that were considerably different from standard optimization, even for specific problem structures such as the optimization of a quadratic cost in the state of nature and the decisions of the team members. In stochastic linear quadratic decision theory, it was believed for a while that separation holds between estimation and optimal decision with complete information, even for team problems. The separation principle can be briefly explained as follows. First assume that every team member has access to the information about the entire state of nature, and find the corresponding optimal decision for each member. Then, each member makes an estimate of the state of nature, which is in turn combined with the optimal decision obtained from the full information assumption. It turns out that this strategy does *not* yield an optimal solution (see [9]).

A general solution to similar stochastic quadratic team problems was presented by Radner [9]. Radner's result gave hope that some related problems of dynamic nature could be solved using similar arguments. But in 1968, Witsenhausen [11] showed in his well known paper that finding the optimal decision can be complex if the decision makers affect each other's information. Witsenhausen considered a dynamic decision problem over two time steps to illustrate that difficulty. The dynamic problem can actually be written as a static team problem:

$$\begin{aligned} &\text{minimize } \mathbf{E}\ \left\{k_0 u_0^2 + (x+u_0-u_1)^2\right\} \\ &\text{subject to } u_0 = \mu_0(x),\ \ u_1 = \mu_1(x+u_0+w), \end{aligned}$$

where x and w are Gaussian with zero mean and variance X and W, respectively. Here, we have two decision makers, one corresponding to u_0, and the other to u_1. Witsenhausen showed that the optimal decisions μ_0 and μ_1 are not linear because of the *coding incentive* of u_0. Decision maker u_1 measures $x+u_0+w$, and hence, its measurement is affected by u_0. Decision maker u_0 tries to *encode* information about x in its decision, which makes the optimal strategy complex. The problem above is actually an information theoretic problem. To see this, consider the slightly modified problem

$$\begin{aligned} &\text{minimize } \mathbf{E}\ (x-u_1)^2 \\ &\text{subject to } u_0 = \mu_0(x),\ \ \mathbf{E}\ u_0^2 \le 1,\ \ u_1 = \mu_1(u_0+w) \end{aligned}$$

The modification made is that we removed u_0 from the objective function, and instead added a constraint $\mathbf{E}\ u_0^2 \le 1$ to make sure that it has a limited variance (of course we could set an arbitrary power limitation on the variance). The modified problem is exactly the Gaussian channel coding/decoding problem (see Figure 1)! The optimal solution to Witsenhausen's counterexample is still unknown. Even if we would restrict the optimization problem to the set of linear decisions, there is still no known polynomial-time algorithm to find optimal solutions.

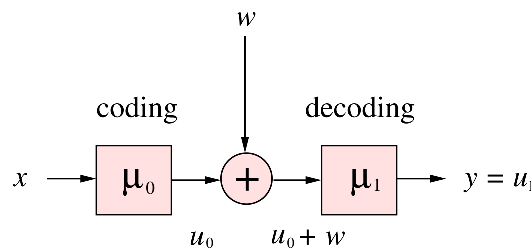

Fig. 1. Coding-decoding diagram over a Gaussian channel.

In this paper, we consider the problem of distributed decision making with information constraints under linear quadratic settings. For instance, information constraints appear naturally when making decisions over networks. These problems can be

Ather Gattami is with the Automatic Control Laboratory, School of Electrical Engineering, KTH, 100 44, Stockholm, Sweden. E-mail: gattami@kth.se.

formulated as team problems. Early results considered static team theory in stochastic settings [8], [9], [6]. In [3], the team problem with two team members was solved. The solution cannot be easily extended to more than two players since it uses the fact that the two members have common information; a property that doesn't necessarily hold for more than two players. Also, a nonlinear team problem with two team members was considered in [1], where one of the team members is assumed to have full information whereas the other member has only access to partial information about the state of the world. Related team problems with exponential cost criterion were considered in [7]. Optimizing team problems with respect to *affine* decisions in a minimax quadratic cost was shown to be equivalent to stochastic team problems with exponential cost, see [4]. The connection is not clear when the optimization is carried out over nonlinear decision functions. In [5], a general solution was given for an arbitrary number of team members, where linear decision were shown to be optimal and can be found by solving a linear matrix inequality. In the deterministic version of Witsenhausen's counterexample, that is minimizing the quadratic cost with respect to the worst case scenario of the state x (instead of the assumption that x is Gaussian), the linear decisions where shown to be optimal in [10].

We will show that for static linear quadratic minimax team problems in general, where the players in the team affect each others information structure through their decisions, linear decisions are optimal, and can be found by solving a linear matrix inequality.

II. Notation

$\mathbb{S}^n$	The set of $n \times n$ symmetric matrices.
$\mathbb{S}^n_+$	The set of $n \times n$ symmetric positive semidefinite matrices.
$\mathbb{S}^n_{++}$	The set of $n \times n$ symmetric positive definite matrices.
$\mathcal{C}$	The set of functions $\mu : \mathbb{R}^p \to \mathbb{R}^m$ with $\mu(y) = (\mu_1(y_1), \mu_2(y_2), ..., \mu_N(y_N))$, $\mu_i : \mathbb{R}^{p_i} \to \mathbb{R}^{m_i}$, $\sum_i m_i = m$, $\sum_i p_i = p$.
$\mathbb{K}$	$\{K \in \mathbb{R}^{m \times p} \mid K = \oplus \sum K_i, K_i \in \mathbb{R}^{m_i \times p_i}\}$
$A^\dagger$	Denotes the pseudo-inverse of the square matrix A.
$A_\perp$	Denotes the matrix orthogonal to A.
A_i	The ith block row of the matrix A.
A_{ij}	The block element of A in position (i,j).
$\succeq$	$A \succeq B \iff A - B \in \mathbb{S}^n_+$.
$\succ$	$A \succ B \iff A - B \in \mathbb{S}^n_{++}$.
Tr	$\mathbf{Tr}[A]$ is the trace of the matrix A.
$\mathcal{N}(m, X)$	The set of Gaussian variables with mean m and covariance X.

III. Stochastic Team Decision Theory

In this section we will review some classical results in stochastic team theory. In the stochastic team decision problem, one would like to solve

$$\begin{aligned} \min_{\mu} \; & \mathbf{E} \begin{pmatrix} x \\ u \end{pmatrix}^T \begin{pmatrix} Q_{xx} & Q_{xu} \\ Q_{ux} & Q_{uu} \end{pmatrix} \begin{pmatrix} x \\ u \end{pmatrix} \\ \text{subject to } & y_i = C_i x \\ & u_i = \mu_i(y_i) \\ & \text{for } i = 1, ..., N, \end{aligned} \tag{1}$$

where $x \sim \mathcal{N}(0, V_{xx})$, u_i takes values in $\mathbb{R}^{m_i}$, $C_i \in \mathbb{R}^{p_i \times n}$, for $i = 1, ..., N$. We assume that $Q_{uu} \in \mathbb{S}^m_{++}$, $m = m_1 + \cdots + m_N$, and

$$\begin{pmatrix} Q_{xx} & Q_{xu} \\ Q_{ux} & Q_{uu} \end{pmatrix} \in \mathbb{S}^{m+n}. \tag{2}$$

$\mu(\cdot)$ is a function, which represents the *decision function* or *decision* of the team.

The following result by Radner [9] shows that linear decisions are optimal:

Proposition 1: Let x be a Gaussian variables with zero mean, taking values in $\mathbb{R}^n$. Also, let u_i be a stochastic variable taking values in $\mathbb{R}^{m_i}$, $Q_{uu} \in \mathbb{S}^m_{++}$, $m = m_1 + \cdots + m_N$, $C_i \in \mathbb{R}^{p_i \times n}$, for $i = 1, ..., N$. Then, the optimal decision μ to the optimization problem

$$\begin{aligned} \min_{\mu} \; & \mathbf{E} \begin{pmatrix} x \\ u \end{pmatrix}^T \begin{pmatrix} Q_{xx} & Q_{xu} \\ Q_{ux} & Q_{uu} \end{pmatrix} \begin{pmatrix} x \\ u \end{pmatrix} \\ \text{subject to } & y_i = C_i x \\ & u_i = \mu_i(y_i) \\ & \text{for } i = 1, ..., N, \end{aligned}$$

is unique and linear in y.

Proof: Consult Radner [9]. ■

IV. Main Results

The deterministic problem considered is a quadratic game between a team of players and nature. Each player has limited information that could be different from the other players in the team. This game is formulated as a minimax problem, where the team is the minimizer and nature is the maximizer. We show that if there is a solution to the static minimax team problem, then linear decisions are optimal, and we show how to find a linear optimal solution by solving a linear matrix inequality.

A. Deterministic Team Decision Problems with a Quadratic Fractional Objective

Consider the following team decision problem

$$
\begin{aligned}
\inf_{\mu} \sup_{0 \neq x \in \mathbb{R}^n} \quad & \frac{J(x,u)}{F(x,u)} \\
\text{subject to} \quad & y_i = C_i x \\
& u_i = \mu_i(y_i) \\
& \text{for } i = 1, ..., N
\end{aligned} \tag{3}
$$

where $u_i \in \mathbb{R}^{m_i}$, $m = m_1 + \cdots + m_N$, $C_i \in \mathbb{R}^{p_i \times n}$. We will consider the case where $J(x,u)$ and $F(x,u)$ are quadratic costs given by

$$
J(x,u) = \begin{pmatrix} x \\ u \end{pmatrix}^T Q \begin{pmatrix} x \\ u \end{pmatrix}, \quad F(x,u) = \begin{pmatrix} x \\ u \end{pmatrix}^T R \begin{pmatrix} x \\ u \end{pmatrix}, \tag{4}
$$

$Q, R \in \mathbb{R}^{(m+n)\times(m+n)}$. The players u_1,..., u_N make up a *team*, which plays against *nature* represented by the vector x, using $\mu \in \mathcal{C}$, that is

$$
\mu(Cx) = \begin{pmatrix} \mu_1(C_1 x) \\ \vdots \\ \mu_N(C_N x) \end{pmatrix}.
$$

The next theorem shows that if the game above has a finite value, then there are linear decisions achieving that value. That is, an optimal decision function $\mu \in \mathcal{C}$ can be taken to be linear:

$$
\mu_i(y_i) = K_i y_i,
$$

for $i = 1, ..., N$.

Theorem 1: Let $J(x,u)$ and $F(x,u)$ be quadratic forms given by (4) with $F(x,u) > 0$ for all $x \neq 0$, and suppose that

$$
\inf_{\mu \in \mathcal{C}} \sup_{0 \neq x \in \mathbb{R}^n} \frac{J(x, \mu(Cx))}{F(x, \mu(Cx))} = \gamma^\star < \infty.
$$

Then, there exist linear decisions $\mu_i(C_i x) = K_i C_i x$, $i = 1, ..., N$, where the value of the game $\gamma^\star$ is achieved.

Proof: See Appendix. ■

B. Computation of Optimal Team Decisions

In Theorem 1, we showed that for the minimax team problem with a quadratic cost, the linear policy $u = KCx$ is optimal, where K is given by $K = \mathrm{diag}(K_1, ..., K_N)$. We will now show how to pose the problem of finding a structured matrix K, where the same technique as that of the proof Theorem 1 will be used. For a given value of γ, we would like to find a $K \in \mathbb{K}$ such that

$$
\frac{\begin{pmatrix} x \\ KCx \end{pmatrix}^T Q \begin{pmatrix} x \\ KCx \end{pmatrix}}{\begin{pmatrix} x \\ KCx \end{pmatrix}^T R \begin{pmatrix} x \\ KCx \end{pmatrix}} \le \gamma,
$$

for all x.

Theorem 2: The feasibility problem

$$
\begin{aligned}
\text{find} \quad & K = \mathrm{diag}(K_1, ..., K_N) \\
\text{subject to} \quad & \frac{\begin{pmatrix} x \\ KCx \end{pmatrix}^T Q \begin{pmatrix} x \\ KCx \end{pmatrix}}{\begin{pmatrix} x \\ KCx \end{pmatrix}^T R \begin{pmatrix} x \\ KCx \end{pmatrix}} \le \gamma,
\end{aligned}
$$

is equivalent to the set of linear matrix inequalities

$$
\text{find } K = \mathrm{diag}(K_1, ..., K_N)
$$

$$
\begin{pmatrix} Z_{xx} & Z_{xu} \\ Z_{ux} & Z_{uu} \end{pmatrix} = Q - \gamma R, \quad Z_{xx} \in \mathbb{R}^{n \times n}
$$

$$
\begin{pmatrix} M_{ww} & M_{wy} & M_{wu} \\ M_{yw} & M_{yy} & M_{yu} \\ M_{uw} & M_{uy} & M_{uu} \end{pmatrix} =
$$

$$
\begin{pmatrix} C_\perp^T Z_{xx} C_\perp & C_\perp^T Z_{xx} C^\dagger & C_\perp^T Z_{xu} \\ (C^\dagger)^T Z_{xx} C_\perp & (C^\dagger)^T Z_{xx} C^\dagger & (C^\dagger)^T Z_{xu} \\ Z_{ux} C_\perp & Z_{ux} C^\dagger & Z_{uu} \end{pmatrix}
$$

$$
\begin{aligned}
Q_\gamma &= \begin{pmatrix} Q_{xx}(\gamma) & Q_{xu}(\gamma) \\ Q_{ux}(\gamma) & Q_{uu}(\gamma) \end{pmatrix} \\
Q_{xx}(\gamma) &= C^T (M_{yy} - M_{yw} M_{ww}^\dagger M_{wy}) C \\
Q_{ux}(\gamma) &= (M_{uy} - M_{uw} M_{ww}^\dagger M_{wy}) C \\
Q_{uu}(\gamma) &= M_{uu} - M_{uw} M_{ww}^\dagger M_{uw}.
\end{aligned}
$$

$$
\begin{pmatrix} Q_{xx}(\gamma) + Q_{xu}(\gamma) KC + C^T K^T Q_{ux}(\gamma) & C^T K^T \\ KC & -Q_{uu}^\dagger(\gamma) \end{pmatrix} \preceq 0.
$$

Proof: See Appendix. ■

Example 1: Consider the min-max team problem

$$
\inf_{\mu_1, \mu_2} \sup_{0 \neq x \in \mathbb{R}^3} \frac{J(x, \mu_1, \mu_2)}{\|x\|^2},
$$

where

$$
J(x, \mu_1, \mu_2) = (\mu_1(x_1 + x_3) + \mu_2(x_2 + x_3) - x_3)^2 + 4\mu_2^2(x_2 + x_3).
$$

Comparing with the problem formulation in (3), we get

$$
Q = \begin{pmatrix} 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 1 & -1 & -1 \\ 0 & 0 & -1 & 1 & 1 \\ 0 & 0 & -1 & 1 & 5 \end{pmatrix},
$$

$R_{xx} = I$, $R_{xu} = 0$, $R_{uu} = 0$, $C_1 = \begin{pmatrix} 1 & 0 & 1 \end{pmatrix}^T$, $C_2 = \begin{pmatrix} 0 & 1 & 1 \end{pmatrix}$. From Theorem 1, we know that it is enough to look for linear decision functions $\mu_i(x_i + x_3) = k_i(x_i + x_3)$, $i = 1, 2$. Hence, we pose it as an LMI, and obtain that an optimal solution is $k_1 = 0.5$, $k_2 = 0.2$, and the value of the game is $\gamma = 0.4$.

V. Deterministic Team Problems with Signaling

Consider the following team decision problem

$$
\begin{aligned}
\inf_{\mu} \sup_{v\in\mathbb{R}^p, 0\neq w\in\mathbb{R}^q} & \frac{L(w,u)}{||v||^2+\|w\|^2} \\
\text{subject to } & y_i = \sum_{j=1}^{N} D_{ij}u_j + E_i w + v_i \qquad (5)\\
& u_i = \mu_i(y_i)\\
& \text{for } i=1,...,N,
\end{aligned}
$$

where $u_i \in \mathbb{R}^{m_i}$ and $E_i \in \mathbb{R}^{p_i\times q}$, for $i=1,...,N$, $p = p_1+\cdots+p_N$.
$L(w,u)$ is a quadratic cost given by

$$L(w,u) = \begin{pmatrix} w\\ u\end{pmatrix}^T \begin{pmatrix} Q_{ww} & Q_{wu}\\ Q_{uw} & Q_{uu}\end{pmatrix}\begin{pmatrix} w\\ u\end{pmatrix},$$

$Q_{uu} \in \mathbb{S}^m_+$, $m = m_1+\cdots+m_N$, and

$$\begin{pmatrix} Q_{ww} & Q_{wu}\\ Q_{uw} & Q_{uu}\end{pmatrix} \in \mathbb{S}^{m+n}.$$

The players u_1,..., u_N make up a *team*, which plays against *nature* represented by the vector w, using $\mu \in \mathcal{C}$. This problem is more complicated than the static team decision problem studied in [5], since it has the same flavor as that of Witsenhausen's counterexample that was presented in the introduction. We see that the measurement y_i of decision maker i could be affected by the other decision makers through the terms $D_{ij}u_j$, $j=1,...,N$.

Theorem 3: Assume that the value of the game (5) is finite, $\gamma^\star < \infty$. Then, there exist linear decisions $\mu_i(y_i) = K_i y_i$, $i=1,...,N$, where the value $\gamma^\star$ is achieved. For any value of the game $\gamma \geq \gamma^\star$, a feasible linear decision Ky, $K\in\mathbb{K}$, that achieves γ can be obtained by solving the linear matrix inequality

$$
\begin{aligned}
\text{find } & K\\
\text{subject to } & K = \text{diag}(K_1,...,K_N)\\
& C = \begin{pmatrix} I & 0\end{pmatrix} \in \mathbb{R}^{p\times(p+q)}, \quad Z_{uu}\in\mathbb{R}^{m\times m}
\end{aligned}
$$

$$
\begin{pmatrix} Z_{xx} & Z_{xu}\\ Z_{ux} & Z_{uu}\end{pmatrix} = \begin{pmatrix} Q_{ww} & 0 & Q_{wu}\\ 0 & 0 & 0\\ Q_{uw} & 0 & Q_{uu}\end{pmatrix} - \gamma \begin{pmatrix} E^TE & -E^T & -E^TD\\ -E & I & -D\\ -D^TE & -D^T & D^TD\end{pmatrix}
$$

$$
\begin{pmatrix} M_{ww} & M_{wy} & M_{wu}\\ M_{yw} & M_{yy} & M_{yu}\\ M_{uw} & M_{uy} & M_{uu}\end{pmatrix} =
\begin{pmatrix} C_\perp^T Z_{xx} C_\perp & C_\perp^T Z_{xx} C^\dagger & C_\perp^T Z_{xu}\\ (C^\dagger)^T Z_{xx} C_\perp & (C^\dagger)^T Z_{xx} C^\dagger & (C^\dagger)^T Z_{xu}\\ Z_{ux}C_\perp & Z_{ux}C^\dagger & Z_{uu}\end{pmatrix}
$$

$$
\begin{pmatrix} Q_{xx}(\gamma) + Q_{xu}(\gamma)KC + C^TK^TQ_{ux}(\gamma) & C^TK^T\\ KC & -Q_{uu}^\dagger(\gamma)\end{pmatrix} \preceq 0,
$$

$$
\begin{aligned}
Q_\gamma &= \begin{pmatrix} Q_{xx}(\gamma) & Q_{xu}(\gamma)\\ Q_{ux}(\gamma) & Q_{uu}(\gamma)\end{pmatrix}\\
Q_{xx}(\gamma) &= C^T(M_{yy} - M_{yw}M_{ww}^\dagger M_{wy})C\\
Q_{ux}(\gamma) &= (M_{uy} - M_{uw}M_{ww}^\dagger M_{wy})C\\
Q_{uu}(\gamma) &= M_{uu} - M_{uw}M_{ww}^\dagger M_{uw}.
\end{aligned}
$$

Proof: First, note that

$$
\begin{aligned}
& y = Du + Ew + v \iff v = y - Du - Ew \Rightarrow\\
& \Rightarrow \frac{L(w,u)}{||v||^2+\|w\|^2} = \frac{L(w,u)}{||y-Du-Ew||^2+\|w\|^2}.
\end{aligned}
$$

Now introduce $x\in\mathbb{R}^n$, $n = p+q$, such that

$$x = \begin{pmatrix} w\\ y\end{pmatrix},$$

and

$$
\begin{aligned}
Q &= \begin{pmatrix} Q_{ww} & 0 & Q_{wu}\\ 0 & 0 & 0\\ Q_{uw} & 0 & Q_{uu}\end{pmatrix},\\
R &= \begin{pmatrix} E^TE & -E^T & -E^TD\\ -E & I & -D\\ -D^TE & -D^T & D^TD\end{pmatrix}.
\end{aligned} \qquad (6)
$$

Then,

$$
\begin{aligned}
J(x,u) &:= \begin{pmatrix} x\\ u\end{pmatrix}^T Q \begin{pmatrix} x\\ u\end{pmatrix} = L(w,u),\\
F(x,u) &:= \begin{pmatrix} x\\ u\end{pmatrix}^T R \begin{pmatrix} x\\ u\end{pmatrix} = ||y-Du-Ew||^2 + \|w\|^2,
\end{aligned}
$$

and $y = Cx$. Hence, we have that

$$\frac{L(w,u)}{||v||^2+\|w\|^2} = \frac{L(w,u)}{||y-Du-Ew||^2+\|w\|^2} = \frac{J(x,u)}{F(x,u)}.$$

Also, $w\neq 0$ implies that $x\neq 0$ and $F(x,u)>0$. Thus, the game in (5) can be formulated as

$$\inf_{\mu\in\mathcal{C}} \sup_{0\neq x\in\mathbb{R}^n} \frac{J(x,\mu(Cx))}{F(x,\mu(Cx))}.$$

Applying Theorem 1, we conclude that linear decisions are optimal, which proves the first part of our theorem. Theorem 2 now gives that the minimizing γ subject to linear decision functions can be posed as a quasi-convex optimization problem given by (3), with respect to Q and R in equation (6), and the proof is complete. ∎

VI. Linear Quadratic Control with Arbitrary Information Constraints

Consider the dynamic team decision problem

$$
\begin{aligned}
\inf_{\mu} \sup_{w,v\neq 0} & \frac{\sum_{k=1}^M \begin{pmatrix} x(k)\\ u(k)\end{pmatrix}^T \begin{pmatrix} Q_{xx} & Q_{xu}\\ Q_{ux} & Q_{uu}\end{pmatrix}\begin{pmatrix} x(k)\\ u(k)\end{pmatrix}}{\sum_{k=1}^M \|w(k)\|^2 + \|v(k)\|^2}\\
\text{subject to } & x(k+1) = Ax(k) + Bu(k) + w(k)\\
& y_i(k) = C_i x(k) + v_i(k)\\
& u_i(k) = [\mu_k]_i(y_i(k)), i=1,...,N.
\end{aligned} \qquad (7)
$$

Now write $x(k)$ and $y(k)$ as

$$x(k) = \sum_{k=1}^{M} A^k Bu(M-k) + \sum_{k=1}^{M} A^k w(M-k),$$
$$y_i(k) = \sum_{k=1}^{M} C_i A^k Bu(M-k) + \sum_{k=1}^{M} C_i A^k w(M-k) + v_i(k).$$

It is easy to see that the optimal control problem above is equivalent to a static team problem of the form (5). Thus, linear controllers are optimal.

VII. Conclusions

We have considered the static team problem in deterministic linear quadratic settings where the team members may affect each others information. We have shown that decisions that are linear in the observations are optimal and can be found by solving a linear matrix inequality.

VIII. Acknowledgements

The author is grateful to Professor Bo Bernhardsson for valuable comments and suggestions.

This work is supported by the Swedish Research Council.

References

[1] P. Bernhard and N. Hovakimyan. Nonlinear robust control and minimax team problems. *International Journal of Robust and Nonlinear Control*, 9(9):239–257, 1999.
[2] S. Boyd and L. Vandenberghe. *Convex Optimization*. Cambridge University Press, 2004.
[3] G. Didinsky and T. Basar. Minimax decentralized controllers for discrete-time linear systems. In *41st Conference on Decision and Control*, 2002.
[4] C. Fan, J. L. Speyer, and C. R. Jaensch. Centralized and decentralized solutions of the linear-exponential-gaussian problem. *IEEE Trans. on Automatic Control*, 39(10):1986–2003, 1994.
[5] A. Gattami, B. Bernhardsson, and A. Rantzer. Robust team decision theory. *IEEE Tran. Automatic Control*, Accepted.
[6] Y.-C. Ho and K.-C. Chu. Team decision theory and information structures in optimal control problems-part i. *IEEE Trans. on Automatic Control*, 17(1), 1972.
[7] J. Krainak, J. L. Speyer, and S. I. Marcus. Static team problems-part i. *IEEE Trans. on Automatic Control*, 27(4):839–848, 1982.
[8] J. Marschak. Elements for a theory of teams. *Management Sci.*, 1:127–137, 1955.
[9] R. Radner. Team decision problems. *Ann. Math. Statist.*, 33(3):857–881, 1962.
[10] M. Rotkowitz. Linear controllers are uniformly optimal for the Witsenhausen counterexample. In *IEEE Conference on Decision and Control*, pages 553–558, 2006.
[11] H. S. Witsenhausen. A counterexample in stochastic optimum control. *SIAM Journal on Control*, 6(1):138–147, 1968.

Appendix

A. Proof of Theorem 1

Proof: From the definition of the infimum, it follows that if $\gamma^\star$ is the value of the game, then for any given real number $\gamma > \gamma^\star$, there exists a decision $\mu \in \mathcal{C}$ such that

$$J(x, \mu(Cx)) - \gamma F(x, \mu(Cx)) \leq 0,$$

for all x. Introduce the matrix $Z = Z(\gamma) = Q - \gamma R$, partitioned according to

$$Z = \begin{pmatrix} Z_{xx} & Z_{xu} \\ Z_{ux} & Z_{uu} \end{pmatrix},$$

where $Z_{xx} \in \mathbb{R}^{n \times n}$. Then for every $\gamma > \gamma^\star$, there is a decision $\mu \in \mathcal{C}$ such that

$$\begin{pmatrix} x \\ \mu(Cx) \end{pmatrix}^T Z \begin{pmatrix} x \\ \mu(Cx) \end{pmatrix} \leq 0, \tag{8}$$

for all x. Write $x = C_\perp w + C^\dagger y$, which implies that $y = Cx$ will serve as the observable part of x to the decision function μ. Then we have that

$$\begin{pmatrix} x \\ u \end{pmatrix}^T Z \begin{pmatrix} x \\ u \end{pmatrix} =$$
$$\begin{pmatrix} w \\ y \\ u \end{pmatrix}^T \begin{pmatrix} C_\perp & C^\dagger & 0 \\ 0 & 0 & I \end{pmatrix}^T Z \begin{pmatrix} C_\perp & C^\dagger & 0 \\ 0 & 0 & I \end{pmatrix} \begin{pmatrix} w \\ y \\ u \end{pmatrix}.$$

To simplify the exposition, define

$$\begin{aligned} M :&= \begin{pmatrix} C_\perp & C^\dagger & 0 \\ 0 & 0 & I \end{pmatrix}^T Z \begin{pmatrix} C_\perp & C^\dagger & 0 \\ 0 & 0 & I \end{pmatrix} \\ &= \begin{pmatrix} C_\perp^T Z_{xx} C_\perp & C_\perp^T Z_{xx} C^\dagger & C_\perp^T Z_{xu} \\ (C^\dagger)^T Z_{xx} C_\perp & (C^\dagger)^T Z_{xx} C^\dagger & (C^\dagger)^T Z_{xu} \\ Z_{ux} C_\perp & Z_{ux} C^\dagger & Z_{uu} \end{pmatrix} \\ &= \begin{pmatrix} M_{ww} & M_{wy} & M_{wu} \\ M_{yw} & M_{yy} & M_{yu} \\ M_{uw} & M_{uy} & M_{uu} \end{pmatrix}. \end{aligned} \tag{9}$$

The inequality (8) is equivalent to

$$\begin{pmatrix} w \\ y \\ u \end{pmatrix}^T M \begin{pmatrix} w \\ y \\ u \end{pmatrix} \leq 0.$$

Since w is unobservable for μ, we must have

$$C_\perp^T Z_{xx} C_\perp = M_{ww} \preceq 0, \tag{10}$$

$$\begin{aligned} &\left(I - C_\perp^T Z_{xx} C_\perp (C_\perp^T Z_{xx} C_\perp)^\dagger\right) \begin{pmatrix} C_\perp^T Z_{xx} C^\dagger & C_\perp^T Z_{xu} \end{pmatrix} = \\ &\left(I - M_{ww} M_{ww}^\dagger\right) \begin{pmatrix} M_{wy} & M_{wu} \end{pmatrix} = 0, \end{aligned} \tag{11}$$

otherwise we can always find a w such that

$$\begin{pmatrix} w \\ y \\ u \end{pmatrix}^T M \begin{pmatrix} w \\ y \\ u \end{pmatrix} > 0,$$

for all u (see e. g. [2], pp. 650-651). If the two conditions (10) and (11) hold, then by standard completion of squares, we get

$$\begin{pmatrix} w \\ y \\ u \end{pmatrix}^T M \begin{pmatrix} w \\ y \\ u \end{pmatrix} = (w - v)^T M_{ww} (w - v) +$$
$$\begin{pmatrix} y \\ u \end{pmatrix}^T \left(\begin{pmatrix} M_{yy} & M_{yu} \\ M_{uy} & M_{uu} \end{pmatrix} - \begin{pmatrix} M_{yw} \\ M_{uw} \end{pmatrix} M_{ww}^\dagger \begin{pmatrix} M_{yw} \\ M_{uw} \end{pmatrix}^T \right) \begin{pmatrix} y \\ u \end{pmatrix}$$

with

$$v = M_{ww}^{\dagger} \begin{pmatrix} M_{wy} & M_{wu} \end{pmatrix} \begin{pmatrix} y \\ u \end{pmatrix}.$$

Considering the completed form above, and substituting $y = Cx$, we get

$$\sup_w \begin{pmatrix} w \\ y \\ u \end{pmatrix}^T M \begin{pmatrix} w \\ y \\ u \end{pmatrix} = \begin{pmatrix} x \\ u \end{pmatrix}^T Q_\gamma \begin{pmatrix} x \\ u \end{pmatrix},$$

where

$$\begin{aligned} Q_\gamma &= \begin{pmatrix} Q_{xx}(\gamma) & Q_{xu}(\gamma) \\ Q_{ux}(\gamma) & Q_{uu}(\gamma) \end{pmatrix} \\ Q_{xx}(\gamma) &= C^T(M_{yy} - M_{yw}M_{ww}^{\dagger}M_{wy})C \\ Q_{ux}(\gamma) &= (M_{uy} - M_{uw}M_{ww}^{\dagger}M_{wy})C \\ Q_{uu}(\gamma) &= M_{uu} - M_{uw}M_{ww}^{\dagger}M_{uw}. \end{aligned} \tag{12}$$

If $Q_{uu}(\gamma)$ is indefinite, then the inequality above is trivial. Therefore, we will assume that there is a $\gamma > \gamma^\star$ for which $Q_{uu}(\gamma) \succ 0$; otherwise the value of the game is $\gamma^\star = -\infty$.

Now for any Gaussian variable x, we have that

$$\mathbf{E} \begin{pmatrix} x \\ \mu(Cx) \end{pmatrix}^T Q_\gamma \begin{pmatrix} x \\ \mu(Cx) \end{pmatrix} \leq 0.$$

Proposition 1 implies that for every Gaussian variable $x \sim \mathcal{N}(0, X)$, the optimal decision $\mu(Cx)$ is linear $\mu(Cx) = K_X Cx$ with $K_X \in \mathbb{K}$. That is

$$\mathbf{E}\left\{ x^T \begin{pmatrix} I \\ K_X C \end{pmatrix}^T Q_\gamma \begin{pmatrix} I \\ K_X C \end{pmatrix} x \right\} \leq 0.$$

Since $\mathbf{E}\{xx^T\} = X$, we get

$$\begin{aligned} 0 &\geq \mathbf{E}\left\{ x^T \begin{pmatrix} I \\ K_X C \end{pmatrix}^T Q_\gamma \begin{pmatrix} I \\ K_X C \end{pmatrix} x \right\} \\ &= \mathbf{Tr} \begin{pmatrix} I \\ K_X C \end{pmatrix}^T Q_\gamma \begin{pmatrix} I \\ K_X C \end{pmatrix} X. \end{aligned}$$

Hence, for every $X \succeq 0$, there is a $K_X \in \mathbb{K}$ such that

$$0 \geq \mathbf{Tr} \begin{pmatrix} I \\ K_X C \end{pmatrix}^T Q_\gamma \begin{pmatrix} I \\ K_X C \end{pmatrix} X. \tag{13}$$

Now introduce the compact set

$$\mathbb{X} = \{X : X \succeq 0, \mathbf{Tr}\{X\} = 1\}.$$

The fact that for every covariance matrix X there is a matrix K_X such that (13) holds implies

$$\alpha := \max_{X \in \mathbb{X}} \min_{K \in \mathbb{K}} \mathbf{Tr} \begin{pmatrix} I \\ KC \end{pmatrix}^T Q_\gamma \begin{pmatrix} I \\ KC \end{pmatrix} X \leq 0.$$

The above max-min problem is convex in K and linear (hence concave and continuous) in X. Thus, there is a saddle point $(K_\gamma, X_\gamma) \in \mathbb{K} \times \mathbb{X}$, and

$$\min_{K \in \mathbb{K}} \max_{X \in \mathbb{X}} \mathbf{Tr} \begin{pmatrix} I \\ KC \end{pmatrix}^T Q_\gamma \begin{pmatrix} I \\ KC \end{pmatrix} X = \alpha \leq 0.$$

This implies that there must be a $K_\gamma \in \mathbb{K}$ such that

$$\begin{pmatrix} I \\ K_\gamma C \end{pmatrix}^T Q_\gamma \begin{pmatrix} I \\ K_\gamma C \end{pmatrix} \preceq \alpha I \preceq 0.$$

Since for each $\gamma > \gamma^\star$ there is K_γ such that the inequality above holds, then by compactness, it must hold for $\gamma = \gamma^\star$ and some $K^\star \in \mathbb{K}$. Hence, there is a $K^\star \in \mathbb{K}$ such that

$$x^T \begin{pmatrix} I \\ K^\star C \end{pmatrix}^T Q \begin{pmatrix} I \\ K^\star C \end{pmatrix} x \leq \gamma^\star x^T \begin{pmatrix} I \\ K^\star C \end{pmatrix}^T R \begin{pmatrix} I \\ K^\star C \end{pmatrix} x$$

for all x. This shows that there exists a linear decision $\mu(Cx) = K^\star Cx$ with $K^\star \in \mathbb{K}$, such that

$$\sup_{x \neq 0} \frac{\begin{pmatrix} x \\ K^\star Cx \end{pmatrix}^T Q \begin{pmatrix} x \\ K^\star Cx \end{pmatrix}}{\begin{pmatrix} x \\ K^\star Cx \end{pmatrix}^T R \begin{pmatrix} x \\ K^\star Cx \end{pmatrix}} \leq \gamma^\star.$$

Since the value of the game is $\gamma^\star$, the inequality above is an equality, and the proof is complete. ■

B. Proof of Theorem 2

Proof: Walking along the same lines as the proof of Theorem 1 we find that the inequality

$$\sup_{x \neq 0} \frac{\begin{pmatrix} x \\ K^\star Cx \end{pmatrix}^T Q \begin{pmatrix} x \\ K^\star Cx \end{pmatrix}}{\begin{pmatrix} x \\ KCx \end{pmatrix}^T R \begin{pmatrix} x \\ K^\star Cx \end{pmatrix}} \leq \gamma,$$

is equivalent to

$$\begin{pmatrix} x \\ KCx \end{pmatrix}^T \begin{pmatrix} Q_{xx}(\gamma) & Q_{xu}(\gamma) \\ Q_{ux}(\gamma) & Q_{uu}(\gamma) \end{pmatrix} \begin{pmatrix} x \\ KCx \end{pmatrix} \leq 0,$$

where $Q_{xx}(\gamma), Q_{xu}(\gamma), Q_{uu}(\gamma)$, are defined by (9) and (12). If $Q_{uu}(\gamma) \not\succeq 0$, the inequality above is trivial. Therefore, assume that $Q_{uu}(\gamma) \succeq 0$. Multiplying the factors in the inequality above yields

$$Q_{xx}(\gamma) + Q_{xu}(\gamma)KC + C^T K^T Q_{ux}(\gamma) + C^T K^T Q_{uu}(\gamma)KC \preceq 0 \tag{14}$$

Using the Schur complement, and the fact that $Q_{uu} \succeq 0$ the inequality in (14) can be written as a linear matrix inequality:

$$\begin{pmatrix} Q_{xx}(\gamma) + Q_{xu}(\gamma)KC + C^T K^T Q_{ux}(\gamma) & C^T K^T \\ KC & -Q_{uu}^{\dagger}(\gamma) \end{pmatrix} \preceq 0.$$

Hence, the search for a block diagonal K and minimal γ can be written as the following semidefinite program:

$$\begin{aligned} &\min_{\gamma, K} \; \gamma \\ &\text{subject to} \quad K = \text{diag}(K_1, ..., K_N) \\ &\begin{pmatrix} Q_{xx}(\gamma) + Q_{xu}(\gamma)KC + C^T K^T Q_{ux}(\gamma) & C^T K^T \\ KC & -Q_{uu}^{\dagger}(\gamma) \end{pmatrix} \preceq 0, \end{aligned}$$

and the proof is complete. ■