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Finite-State Simulations and Bisimulations for Discrete-Time Piecewise Affine Systems

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Abstract— **A procedure for the stability analysis of discrete-time piecewise affine systems is proposed, where a sequence of finite-state symbolic models is constructed such that each of these symbolic models simulates the given piecewise affine system. For a fairly large class of piecewise affine systems, the proposed procedure does not suffer from inherent conservatism, and thus the sequence of symbolic models converges to the original system. For a smaller, restricted class of systems, the sequence of symbolic models is finite, and hence a finite-state bisimulation of the original system is obtained.**

I. Introduction

Hybrid systems constitute an important modeling framework describing a large variety of situations where both discrete and continuous dynamics interact with each other [1], [2]. Important problems regarding hybrid systems include verification, reachability, and stability analysis problems [3], [4]. These problems are known to be NP hard or undecidable [5], and there remain major challenges in hybrid systems analysis despite the progress made over the past decade.

A common approach to solving verification and reachability problems is to build a finite-state symbolic model that abstracts the original infinite-state hybrid system [3], [6]–[8]. The main objective of such an approach is to discover a symbolic model, called a *bisimulation*, which possesses the same reachability properties as the original hybrid system. The existence of a bisimulation guarantees an efficient and exact reachability analysis, but only a small, restricted class of hybrid systems admits bisimulations. In the context of stability analysis, on the other hand, the usual approach is to construct a single or multiple quadratic Lyapunov functions, whose decay properties along the system trajectory can be used to deduce the stability of the original system [9]–[11]. However, not every hybrid system admits such Lyapunov functions. What is common in these approaches to reachability and stability analysis is that they are inherently conservative. That is, if a given hybrid system does not admit a bisimulation or common/multiple quadratic Lyapunov functions, these approaches do not suggest an alternative, potentially better, solution approach even if one is willing to pay more computational cost.

In this paper, we focus on stability analysis of discrete-time piecewise affine systems, which are a typical example of hybrid systems. We present a nonconservative stability analysis for a large class of such systems by following the lead taken in [12]–[15], where the connection between symbolic models and Lyapunov functions is discovered and exploited. The basic idea is to obtain a nested sequence of finite-state symbolic models that *simulate* the original piecewise affine system, so that the stability of each symbolic model within the sequence is sufficient not only for the stability of the next symbolic model within the sequence but also for the stability of the original piecewise affine system. Our algorithm can be considered as the classical bisimulation algorithm [6], [16] enhanced and specialized for the stability analysis of piecewise affine systems, and hence yields stronger results than general algorithms. The contributions of the paper are twofold:

- First, we show for a fairly large class of piecewise affine systems that our sequence of simulations does not suffer from inherent conservatism. That is, if one moves along this sequence, one symbolic model at a time, then in the limit one will obtain the largest subset of the state space within which the original system is stable.
- Next, we show for a smaller, restricted class of piecewise affine systems that our sequence of symbolic models is finite and hence a bisimulation is guaranteed to exist. The existence of a bisimulation was demonstrated for, e.g., timed automata [17], [18], O-minimal hybrid automata [19], and STORMED hybrid systems [20]. However, these systems define restricted classes of hybrid systems in the continuous-time domain.

Notation: The sets of real numbers, positive integers, and nonnegative integers are denoted by $\mathbb{R}$, $\mathbb{N}$, and $\mathbb{N}_0$, respectively. For vectors $\mathbf{x} \in \mathbb{R}^n$, denoted by $\|\mathbf{x}\|$ is the Euclidean norm of $\mathbf{x}$. For matrices $\mathbf{X} \in \mathbb{R}^{n\times n}$, we write $\mathbf{X} \prec \mathbf{0}$ to mean $\mathbf{X}$ is symmetric and negative definite.

II. Problem Formulation

A. Definitions: Stability of Piecewise Affine Systems

Given $\mathbf{A}_1, \ldots, \mathbf{A}_N \in \mathbb{R}^{n\times n}$ and $\mathbf{b}_1, \ldots, \mathbf{b}_N \in \mathbb{R}^n$, let

$$\mathcal{S} = \{(\mathbf{A}_1, \mathbf{b}_1), \ldots, (\mathbf{A}_N, \mathbf{b}_N)\}.$$

Let

$$\mathcal{D} = \{D_1, \ldots, D_N\}$$

be a partition of the state space $\mathbb{R}^n$ into N nonempty polyhedral cells; that is, each D_i, $i = 1, \ldots, N$, is convex, but not necessarily closed or bounded, polyhedron (i.e. an intersection of either closed or open half-spaces) such that $\bigcup_{i=1}^N D_i = \mathbb{R}^n$ and $D_i \cap D_j = \varnothing$ whenever $i \neq j$. Then the pair $(\mathcal{S}, \mathcal{D})$ defines the discrete-time *piecewise affine system* represented by

$$\mathbf{x}(t+1) = \mathbf{A}_{\theta(t)}\mathbf{x}(t) + \mathbf{b}_{\theta(t)}, \tag{2a}$$

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for $t \in \mathbb{N}_0$ and $\mathbf{x}(0) \in \mathbb{R}^n$, where the *switching sequence* $\theta = (\theta(0), \theta(1), \dots)$ is such that

$$\theta(t) = i \quad \text{if } \mathbf{x}(t) \in D_i. \tag{2b}$$

Each initial state $\mathbf{x}(0) \in \mathbb{R}^n$ generates a switching sequence $\theta = (\theta(0), \theta(1), \dots)$ and a state sequence $\mathbf{x} = (\mathbf{x}(0), \mathbf{x}(1), \dots)$, where $\theta(t) \in \{1, \dots, N\}$ and $\mathbf{x}(t) \in \mathbb{R}^n$ for $t \in \mathbb{N}_0$.

Given a switching sequence θ for $(\mathcal{S}, \mathcal{D})$, we have

$$\mathbf{x}(t) = \mathbf{\Phi}_\theta(t, t_0)\mathbf{x}(t_0) + \mathbf{f}_\theta(t, t_0)$$

for t, $t_0 \in \mathbb{N}_0$ with $t \geq t_0$, where the state transition matrix $\mathbf{\Phi}_\theta(t, t_0) \in \mathbb{R}^{n \times n}$ is defined as the matrix product

$$\mathbf{\Phi}_\theta(t, t_0) = \begin{cases} \mathbf{I}_n & \text{if } t = t_0; \\ \mathbf{A}_{\theta(t-1)} \cdots \mathbf{A}_{\theta(t_0)} & \text{if } t > t_0, \end{cases}$$

and the vector $\mathbf{f}_\theta(t, t_0) \in \mathbb{R}^n$ as the vector sum

$$\mathbf{f}_\theta(t, t_0) = \sum_{s=t_0}^{t-1} \mathbf{\Phi}_\theta(t, s+1)\mathbf{b}_{\theta(s)}.$$

Definition 1: Let $P \subseteq \mathbb{R}^n$. The piecewise affine system $(\mathcal{S}, \mathcal{D})$ is said to be *uniformly exponentially stable on* P if there exist $c \geq 1$ and $\lambda \in (0, 1)$ such that, for all switching sequences θ generated by initial states $\mathbf{x}(0) \in P$ according to (2), we have

$$\|\mathbf{\Phi}_\theta(t, t_0)\| \leq c\lambda^{t-t_0}, \tag{3a}$$

$$\|\mathbf{f}_\theta(t, t_0)\| \leq c \tag{3b}$$

for t, $t_0 \in \mathbb{N}_0$ with $t \geq t_0$, and

$$\|\mathbf{f}_\theta(t, t_0)\| \to 0 \tag{3c}$$

as $t - t_0 \to \infty$.

Definition 2: A set Θ of switching sequences is said to be *uniformly exponentially stabilizing* for $\mathcal{S}$ if there exist $c \geq 1$ and $\lambda \in (0, 1)$ such that (3a) is satisfied for all $\theta = (\theta(0), \theta(1), \dots) \in \Theta$ and for all t_0, $t \in \mathbb{N}_0$ with $t \geq t_0$. In this case, each $\theta \in \Theta$ will be said to be *uniformly exponentially stabilizing* for $\mathcal{S}$.

Our notion of stability, in a sense, imposes an additional robustness requirement on the switching sequences against unforeseen perturbations in the initial state. See [21] for an example that illustrates this point.

B. Definitions: Simulation and Bisimulation

We will now construct a sequence of finite-state symbolic models, and define the notion of simulation and bisimulation with respect to these models. To do so, we will first obtain a family of state-space partitions. For $L \in \mathbb{N}$ and for *switching paths* $(i_0, \dots, i_L) \in \{1, \dots, N\}^{L+1}$ of length L, define sets $D_{(i_0,\dots,i_L)} \subset \mathbb{R}^n$ recursively by

$$D_{(i_0,\dots,i_L)} = \left\{ \mathbf{x} \in D_{(i_0,\dots,i_{L-1})} \colon \mathbf{A}_{i_0}\mathbf{x} + \mathbf{b}_{i_0} \in D_{(i_1,\dots,i_L)} \right\}.$$

Each polyhedron $D_{(i_0,\dots,i_L)}$ is the set of all states in $D_{(i_0,\dots,i_{L-1})}$ which will evolve to a state in $D_{(i_1,\dots,i_L)}$ in one step. Define the indexed family $\mathcal{D}_L$, $L \in \mathbb{N}_0$, where $\mathcal{D}_0 = \mathcal{D} = \{D_1, \dots, D_N\}$ and

$$\mathcal{D}_L = \left\{ D_{(i_0,\dots,i_L)} \colon (i_0, \dots, i_L) \in \{1, \dots, N\}^{L+1} \right\}$$

for each $L \in \mathbb{N}$. Each $\mathcal{D}_L$ is a family of polyhedral cells that form a partition of the state space and it is potentially finer than $\mathcal{D}_{L-1}$. The polyhedral cells $D_{(i_0,\dots,i_L)}$ are intersections of half-spaces, and hence they are convex and their interiors are obtained simply by solving linear vector inequalities.

Let us associate a directed graph G_L to each of the families $\mathcal{D}_L$, so that each node of G_L is a switching path of length L. For path length $L = 0$, the nodes of G_0 are the modes $i \in \{1, \dots, N\}$ such that $D_i \in \mathcal{D}_0$ is nonempty, and there is a directed edge from node i to node j in G_0 if and only if $D_{(i,j)} \in \mathcal{D}_1$ is nonempty. For path lengths $L \in \mathbb{N}$, a switching path $(i_0, \dots, i_L) \in \{1, \dots N\}^{L+1}$ is a node of G_L if and only if $D_{(i_0,\dots,i_L)} \in \mathcal{D}_L$ is nonempty, and there is a directed edge from node $(i_0, \dots, i_L)$ to node $(j_0, \dots, j_L)$ in G_L if and only if $(i_1, \dots, i_L) = (j_0, \dots, j_{L-1})$ and $D_{(i_0,\dots,i_L,j_L)} \in \mathcal{D}_{L+1}$ is nonempty.

Definition 3: For $L \in \mathbb{N}_0$, a switching sequence $\theta = (\theta(0), \theta(1), \dots)$ is said to be generated by G_L if there is a directed edge from $(\theta(t), \dots, \theta(t+L))$ to $(\theta(t+1), \dots, \theta(t+L+1))$ in G_L for every $t \in \mathbb{N}_0$. The set of such switching sequences is denoted by $\Theta(G_L)$.

The sequences of state-space partitions $\mathcal{D}_L$ and directed graphs G_L define a nested sequence of symbolic models defined by the pairs $(\mathcal{D}_L, G_L)$, $L \in \mathbb{N}_0$. For each $L \in \mathbb{N}_0$, it is readily seen that $\Theta(G_{L+1}) \subseteq \Theta(G_L)$ and more importantly, G_L generates all the switching sequences that initial states $\mathbf{x}(0) \in \mathbb{R}^n$ of the piecewise affine system $(\mathcal{S}, \mathcal{D})$ generate according to (2). Therefore, in view of terminology in algorithmic approaches to the analysis of dynamical systems [22], we shall say that, for each $L \in \mathbb{N}_0$, the symbolic model $(\mathcal{D}_L, G_L)$ is a *simulation* of the piecewise affine system $(\mathcal{S}, \mathcal{D})$.

The sequence of simulations $(\mathcal{D}_L, G_L)$ of the piecewise affine system $(\mathcal{S}, \mathcal{D})$ is not finite in general. However, if for some $L \in \mathbb{N}_0$, the state-space partition $\mathcal{D}_{L+1}$ turns out to be as fine as the state-space partition $\mathcal{D}_L$ in the following sense, this sequence becomes finite.

Definition 4: Let $L \in \mathbb{N}_0$. A cell $D_{(i_0,\dots,i_L)} \in \mathcal{D}_L$ is said to be $(\mathcal{S}, \mathcal{D})$-*invariant* if $D_{(i_0,\dots,i_L)} \in \mathcal{D}_L$ is equal to $D_{(i_0,\dots,i_{L+1})} \in \mathcal{D}_{L+1}$ for some $i_{L+1} \in \{1, \dots, N\}$. If every $D_{(i_0,\dots,i_L)} \in \mathcal{D}_L$ is $(\mathcal{S}, \mathcal{D})$-invariant, then the state-space partition $\mathcal{D}_L$ is said to be $(\mathcal{S}, \mathcal{D})$-*invariant*.

It is shown in [14] that, if $\mathcal{D}_L$ is $(\mathcal{S}, \mathcal{D})$-invariant for some $L \in \mathbb{N}_0$, then each switching sequence in $\Theta(G_L)$ is generated by some initial state $\mathbf{x}(0) \in \mathbb{R}^n$ for the piecewise affine system $(\mathcal{S}, \mathcal{D})$. Hence, in view of terminology in algorithmic approaches to the analysis of dynamical systems [3], [22], we shall say that the symbolic model $(\mathcal{D}_L, G_L)$ is a *(finite-state) bisimulation* of the piecewise affine system $(\mathcal{S}, \mathcal{D})$ if $\mathcal{D}_L$ is $(\mathcal{S}, \mathcal{D})$-invariant. It has been reported in [14], [15] that, if $(\mathcal{D}_L, G_L)$ is a bisimulation of $(\mathcal{S}, \mathcal{D})$ for some $L \in \mathbb{N}_0$, one can carry out exact analysis of the stability and performance of the piecewise affine system by checking the feasibility of a system of linear matrix inequalities.

C. Problem Statement

In this paper, two problems are addressed:

- The first problem is to demonstrate that, for a large class of piecewise affine systems, a stability analysis based on our sequence of simulations does not suffer inherent conservatism.
- The second problem is to present a condition under which the piecewise affine system admits a bisimulation; one should be able to check such a condition before actually constructing a sequence of simulations.

III. Nonconservative Sequence of Simulations

In this section, we will establish that the stability analysis of piecewise affine systems $(\mathcal{S}, \mathcal{D})$ based on the sequence of simulations $(\mathcal{D}_L, G_L)$, $L \in \mathbb{N}_0$, is nonconservative.

Definition 5: Let $L \in \mathbb{N}_0$. A nonempty subset $\mathcal{N}$ of $\{1, \ldots, N\}^{L+1}$ is said to be an *admissible set of L-paths* if for each $(i_0, \ldots, i_L) \in \mathcal{N}$ there exists an integer $M > L$ and a switching path $(i_{L+1}, \ldots, i_M)$ such that $(i_{M-L}, \ldots, i_M) = (i_0, \ldots, i_L)$ and $(i_t, \ldots, i_{t+L}) \in \mathcal{N}$ for $0 \leq t \leq M - L$. Moreover, if there exists an indexed family $\{\mathbf{X}_{(j_0,\ldots,j_{L-1})} : (j_0, \ldots, j_L) \in \mathcal{N}\}$ of symmetric and positive definite matrices $\mathbf{X}_{(j_0,\ldots,j_{L-1})} \in \mathbb{R}^{n \times n}$ such that

$$\mathbf{A}_{i_0}^T \mathbf{X}_{(i_1,\ldots,i_L)} \mathbf{A}_{i_0} - \mathbf{X}_{(i_0,\ldots,i_{L-1})} \prec \mathbf{0}$$

for all switching paths $(i_0, \ldots, i_L) \in \mathcal{N}$ of length L, then $\mathcal{N}$ is said to be an *$\mathcal{S}$-admissible set of L-paths*.

If $\mathcal{N}$ is an admissible set of L-paths, then each switching path in $\mathcal{N}$ leads to itself via the switching paths in $\mathcal{N}$. The stability of a switching sequence θ is determined by the switching paths that occur infinitely many times in θ, and the set of all such switching paths of any length L is necessarily admissible. If this set is $\mathcal{S}$-admissible, then θ is uniformly exponentially stabilizing for $\mathcal{S}$ [23]. Conversely, if the directed graph G_L is given for some $L \in \mathbb{N}_0$, and if $\Theta(G_L)$ is uniformly exponentially stabilizing for $\mathcal{S}$, then there exists a path length $M \in \mathbb{N}_0$ such that the largest admissible subset of

$$\{(\theta(t), \ldots, \theta(t + M)) : \theta \in \Theta(G_L), t \in \mathbb{N}_0\}$$

is $\mathcal{S}$-admissible [24]. Therefore, one can try to obtain all admissible sets of M-paths that appear in $\Theta(G_L)$ over all $M \in \mathbb{N}_0$ to determine the stability of $\Theta(G_L)$. We shall see, however, that considering the case of $M = L+1$ suffices for our purposes.

Definition 6: Let $L \in \mathbb{N}_0$. Let $\mathcal{N}$ be an admissible set of L-paths. If the only admissible set $\widetilde{\mathcal{N}}$ of L-paths satisfying $\widetilde{\mathcal{N}} \subseteq \mathcal{N}$ is $\mathcal{N}$ itself, then $\mathcal{N}$ is said to be a *minimal set of L-paths*. Moreover, if $\mathcal{N}$ is minimal and $\mathcal{S}$-admissible, then $\mathcal{N}$ is said to be an *$\mathcal{S}$-minimal set of L-paths*.

It is readily seen that each admissible (resp. S-admissible) set of L-paths is a finite union of minimal (resp. $\mathcal{S}$-minimal) sets. Hence, in order to obtain admissible sets, it suffices to identify minimal sets. It is shown in [13] that, if $M = L+1$, then there exists a one-to-one correspondence between the family of minimal sets of M-paths that occur in $\Theta(G_L)$ and the family of elementary cycles in G_L. A procedure to obtain the minimal sets is given in [13].

Given an $L \in \mathbb{N}_0$ and $D_{(i_0,\ldots,i_L)} \in \mathcal{D}_L$, define

$$\begin{aligned}\mathcal{R}_{(i_0,\ldots,i_L)} = \{&(\theta(t), \ldots, \theta(t + L + 1)) : \\ &\theta \in \Theta(G_L), t \in \mathbb{N}_0, (\theta(0), \ldots, \theta(L)) = (i_0, \ldots, i_L)\}.\end{aligned}$$

Then $\mathcal{R}_{(i_0,\ldots,i_L)}$ can be said to be the set of all switching paths of length $L + 1$ that are reachable from node $(i_0, \ldots, i_L)$ in G_L. Define

$$\begin{aligned}\mathcal{I}_{(i_0,\ldots,i_L)} = \bigcup\{&\mathcal{N} \subseteq \mathcal{R}_{(i_0,\ldots,i_L)} : \\ &\mathcal{N} \text{ is a minimal set of } (L + 1)\text{-paths}\},\end{aligned}$$

so that $\mathcal{I}_{(i_0,\ldots,i_L)}$ is the maximal admissible set of $(L + 1)$-paths reachable from node $(i_0, \ldots, i_L)$ in G_L. All $(L + 1)$-paths which occur infinitely many times in switching sequences $\theta \in \Theta(G_L)$ with $(\theta(0), \ldots, \theta(L)) = (i_0, \ldots, i_L)$ are contained in $\mathcal{I}_{(i_0,\ldots,i_L)}$.

Lemma 7: Let $L \in \mathbb{N}_0$ and $D_{(i_0,\ldots,i_L)} \in \mathcal{D}_L$. The piecewise affine system $(\mathcal{S}, \mathcal{D})$ is uniformly exponentially stable on $D_{(i_0,\ldots,i_L)}$ if $\mathcal{I}_{(i_0,\ldots,i_L)}$ is $\mathcal{S}$-admissible and if $\mathbf{b}_{j_0} = \cdots = \mathbf{b}_{j_L} = \mathbf{0}$ for all $(j_0, \ldots, j_L) \in \mathcal{I}_{(i_0,\ldots,i_L)}$.

Proof: See [13, Theorem 2]. ■

For $L \in \mathbb{N}_0$, let P_L denote the union of all cells $D_{(i_0,\ldots,i_L)} \in \mathcal{D}_L$ such that the piecewise affine system $(\mathcal{S}, \mathcal{D})$ is uniformly exponentially stable on $D_{(i_0,\ldots,i_L)}$. Then we necessarily have $P_L \subseteq P_{L+1}$ for all $L \in \mathbb{N}_0$. The following theorem says that, for a large class of piecewise affine systems $(\mathcal{S}, \mathcal{D})$, the union of P_L over all $L \in \mathbb{N}_0$ is the largest set, outside of which the system $(\mathcal{S}, \mathcal{D})$ is not uniformly exponentially stable.

Theorem 8: Suppose that, for some $i \in \{1, \ldots, N\}$, the polyhedral cell D_i is bounded, the origin belongs to the interior of D_i, the spectral radius of $\mathbf{A}_i$ is less than one, and the affine term $\mathbf{b}_i = \mathbf{0}$. Then the piecewise affine system $(\mathcal{S}, \mathcal{D})$ is not uniformly exponentially stable on any subset of $\mathbb{R}^n \setminus \left(\bigcup_{L=0}^{\infty} P_L\right)$.

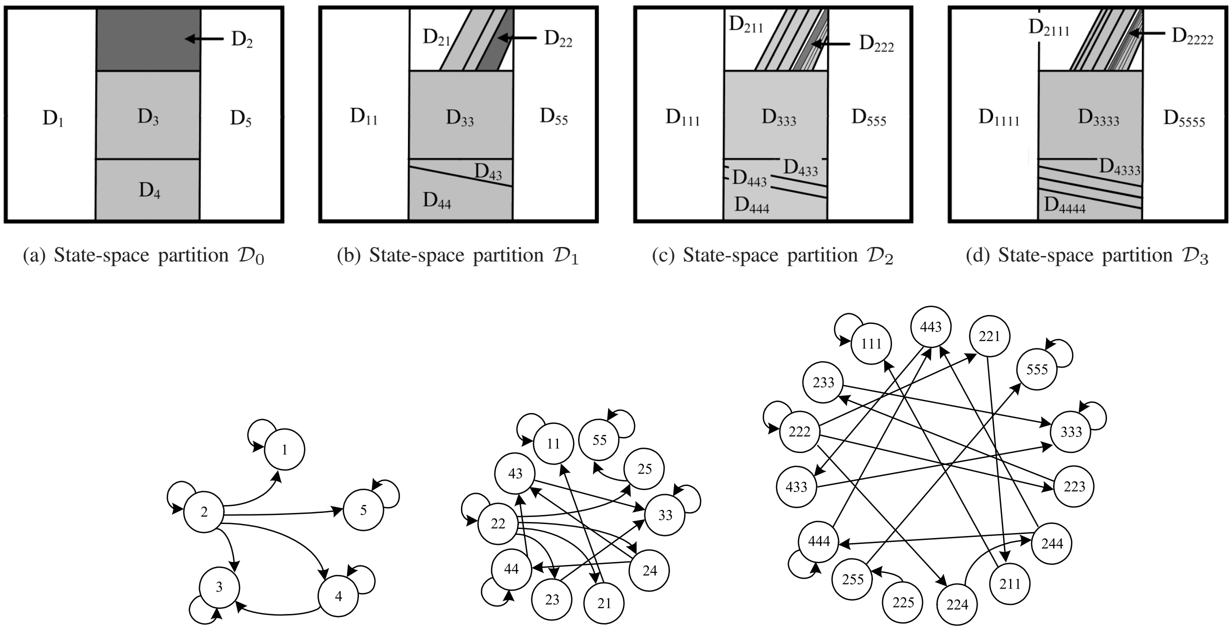
The proof of this theorem is omitted due to space constraints. The significance of the theorem is that, for any path length $L \in \mathbb{N}_0$, if a simulation $(\mathcal{D}_L, G_L)$ of the piecewise affine system $(\mathcal{S}, \mathcal{D})$ does not give a satisfactory stability analysis, then one has the option to pay more computational cost (whenever one is able and willing to do so) and use the next simulation $(\mathcal{D}_{L+1}, G_{L+1})$ to obtain a potentially better stability analysis. This point is illustrated by the following example.

Example 9: Consider the piecewise affine system $(\mathcal{S}, \mathcal{D})$ with $N = 5$,

$$\mathbf{A}_1 = \begin{bmatrix} 2 & 0 \\ 4 & 0 \end{bmatrix}, \ \mathbf{A}_2 = \begin{bmatrix} 4 & -\frac{1}{2} \\ 16 & -1 \end{bmatrix}, \ \mathbf{A}_3 = \begin{bmatrix} \frac{1}{4} & 0 \\ 0 & \frac{1}{2} \end{bmatrix},$$

$$\mathbf{A}_4 = \begin{bmatrix} 0 & 0 \\ \frac{1}{4} & \frac{1}{2} \end{bmatrix}, \ \mathbf{A}_5 = \begin{bmatrix} 1 & 0 \\ 1 & \frac{3}{2} \end{bmatrix},$$

$$\mathbf{b}_1 = \begin{bmatrix} 1 \\ 0 \end{bmatrix}, \ \mathbf{b}_2 = \mathbf{b}_3 = \mathbf{b}_4 = \begin{bmatrix} 0 \\ 0 \end{bmatrix}, \ \mathbf{b}_5 = \begin{bmatrix} 0 \\ 1 \end{bmatrix},$$



(a) State-space partition $\mathcal{D}_0$ (b) State-space partition $\mathcal{D}_1$ (c) State-space partition $\mathcal{D}_2$ (d) State-space partition $\mathcal{D}_3$

(e) Directed graph G_0 (f) Directed graph G_1 (g) Directed graph G_2

Fig. 1. The sequence of simulations for Example 9. Directed graphs G_L are obtained from $\mathcal{D}_L$ and $\mathcal{D}_{L+1}$. Each symbolic model $(\mathcal{D}_L, G_L)$ simulates the piecewise affine system $(\mathcal{S}, \mathcal{D})$. The system $(\mathcal{S}, \mathcal{D})$ is uniformly exponentially stable on the cells shaded in light grey in $\mathcal{D}_L$, and not uniformly exponentially stable on any subset of the areas not shaded in $\mathcal{D}_L$. The stability of the system on the area shaded in dark grey in $\mathcal{D}_L$ is not determined based on $(\mathcal{D}_L, G_L)$.

and

$$\begin{aligned}
D_1 &= \left\{ [x_1 \; x_2]^T \in \mathbb{R}^2 \colon x_1 < -1 \right\}, \\
D_2 &= \left\{ [x_1 \; x_2]^T \in \mathbb{R}^2 \colon -1 \leq x_1 < 1, 1 \leq x_2 \right\}, \\
D_3 &= \left\{ [x_1 \; x_2]^T \in \mathbb{R}^2 \colon -1 \leq x_1 < 1, -1 < x_2 < 1 \right\}, \\
D_4 &= \left\{ [x_1 \; x_2]^T \in \mathbb{R}^2 \colon -1 \leq x_1 < 1, x_2 \leq -1 \right\}, \\
D_5 &= \left\{ [x_1 \; x_2]^T \in \mathbb{R}^2 \colon 1 \leq x_1 \right\}.
\end{aligned}$$

The first few state-space partitions $\mathcal{D}_L$ and directed graphs G_L are shown in Fig. 1. The partitions $\mathcal{D}_0$ and $\mathcal{D}_1$ give rise to the graph G_0, from which we obtain $\mathcal{I}_1 = \{11\}$, $\mathcal{I}_2 = \{11, 22, 33, 44, 55\}$, $\mathcal{I}_3 = \{33\}$, $\mathcal{I}_4 = \{33, , 44\}$, and $\mathcal{I}_5 = \{55\}$. Among these sets, only $\mathcal{I}_3$ and $\mathcal{I}_4$ are $\mathcal{S}$-admissible. Thus we determine that the system $(\mathcal{S}, \mathcal{D})$ is uniformly exponentially stable on $P_0 = D_3 \cup D_4$. On the other hand, we can also determine that the system is not uniformly exponentially stable on $D_1 \cup D_5$ because the spectral radii of $\mathbf{A}_1$ and $\mathbf{A}_5$ are not less than one and because $\mathcal{I}_1 = \{11\}$ and $\mathcal{I}_5 = \{55\}$. The stability of the system on D_2 cannot be determined at the moment because $\mathcal{I}_2$ tells us that some initial states in D_2 will reach P_0 and some other initial states in D_2 will reach $D_1 \cup D_5$. In order to obtain a better stability analysis, let us consider the case of $L = 1$ (i.e., the directed graph G_1 obtained from $\mathcal{D}_1$ and $\mathcal{D}_2$), and deduce that $\mathcal{I}_{21} = \{111\}$, $\mathcal{I}_{22} = \{111, 222, 333, 444, 555\}$, $\mathcal{I}_{23} = \{333\}$, $\mathcal{I}_{24} = \{333, 444\}$, and $\mathcal{I}_{25} = \{555\}$. It is readily seen that the sets $\mathcal{I}_{23}$ and $\mathcal{I}_{24}$ are $\mathcal{S}$-admissible, and hence the system $(\mathcal{S}, \mathcal{D})$ is uniformly exponentially stable on $P_1 = P_0 \cup D_{23} \cup D_{24}$. Similarly, we find that the system is not uniformly exponentially stable on $D_{21} \cup D_{25}$. Thus, we are now able to determine the stability of the system on a large portion of D_2. Nevertheless, whether the system is uniformly exponentially stable on the D_{22}, which is a small bounded subset of D_2 shown in Fig. 1(b), remains undetermined. Incrementing the path length L once more, we consider the case of $L = 2$ and obtain $\mathcal{I}_{221} = \{1111\}$, $\mathcal{I}_{222} = \{1111, 2222, 3333, 4444\}$, $\mathcal{I}_{223} = \{3333\}$, $\mathcal{I}_{224} = \{3333, 4444\}$, and $\mathcal{I}_{225} = \{5555\}$, where $\mathcal{I}_{223}$ and $\mathcal{I}_{224}$ are $\mathcal{S}$-admissible. Hence the system is uniformly exponentially stable on $P_2 = P_1 \cup D_{223} \cup D_{224}$. Also, we can determine that the system is not uniformly exponentially stable on $D_{221} \cup D_{225}$. If it is deemed that the current result is satisfactory or the computational burden to proceed further is great, then we stop at this point and conclude that the piecewise affine system is uniformly exponentially stable on $D_3 \cup D_4 \cup D_{23} \cup D_{24} \cup D_{223} \cup D_{224}$, and that it is not uniformly exponentially stable on any subset of $D_1 \cup D_5 \cup D_{21} \cup D_{25} \cup D_{221} \cup D_{225}$. The only part of the state space on which the stability of the system is not determined for $L = 2$ is the thin bounded tube D_{222} shown in Fig. 1(c).

IV. Existence of Finite-State Bisimulation

In this section, we present a sufficient condition under which the piecewise affine system $(\mathcal{S}, \mathcal{D})$ admits a finite-state bisimulation $(\mathcal{D}_L, G_L)$.

Definition 10: Let $\theta = (\theta(0), \theta(1), \dots)$ be a switching sequence; let $t \in \mathbb{N}_0$. The pair $(\theta(t), \theta(t+1))$ is called a *discrete transition in* θ if $\theta(t) \neq \theta(t+1)$. If the number of discrete transitions in θ is finite, then this number is denoted by d_θ.

The following theorem provides a condition under which there is a uniform bound on d_θ over all switching sequences θ generated by a piecewise affine system. This condition is the discrete-time counterpart of the bounded-horizon property for STORMED hybrid systems [20].

Theorem 11: Let $Q \subset \mathbb{R}^n$ be a bounded polyhedron such that all state sequences generated by initial states in Q remain in Q. Suppose the following hold true:

(a) There exists $\gamma > 0$ such that

$$\|\mathbf{A}_i\mathbf{x} + \mathbf{b}_i - \mathbf{x}\| \geq \gamma$$

whenever $\mathbf{x} \in D_i \cap Q$ and $\mathbf{A}_i\mathbf{x} + \mathbf{b}_i \in D_j \cap Q$ with $i \neq j$.

(b) There exist $\epsilon > 0$, a_-, $a_+ \in \mathbb{R}$, and $\boldsymbol{\phi} \in \mathbb{R}^n$ such that

$$\boldsymbol{\phi}^T(\mathbf{A}_i\mathbf{x} + \mathbf{b}_i - \mathbf{x}) \geq \epsilon\|\mathbf{A}_i\mathbf{x} + \mathbf{b}_i - \mathbf{x}\|$$

whenever $\mathbf{x} \in D_i \cap Q$, and such that $a_- \leq \boldsymbol{\phi}^T\mathbf{x} \leq a_+$ for all $\mathbf{x} \in Q$.

Then

$$d_\theta \leq \frac{a_+ - a_-}{\epsilon\gamma}$$

for all switching sequences θ generated by initial states $\mathbf{x}(0) \in Q$ according to (2).

The proof of this theorem is similar to that of [25, Lemma 10]; the details are omitted due to space constraints. If the conditions of the theorem are satisfied, then the number of discrete transitions in any switching sequence generated by the initial states in a bounded polyhedron is uniformly bounded above. Consequently, every switching sequence will be eventually constant in this case.

Example 12: Consider the system $(\mathcal{S}, \mathcal{D})$ with $N = 4$,

$$\mathbf{A}_1 = \begin{bmatrix} 1 & 0 \\ 0 & 1/4 \end{bmatrix},\ \mathbf{A}_2 = \begin{bmatrix} 1 & 0 \\ 1/2 & 1/4 \end{bmatrix},$$
$$\mathbf{A}_3 = \begin{bmatrix} 1 & 0 \\ 1/4 & 1/2 \end{bmatrix},\ \mathbf{A}_4 = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix},$$
$$\mathbf{b}_1 = \begin{bmatrix} 1 \\ 0 \end{bmatrix},\ \mathbf{b}_2 = \begin{bmatrix} 3/2 \\ 1/4 \end{bmatrix},\ \mathbf{b}_3 = \begin{bmatrix} 1 \\ 1/4 \end{bmatrix},\ \mathbf{b}_4 = \begin{bmatrix} 0 \\ 0 \end{bmatrix},$$

and

$$D_1 = \left\{[x_1\ x_2]^T \in \mathbb{R}^2 \colon x_1 < -1\right\},$$
$$D_2 = \left\{[x_1\ x_2]^T \in \mathbb{R}^2 \colon -1 \leq x_1 < 0\right\},$$
$$D_3 = \left\{[x_1\ x_2]^T \in \mathbb{R}^2 \colon 0 \leq x_1 < 1\right\},$$
$$D_4 = \left\{[x_1\ x_2]^T \in \mathbb{R}^2 \colon 1 \leq x_1\right\}.$$

Let $Q \subset \mathbb{R}^2$ be the box whose vertices are at $[-2\ 1]^T$, $[-2\ -1]^T$, $[2\ 1]^T$ and $[2\ -1]^T$. It is readily seen that all state sequences $\mathbf{x}(t)$, $t \in \mathbb{N}_0$, generated by initial states in Q remain in Q. For $\mathbf{x} \in D_1 \cap Q$, we have $\mathbf{A}_1\mathbf{x} + \mathbf{b}_1 \in D_2 \cap Q$ and $1 \leq \|\mathbf{A}_1\mathbf{x} + \mathbf{b}_1 - \mathbf{x}\| \leq 5/4$. For $\mathbf{x} \in D_2 \cap Q$, we have $\mathbf{A}_2\mathbf{x} + \mathbf{b}_2 \in (D_3 \cup D_4) \cap Q$ and $3/2 \leq \|\mathbf{A}_2\mathbf{x} + \mathbf{b}_2 - \mathbf{x}\| \leq 2$. Also, for $\mathbf{x} \in D_3 \cap Q$, we have $\mathbf{A}_3\mathbf{x} + \mathbf{b}_3 \in D_4 \cap Q$ and $1 \leq \|\mathbf{A}_3\mathbf{x} + \mathbf{b}_3 - \mathbf{x}\| \leq 2$. Lastly, we have $\mathbf{A}_4\mathbf{x} + \mathbf{b}_4 \in D_4 \cap Q$ whenever $\mathbf{x} \in D_4 \cap Q$. Thus condition (a) in Theorem 11 is satisfied with $\gamma = 1$. Let $\boldsymbol{\phi} = [1\ 1/2]^T$. Then

$$\boldsymbol{\phi}^T(\mathbf{A}_1\mathbf{x} + \mathbf{b}_1 - \mathbf{x}) \geq 5/8 \geq 0.5\|\mathbf{A}_1\mathbf{x} + \mathbf{b}_1 - \mathbf{x}\|$$

for $\mathbf{x} \in D_1 \cap Q$,

$$\boldsymbol{\phi}^T(\mathbf{A}_2\mathbf{x} + \mathbf{b}_2 - \mathbf{x}) \geq 1/2 \geq 0.25\|\mathbf{A}_2\mathbf{x} + \mathbf{b}_2 - \mathbf{x}\|$$

for $\mathbf{x} \in D_2 \cap Q$,

$$\boldsymbol{\phi}^T(\mathbf{A}_3\mathbf{x} + \mathbf{b}_3 - \mathbf{x}) \geq 7/8 \geq 0.44\|\mathbf{A}_3\mathbf{x} + \mathbf{b}_3 - \mathbf{x}\|$$

for $\mathbf{x} \in D_3 \cap Q$, and $\|\mathbf{A}_4\mathbf{x} + \mathbf{b}_4 - \mathbf{x}\| = 0$ for $\mathbf{x} \in D_4 \cap Q$. Moreover, $-5/2 \leq \boldsymbol{\phi}^T\mathbf{x} \leq 5/2$ for all $\mathbf{x} \in Q$. Thus condition (b) in Theorem 11 is satisfied with $\epsilon = 0.25$, $a_- = -5/2$, and $a_+ = 5/2$. Therefore, we conclude that $d_\theta \leq (a_+ - a_-)/(\epsilon\gamma) = 20$ for every switching sequence θ generated by initial states $\mathbf{x}(0) \in Q$ according to (2).

Theorem 13: Suppose the number of discrete transitions in any switching sequence generated by $(\mathcal{S}, \mathcal{D})$ is uniformly bounded above by $K \in \mathbb{N}_0$. Suppose there exists an $\widetilde{L} \in \mathbb{N}_0$ such that for all $i \in \{1, \ldots, N\}$ the cell $D_{(i,\ldots,i)} \in \mathcal{D}_{\widetilde{L}}$ is either empty or equal to $D_{(i,\ldots,i,i)} \in \mathcal{D}_{\widetilde{L}+1}$. Then there exists an $L \in \mathbb{N}_0$ with $L \leq K\widetilde{L}$ such that $(\mathcal{D}_L, G_L)$ is a bisimulation of $(\mathcal{S}, \mathcal{D})$.

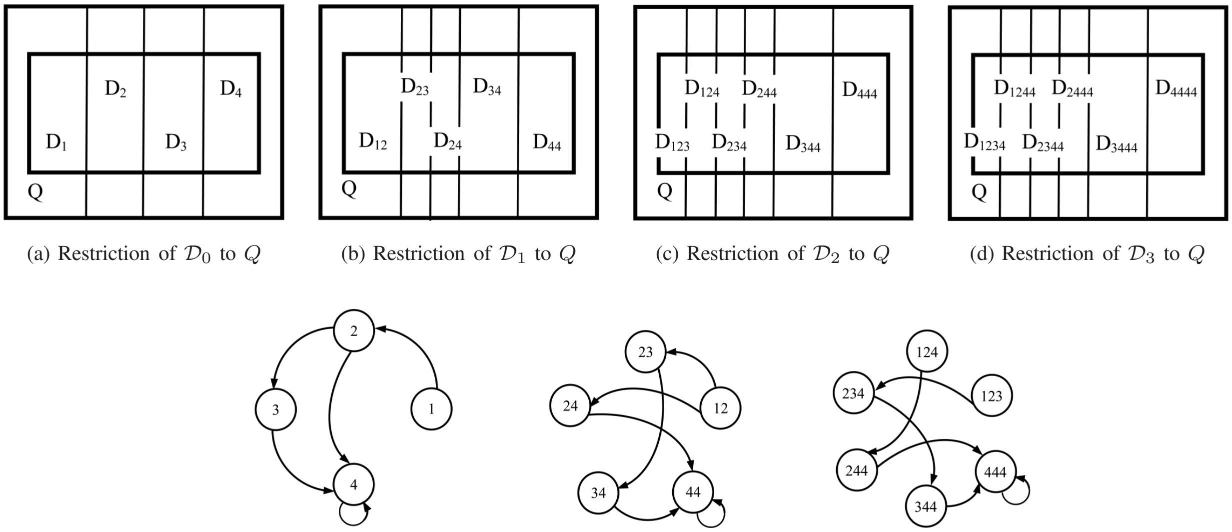
Proof: Due to space constraints, we only sketch the proof. Since $d_\theta \leq K$, the switching sequence θ is eventually constant and $\theta(t) = \theta(t+1)$ for all $t \geq K\widetilde{L}$. Let $L = K\widetilde{L}$. Then clearly every cell in $\mathcal{D}_L$ is $(\mathcal{S}, \mathcal{D})$-invariant and hence $(\mathcal{D}_L, G_L)$ is a bisimulation of $(\mathcal{S}, \mathcal{D})$. ■

Example 14: Let the system $(\mathcal{S}, \mathcal{D})$ be as in Example 12. It has been shown in Example 12 that the number of discrete transitions in any switching sequence generated by initial states in Q is uniformly bounded by 20. Moreover, we have also seen in Example 12 that $\mathbf{A}_1\mathbf{x} + \mathbf{b}_1 \notin D_1 \cap Q$ for $\mathbf{x} \in D_1 \cap Q$, $\mathbf{A}_2\mathbf{x} + \mathbf{b}_2 \notin D_2 \cap Q$ for $\mathbf{x} \in D_2 \cap Q$, and $\mathbf{A}_3\mathbf{x} + \mathbf{b}_3 \notin D_3 \cap Q$ for $\mathbf{x} \in D_3 \cap Q$, but that $\mathbf{A}_4\mathbf{x} + \mathbf{b}_4 \in D_4 \cap Q$ for $\mathbf{x} \in D_4 \cap Q$. Thus the conditions in Theorem 13 are satisfied with $\widetilde{L} = 1$ as well as $K = 20$; that is, the symbolic model $(\mathcal{D}_L, G_L)$ is a bisimulation of $(\mathcal{S}, \mathcal{D})$ for some $L \leq K\widetilde{L} = 20$. The symbolic models $(\mathcal{D}_L, G_L)$ of $(\mathcal{S}, \mathcal{D})$ for $L = 0$, 1, and 2 are shown in Fig. 2. Indeed, the state-space partition $\mathcal{D}_2$ (restricted to Q) is $(\mathcal{S}, \mathcal{D})$-invariant and hence the symbolic model $(\mathcal{D}_2, G_2)$ is a bismulation of $(\mathcal{S}, \mathcal{D})$ (restricted to Q).

V. Conclusions

We presented a stability analysis of piecewise affine systems based on a nested sequence of finite-state simulations. It was shown that this analysis is nonconservative for a large class of piecewise affine systems. Moreover, for the restricted class of systems for which the number and time of discrete transitions between affine models are uniformly bounded, it was shown that the existence of bisimulations is guaranteed. Considering that even simple problems regarding hybrid systems are known to be undecidable, it is not surprising that our class of systems admitting bisimulations is restricted.

Future research directions include application and extension of the results to realistic examples and controller synthesis, respectively. Potential real-world examples include power systems and power electronic circuits. On the other hand, a starting point toward the extension to controller



(a) Restriction of $\mathcal{D}_0$ to Q (b) Restriction of $\mathcal{D}_1$ to Q (c) Restriction of $\mathcal{D}_2$ to Q (d) Restriction of $\mathcal{D}_3$ to Q

(d) Restriction of G_0 to Q (e) Restriction of G_1 to Q (f) Restriction of G_2 to Q

Fig. 2. The sequence of simulations for Example 14. All state sequences generated by initial states in the box Q remain in Q. The restriction of state-space partition $\mathcal{D}_2$ to Q is as fine as the restriction of the next partition $\mathcal{D}_3$ to Q. Thus, restricted to Q, the symbolic model $(\mathcal{D}_2, G_2)$ is a bisimulation of the piecewise affine system $(\mathcal{S}, \mathcal{D})$.

synthesis is to characterize the set of controllers under which the closed-loop system admits a finite-state bisimulation.

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