2011 50th IEEE Conference on Decision and Control and
European Control Conference (CDC-ECC)
Orlando, FL, USA, December 12-15, 2011

Alternating approximately bisimilar symbolic models for nonlinear control systems affected by disturbances

Alessandro Borri, Giordano Pola and Maria Domenica Di Benedetto

***Abstract*— Symbolic models of continuous and hybrid systems provide a formal approach to solve control problems where software and hardware interact with the physical world. Symbolic models are abstract descriptions of continuous systems in which one symbol corresponds to an "aggregate" of continuous states. In this paper, we address the construction of symbolic models for nonlinear control systems affected by disturbances. The main contribution of this paper is in proposing symbolic models that can be effectively constructed and that are alternating approximately bisimilar to incrementally stable nonlinear control systems, with arbitrarily good accuracy.**

I. INTRODUCTION

An emerging trend in the control systems community is the use of symbolic models for the analysis and control design of continuous and hybrid systems [1]. Symbolic models are abstract descriptions of continuous systems in which each symbol corresponds to an "aggregate" of continuous states [2]. The use of symbolic models provides a formal approach to solve problems of control in which software and hardware interact with the physical world. Moreover, it provides the designer with a systematic method to address a wide spectrum of novel specifications, that are difficult to enforce by means of conventional control design paradigms. Examples of such specifications include logic specifications expressed in linear temporal logic or automata on infinite strings.

During the last years, several classes of dynamical and control systems admitting symbolic models were identified. We recall from [3] timed, multi–rate, rectangular automata, and o-minimal hybrid systems in the class of hybrid automata. Control systems were addressed in [4], [5] and [6], where symbolic models were shown to exist for controllable discrete–time linear systems, piecewise–affine systems and multi–affine systems, respectively. Most of the aforementioned work is based on the notions of simulation and bisimulation relations, as introduced by Milner [7] and Park [8]. Insights into the construction of symbolic models for continuous and hybrid systems have been recently gained by the notion of approximate bisimulation [9]. Based on this notion, incrementally stable nonlinear control systems were shown in [10] to admit symbolic models. This result has been further generalized to nonlinear switched systems in [11] and nonlinear time–delay systems in [12], [13]. In the aforementioned work, control systems are supposed to be not affected by exogenous disturbance inputs. However, in many realistic situations, physical processes are characterized by a certain degree of uncertainty which is often modeled by disturbance inputs.

A. Borri, G. Pola and M.D. Di Benedetto are with the Department of Electrical and Information Engineering, Center of Excellence DEWS, University of L'Aquila, 67100 L'Aquila, Italy, {alessandro.borri,giordano.pola,mariadomenica.dibenedetto}@univaq.it.

The research leading to these results has been partially supported by the Center of Excellence DEWS and received funding from the European Union Seventh Framework Programme [FP7/2007-2013] under grant agreement n.257462 HYCON2 Network of excellence.

In this paper, we face the problem of studying symbolic models for nonlinear control systems affected by disturbances. The presence of disturbances requires us to replace the notion of approximate bisimulation employed in [10], [11], [12] with the notion of alternating approximate bisimulation, as introduced in [14] and inspired by Alur and coworkers' alternating bisimulation [15]. As discussed in [14], [2], this notion is a key ingredient when constructing symbolic models of systems affected by disturbances because it guarantees that control strategies synthesized on the symbolic models can be readily transferred to the original model. The existence of alternating approximately bisimilar symbolic models for incrementally stable nonlinear control systems affected by disturbances has been proved in [14]. However, the results of [14] cannot easily be used in practice because they rely upon the computation of the set of reachable states, which is a difficult task in general.

This work proposes alternative symbolic models to the ones proposed in [14], which are proved to be effectively computable. The key ingredient in our results is in deriving a finite approximation of the disturbance input functional space, by resorting to spline analysis [16]. Spline analysis has been also employed in [12], [13] for constructing symbolic models of time–delay systems. However, the approximation scheme proposed in these papers does not guarantee a proper approximation of the disturbance input functional space, which leads us to propose an alternative approximation scheme.

The main contribution of this paper lies in showing that *if the disturbance input functional space is bounded and Lipschitz continuous with uniform Lipschitz constant, and if the control system is incrementally stable, then symbolic models can be effectively constructed, which are shown to be alternating approximately bisimilar to the original control systems.*

This paper is organized as follows. Preliminary definitions are recalled in Section II. In Section III, we propose a spline–based approximation scheme for the disturbance input functional space. In Section IV, we present symbolic models and show how they are related to nonlinear control systems affected by disturbances, in terms of alternating approximate bisimulation. In Section V, we present an illustrative example. Finally, Section VI offers some concluding remarks.

II. Preliminary definitions

A. Notation

The identity map on a set A is denoted by 1_A. Given two sets A and B, if A is a subset of B we denote by $\imath_A : A \hookrightarrow B$ or simply by $\imath$ the natural inclusion map taking any $a \in A$ to $\imath(a) = a \in B$. Given a function $f : A \to B$ the symbol $f(A)$ denotes the image of A through f, i.e. $f(A) := \{b \in B : \exists a \in A \text{ s.t. } b = f(a)\}$; if $C \subset A$ we denote by $f|_C$ the restriction of f to C, i.e. $f|_C(x) := f(x)$ for any $x \in C$. Given a relation $\mathcal{R} \subseteq A \times B$, $\mathcal{R}^{-1}$ denotes the inverse relation of $\mathcal{R}$, i.e. $\mathcal{R}^{-1} := \{(b, a) \in B \times A : (a, b) \in \mathcal{R}\}$. The symbols $\mathbb{N}$, $\mathbb{Z}$, $\mathbb{R}$, $\mathbb{R}^+$ and $\mathbb{R}_0^+$ denote the set of natural, integer, real, positive real, and nonnegative real numbers, respectively. Given a vector $x \in \mathbb{R}^n$, we denote by $\|x\|$ the infinity norm of x. Given a measurable function $f : \mathbb{R}_0^+ \to \mathbb{R}^n$, the (essential) supremum of f is denoted by $\|f\|_\infty$. Given $\mu \in \mathbb{R}^+$ and $A \subseteq \mathbb{R}^n$, we denote by μA the set $\{b \in \mathbb{R}^n \,|\, \exists a \in A \text{ s.t. } b = \mu a\}$. A set $A \subseteq \mathbb{R}^n$ is radial if $\mu A \subseteq A$ for any $\mu \in [0, 1]$. A continuous function $\gamma : \mathbb{R}_0^+ \to \mathbb{R}_0^+$ is said to belong to class $\mathcal{K}$ if it is strictly increasing and $\gamma(0) = 0$; γ is said to belong to class $\mathcal{K}_\infty$ if $\gamma \in \mathcal{K}$ and $\gamma(r) \to \infty$ as $r \to \infty$. A continuous function $\beta : \mathbb{R}_0^+ \times \mathbb{R}_0^+ \to \mathbb{R}_0^+$ is said to belong to class $\mathcal{KL}$ if, for each fixed s, the map $\beta(r, s)$ belongs to class $\mathcal{K}_\infty$ with respect to r and, for each fixed r, the map $\beta(r, s)$ is decreasing with respect to s and $\beta(r, s) \to 0$ as $s \to \infty$. The symbol $C^0([0, \tau]; Y)$ denotes the set of continuous functions from the closed interval $[0, \tau]$ with $\tau \in \mathbb{R}^+$ to the set $Y \subseteq \mathbb{R}^m$.

B. Control systems and incremental stability

In this paper, we consider the following nonlinear control system:

$$\dot{x} = f(x, u, d), \tag{1}$$

where $x \in X \subseteq \mathbb{R}^n$ is the state, $u \in U \subseteq \mathbb{R}^m$ and $d \in D \subseteq \mathbb{R}^l$ are the control input and the disturbance input. We suppose that U and D are compact, convex sets with the origin as an interior point. Control and disturbance input functions are supposed to belong to the sets $\mathcal{U}$ and $\mathcal{D}$ of continuous functions of time from intervals of the form $]a, b[\subseteq \mathbb{R}$ to U and D, respectively, and $f : \mathbb{R}^n \times U \times D \to \mathbb{R}^n$ is a continuous map satisfying the following Lipschitz assumption: for every compact set $K \subset \mathbb{R}^n$, there exists a constant $\kappa \in \mathbb{R}^+$ such that

$$\|f(x, u, d) - f(y, u, d)\| \leq \kappa \|x - y\|,$$

for all $x, y \in K$, $u \in U$ and $d \in D$. In the sequel, we refer to the nonlinear control system in (1) by means of the tuple:

$$\Sigma = (X, \mathcal{U}, \mathcal{D}, f), \tag{2}$$

where each entity has been defined above. A curve $\xi :]a, b[\to \mathbb{R}^n$ is said to be a *trajectory* of Σ if there exist $u \in \mathcal{U}$ and $d \in \mathcal{D}$, satisfying

$$\dot{\xi}(t) = f(\xi(t), u(t), d(t))$$

for almost all $t \in]a, b[$. Although we have defined trajectories over open domains, we shall refer to trajectories $\xi : [0, T] \to \mathbb{R}^n$ defined on closed domains $[0, T]$, $T \in \mathbb{R}^+$ with the understanding of the existence of a trajectory $\xi' :]a, b[\to \mathbb{R}^n$ such that $\xi = \xi'|_{[0,T]}$. We also write $\xi_{xud}(t)$ to denote the point reached at time t under the control input u and the disturbance input d from the initial condition x; this point is uniquely determined, since the assumptions on f ensure existence and uniqueness of trajectories [17]. A control system Σ is said to be forward complete if every trajectory is defined on an interval of the form $]a, \infty[$. Sufficient and necessary conditions for a system to be forward complete can be found in [18]. In the sequel, we will make use of the following stability notion.

Definition 2.1: [19] A control system Σ is incrementally input–to–state stable (δ–ISS) if it is forward complete and there exist a $\mathcal{KL}$ function β and two $\mathcal{K}_\infty$ functions γ_u and γ_d such that for any $t \in \mathbb{R}_0^+$, any $x_1, x_2 \in \mathbb{R}^n$, any $u_1, u_2 \in \mathcal{U}$ and any $d_1, d_2 \in \mathcal{D}$, the following condition is satisfied:

$$\begin{aligned} &\|\xi_{x_1 u_1 d_1}(t) - \xi_{x_2 u_2 d_2}(t)\| \leq \\ &\beta(\|x_1 - x_2\|, t) + \gamma_u(\|u_1 - u_2\|_\infty) + \gamma_d(\|d_1 - d_2\|_\infty). \end{aligned} \tag{3}$$

The above incremental stability notion can be characterized in terms of dissipation inequalities, as follows.

Definition 2.2: [19] A smooth function

$$V : \mathbb{R}^n \times \mathbb{R}^n \to \mathbb{R}$$

is called a δ–ISS Lyapunov function for a control system $\Sigma = (X, \mathcal{U}, \mathcal{D}, f)$ if there exist $\lambda \in \mathbb{R}^+$ and $\mathcal{K}_\infty$ functions $\underline{\alpha}$, $\overline{\alpha}$, σ_u and σ_d such that, for any $x_1, x_2 \in \mathbb{R}^n$, any $u_1, u_2 \in U$, and any $d_1, d_2 \in D$, the following conditions hold true:

(i) $\underline{\alpha}(\|x_1 - x_2\|) \leq V(x_1, x_2) \leq \overline{\alpha}(\|x_1 - x_2\|)$,

(ii) $\frac{\partial V}{\partial x_1} f(x_1, u_1, d_1) + \frac{\partial V}{\partial x_2} f(x_2, u_2, d_2) \leq$
$- \lambda V(x_1, x_2) + \sigma_u(\|u_1 - u_2\|) + \sigma_d(\|d_1 - d_2\|)$.

The following result completely characterizes δ–ISS in terms of existence of δ–ISS Lyapunov functions.

Theorem 2.3: [19] A control system $\Sigma = (X, \mathcal{U}, \mathcal{D}, f)$ is δ–ISS if and only if it admits a δ–ISS Lyapunov function.

C. Systems and approximate equivalence notions

We will use systems to describe both control systems as well as their symbolic models.

Definition 2.4: [2] A system S is a quintuple

$$S = (X, L, \longrightarrow, Y, H),$$

consisting of:

- a set of states X;
- a set of inputs $L = A \times B$, where A is the set of control inputs and B is the set of disturbance inputs;
- a transition relation $\longrightarrow \subseteq X \times L \times X$;
- a set of outputs Y;
- an output function $H : X \to Y$.

A transition $(x, (a, b), x') \in \longrightarrow$ is denoted by $x \xrightarrow{(a,b)} x'$. System S is said to be *countable* if X and L are countable sets, *symbolic* if X and L are finite sets, and *metric* if the output set Y is equipped with a metric $\mathbf{d}_Y : Y \times Y \to \mathbb{R}_0^+$.

In the sequel, we consider bisimulation relations [7], [8] to relate properties of control systems and symbolic models.

Intuitively, a bisimulation relation between a pair of systems S_1 and S_2 is a relation between the corresponding state sets explaining how a state trajectory s_1 of S_1 can be transformed into a state trajectory s_2 of S_2 and vice versa. While typical bisimulation relations require that s_1 and s_2 are observationally indistinguishable, the notion of approximate bisimulation, introduced in [9], relaxes this condition by requiring the outputs of s_1 and s_2, to simply be close, where closeness is measured with respect to the metric on the output set. In this work we consider a generalization of approximate bisimulation, called alternating approximate bisimulation, which has been introduced in [14] as an appropriate notion to relate properties of control systems affected by disturbances and their symbolic models.

Definition 2.5: [14] Consider a pair of metric systems $S_1 = (X_1, A_1 \times B_1, \underset{1}{\longrightarrow}, Y_1, H_1)$ and $S_2 = (X_2, A_2 \times B_2, \underset{2}{\longrightarrow}, Y_2, H_2)$ with the same output set $Y_1 = Y_2$ and metric $\mathbf{d}_Y$ and consider a precision $\varepsilon \in \mathbb{R}_0^+$. A relation $\mathcal{R} \subseteq X_1 \times X_2$ is said to be an alternating ε–approximate ($A\varepsilon A$) bisimulation relation between S_1 and S_2 if for every $(x_1, x_2) \in \mathcal{R}$ the following conditions are satisfied:

(i) $\mathbf{d}_Y(H_1(x_1), H_2(x_2)) \leq \varepsilon$;

(ii) $\forall a_1 \in A_1\ \exists a_2 \in A_2\ \forall b_2 \in B_2\ \exists b_1 \in B_1$ such that $x_1 \xrightarrow[1]{(a_1,b_1)} x_1'$, $x_2 \xrightarrow[2]{(a_2,b_2)} x_2'$ and $(x_1', x_2') \in \mathcal{R}$.

(iii) $\forall a_2 \in A_2\ \exists a_1 \in A_1\ \forall b_1 \in B_1\ \exists b_2 \in B_2$ such that $x_1 \xrightarrow[1]{(a_1,b_1)} x_1'$, $x_2 \xrightarrow[2]{(a_2,b_2)} x_2'$ and $(x_1', x_2') \in \mathcal{R}$.

Systems S_1 and S_2 are alternating ε–approximately ($A\varepsilon A$) bisimilar if there exists an $A\varepsilon A$ bisimulation relation so that $\mathcal{R}(X_1) = X_2$ and $\mathcal{R}^{-1}(X_2) = X_1$.

When the sets B_1 and B_2 are singleton, the above notion boils down to the one of approximate bisimulation [9]. When $\varepsilon = 0$, the above notion can be viewed as the two-player version of the notion of alternating bisimulation [15]. For a detailed discussion on the above notion of bisimulation the reader is referred to [14], [2].

III. Spline approximation of functional spaces

In this paper, we approximate the disturbance input functional space through spline analysis [16]. Given a time parameter $\tau \in \mathbb{R}^+$ define:

$$\mathcal{D}_\tau = \{d \in \mathcal{D} |\ \textit{the domain of } d \textit{ is } [0, \tau]\},$$

and suppose that:

Assumption 3.1:

(i) The set $D \subseteq \mathbb{R}^l$ is radial;

(ii) there exists $\kappa \in \mathbb{R}^+$ such that $\|d(b) - d(a)\| \leq \kappa |b - a|$, for any $a, b \in [0, \tau]$ and $d \in \mathcal{D}_\tau$.

Let $M \in \mathbb{R}^+$ so that $\|d\|_\infty \leq M$ for any $d \in \mathcal{D}_\tau$. In the sequel, we propose an approximation of the functional space $\mathcal{D}_\tau$ in the sense of the following definition.

Definition 3.2: A map

$$\mathcal{A} : \mathbb{R}^+ \rightarrow 2^{C^0([0,\tau];D)}$$

is a finite inner approximation of $\mathcal{D}_\tau$ if for any desired precision $\lambda \in \mathbb{R}^+$:

(i) $\mathcal{A}(\lambda)$ is a finite set;

(ii) $\mathcal{A}(\lambda) \subseteq \mathcal{D}_\tau$;

(iii) $\forall y \in \mathcal{D}_\tau$, $\exists z \in \mathcal{A}(\lambda)$ such that $\|y - z\|_\infty \leq \lambda$.

We start by recalling from [16] the notion of spline. Given $N \in \mathbb{N}$, consider the following functions:

$$s_0(t) = \begin{cases} 1 - t/h, & t \in [0, h], \\ 0, & \text{otherwise}, \end{cases}$$

$$s_i(t) = \begin{cases} 1 - i + t/h, & t \in [(i-1)h, ih], \\ 1 + i - t/h, & t \in [ih, (i+1)h], \\ 0, & \text{otherwise}, \end{cases}$$
$$i = 1, 2, ..., N,$$

$$s_{N+1}(t) = \begin{cases} 1 + (t - \tau)/h, & t \in [\tau - h, \tau], \\ 0, & \text{otherwise}, \end{cases}$$

where $h = \tau/(N+1)$. Functions s_i, called *splines*, are used to approximate $\mathcal{D}_\tau$. The approximation scheme that we propose is based on three steps:

- We first scale the function $d \in \mathcal{D}$ (Figure 1; first panel) to get the function $d_1 = \rho_{\kappa,\tau,M}(N, \mu) d$, where:

$$\rho_{\kappa,\tau,M}(N, \mu) := 1 - \max\left\{\frac{\mu}{M}, \frac{2\mu(N+1)}{\kappa\tau}\right\}, \quad (4)$$

  and $\mu \in \mathbb{R}^+$ is a suitable quantization parameter whose role will appear clear in the sequel. For notational simplicity in the following we write $\rho_{\kappa,\tau,M}(N, \mu) = \rho$.
- We then approximate the function $d_1 \in \mathcal{D}$ (Figure 1; second panel) by means of the piecewise–linear function d_2 (Figure 1; third panel), obtained by the linear combination of the $N+2$ splines s_i with coefficients[1] $d_1(ih)$.
- We finally approximate the function d_2 by means of function d_3 (Figure 1; fourth panel), obtained by the linear combination of the $N+2$ splines s_i with coefficients d_3^i, chosen in the set $2\mu\mathbb{Z}^l \cap \rho D$ and minimizing the distance from[2] $d_2(ih)$, i.e.

$$d_3^i = \arg \min_{d_i \in 2\mu\mathbb{Z}^l \cap \rho D} \|d_i - d_2(ih)\|.$$

Given $N \in \mathbb{N}$ and $\mu \in \mathbb{R}^+$, define the following function:

$$\Lambda_{\kappa,\tau,M}(N, \mu) = (1 - \rho)M + (1 + \rho)\kappa h + \mu. \quad (5)$$

Function Λ will be shown to be an upper bound of the error associated to the approximation scheme that we propose. It is readily seen that:

Lemma 3.3: For any $\lambda \in \mathbb{R}^+$, there exist $N \in \mathbb{N}$ and $\mu \in \mathbb{R}^+$ so that $\Lambda_{\kappa,\tau,M}(N, \mu) \leq \lambda$ and $\rho > 0$.

Let N_λ and μ_λ satisfy the inequalities in the above lemma for a given $\lambda \in \mathbb{R}^+$, and write $\rho_{\kappa,\tau,M}(N_\lambda, \mu_\lambda) = \rho$ for a shorter notation.

Definition 3.4: Consider the map

$$\mathcal{A}_{\mathcal{D}_\tau} : \mathbb{R}^+ \rightarrow 2^{C^0([0,\tau];D)}$$

[1] This second step allows us to approximate the *infinite*-dimensional space $\mathcal{D}$ by means of the *finite*-dimensional space D^{N+2}.

[2] This third step allows us to approximate the *finite*-dimensional space D^{N+2} by means of the *finite* set $(2\mu\mathbb{Z}^l \cap \rho D)^{N+2}$.

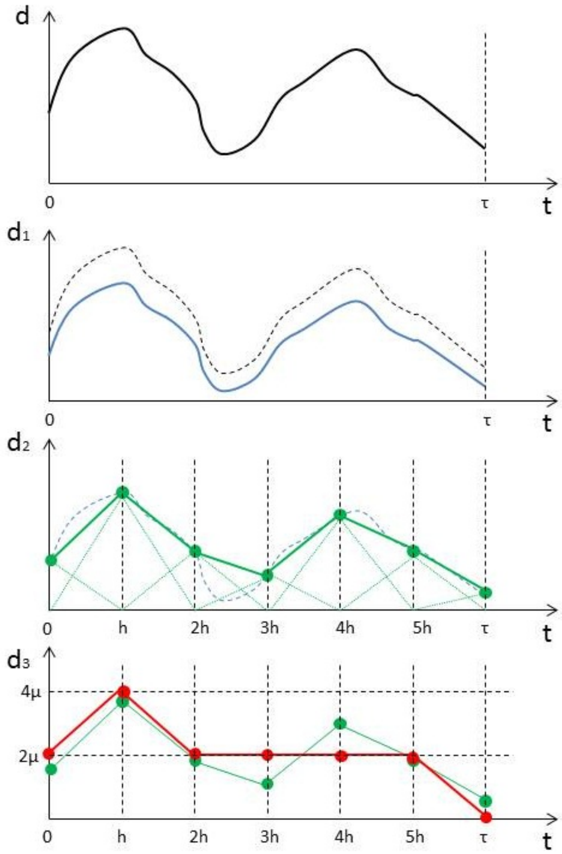

Fig. 1. Spline–based inner approximation scheme of the disturbance input space.

that associates to any precision $\lambda \in \mathbb{R}^+$ the set $\mathcal{A}_{\mathcal{D}_\tau}(\lambda)$ consisting of the collection of all functions:

$$z(t) := \sum_{i=0}^{N_\lambda+1} z_i s_i(t), \qquad t \in [0,\tau], \tag{6}$$

satisfying the following conditions:

(i) $z_i \in 2\mu_\lambda \mathbb{Z}^l \cap \rho D$, for any $i = 0, 1, ..., N_\lambda + 1$;
(ii) $\|z_{i+1} - z_i\| \leq \kappa\tau/(N_\lambda + 1)$, for any $i = 0, 1, ..., N_\lambda$.

Remark 3.5: Since the set D is bounded then the set $2\mu_\lambda \mathbb{Z}^l \cap \rho D$ is finite. Therefore the set $\mathcal{A}_{\mathcal{D}_\tau}(\lambda)$ is composed of a finite number of functions which can be effectively computed.

It is readily seen that:

Lemma 3.6: For any $\lambda \in \mathbb{R}^+$, $\mathcal{A}_{\mathcal{D}_\tau}(\lambda) \subseteq \mathcal{D}_\tau$.

We are now ready to present the main result of this section.

Theorem 3.7: Map $\mathcal{A}_{\mathcal{D}_\tau}$ in Definition 3.4 is a finite inner approximation of $\mathcal{D}_\tau$.

Proof: Consider any precision $\lambda \in \mathbb{R}^+$. By Assumption 3.1 (i), the set $\mathcal{A}_{\mathcal{D}_\tau}(\lambda)$ is finite. Hence, condition (i) in Definition 3.2 is satisfied. Condition (ii) in Definition 3.2 is implied by Lemma 3.6. We now show that also condition (iii) in Definition 3.2 is satisfied. For any function $d \in \mathcal{D}_\tau$, consider a function z as in (6), with $z_i \in 2\mu_\lambda \mathbb{Z}^l \cap \rho D$ for any $i = 0, 1, ..., N_\lambda + 1$ and

$$\|z_i - \rho\, d(ih)\| \leq \mu_\lambda, \tag{7}$$

for any $i = 0, 1, ..., N_\lambda$. Note that such values z_i always exist. We first show that function $z \in \mathcal{A}_{\mathcal{D}_\tau}(\lambda)$. By definition of z, condition (i) in Definition 3.4 is satisfied. We now show that also condition (ii) is satisfied. From (7), the following chain of inequalities holds:

$$\begin{array}{rcl} \|z_{i+1} - z_i\| & \leq & \|z_{i+1} - \rho d((i+1)h)\| \\ & + & \|\rho d((i+1)h) - \rho d(ih)\| + \|\rho d(ih) - z_i\| \\ & \leq & \rho\|d((i+1)h) - d(ih)\| + 2\mu_\lambda \\ & \leq & \rho\kappa h + 2\mu_\lambda \leq (1 - \frac{2\mu_\lambda}{\kappa h})\kappa h + 2\mu_\lambda = \kappa h, \end{array}$$

where $h = \tau/(N_\lambda + 1)$ and the last inequality holds by definition of ρ. Hence, condition (ii) is satisfied and $z \in \mathcal{A}_{\mathcal{D}_\tau}(\lambda)$. In order to conclude the proof of condition (iii) in Definition 3.2, we need to show that $\|d - z\|_\infty \leq \lambda$. By Assumption 3.1, the following chain of inequalities holds:

$$\begin{aligned} \|d - z\|_\infty &= \max_{\substack{i=0,1,...,N_\lambda \\ t\in[0,h]}} \|d(ih+t) - z(ih+t)\| \\ &\leq \max_{\substack{i=0,1,...,N_\lambda \\ t\in[0,h]}} (\|d(ih+t) - \rho\, d(ih+t)\| \\ &\quad + \|\rho\, d(ih+t) - \rho\, d(ih)\| \\ &\quad + \|\rho\, d(ih) - z(ih)\| + \|z(ih) - z(ih+t)\|) \\ &\leq (1-\rho)M + (1+\rho)\kappa h + \mu_\lambda \\ &= \Lambda_{\kappa,\tau,M}(N_\lambda, \mu_\lambda) \leq \lambda, \end{aligned}$$

where the last step holds by Eq. (5) and by definition of N_λ and μ_λ. From the above chain of inequalities, condition (iii) in Definition 3.2 is satisfied, concluding the proof. ■

We conclude this section by stressing that while the spline approximation scheme here proposed guarantees that the map $\mathcal{A}_{\mathcal{D}_\tau}$ is a finite inner approximation of $\mathcal{D}_\tau$, the scheme proposed in [12] does not. This feature is of key importance in the subsequent results.

IV. Alternating approximate bisimilar symbolic models

In this section, we propose symbolic models that approximate nonlinear control systems with disturbances in the sense of alternating approximate bisimulation.

Given the control system $\Sigma = (X, \mathcal{U}, \mathcal{D}, f)$ and a sampling time parameter $\tau \in \mathbb{R}^+$, consider the following system:

$$S_\tau(\Sigma) := (X, \mathcal{U}_\tau \times \mathcal{D}_\tau, \underset{\tau}{\longrightarrow}, Y, H), \tag{8}$$

where:

- $\mathcal{U}_\tau = \{u \in \mathcal{U} |$ the domain of u is $[0,\tau]$ and $u(t) = u(0), t \in [0,\tau]\}$;
- $\mathcal{D}_\tau = \{d \in \mathcal{D} |$ the domain of d is $[0,\tau]\}$;
- $x \xrightarrow[\tau]{(u,d)} x'$ if there exists a trajectory $\xi : [0,\tau] \to X$ of Σ satisfying $\xi_{xud}(\tau) = x'$;
- $Y = X$;
- $H = 1_X$.

System $S_\tau(\Sigma)$ is metric when we regard $Y = X$ as being equipped with the metric $\mathbf{d}_Y(p,q) = \|p - q\|$. System $S_\tau(\Sigma)$

can be thought of as the time discretization of the control system Σ. Given a nonlinear control system Σ, suppose that $\mathcal{D}_\tau$ satisfies Assumption 3.1 with $\kappa = \kappa_d \in \mathbb{R}^+$ and let $M_d \in \mathbb{R}^+$ s.t. $\|d\|_\infty \leq M_d$ for any $d \in \mathcal{D}_\tau$. Consider a vector of (positive) quantization parameters

$$\mathbb{Q} = (\tau, \eta, \mu_u, N_d, \mu_d), \tag{9}$$

and define the following system:

$$S_{\mathbb{Q}}(\Sigma) := (X_{\mathbb{Q}}, L_{\mathbb{Q}}, \underset{\mathbb{Q}}{\longrightarrow}, Y_{\mathbb{Q}}, H_{\mathbb{Q}}), \tag{10}$$

where:

- $X_{\mathbb{Q}} = 2\eta\mathbb{Z}^n \cap X$;
- $L_{\mathbb{Q}} = (2\mu_u\mathbb{Z}^m \cap U) \times \mathcal{A}_{\mathcal{D}_\tau}(\Lambda_{\kappa_d,\tau,M_d}(N_d, \mu_d))$, where $\mathcal{A}_{\mathcal{D}_\tau}$ is a finite inner approximation of $\mathcal{D}_\tau$, as in Definition 3.4 and function Λ is defined as in (5);
- $x \xrightarrow[\mathbb{Q}]{(u,d)} y$ if $\|\xi_{xud}(\tau) - y\| \leq \eta$;
- $Y_{\mathbb{Q}} = X$;
- $H_{\mathbb{Q}} = \imath : X_{\mathbb{Q}} \hookrightarrow Y_{\mathbb{Q}}$.

Remark 4.1: It is readily seen that the system $S_{\mathbb{Q}}(\Sigma)$ is countable and it becomes symbolic when the set of states X is bounded. As stressed in Remark 3.5, the set of control and disturbance inputs $L_{\mathbb{Q}}$ can be effectively computed, from which the system $S_{\mathbb{Q}}(\Sigma)$ can be effectively computed.

We now have all the ingredients to present the main result of this paper.

Theorem 4.2: Consider a control system $\Sigma = (X, \mathcal{U}, \mathcal{D}, f)$ and suppose that:

(A1) There exists a δ–ISS Lyapunov function satisfying the inequality (ii) in Definition 2.2 for some $\lambda \in \mathbb{R}^+$.

(A2) There exists a $\mathcal{K}_\infty$ function γ such that[3]:

$$V(x, x') - V(x, x'') \leq \gamma(\|x' - x''\|),$$

for every $x, x', x'' \in X$.

(A3) The disturbance input space $\mathcal{D}_\tau$ satisfies Assumption 3.1.

Then, for any desired precision $\varepsilon \in \mathbb{R}^+$ and any quantization parameters in the vector $\mathbb{Q}$ in (9) satisfying the following inequality:

$$\frac{\max\{\sigma_u(\mu_u), \sigma_d(\Lambda_{\kappa_d,\tau,M_d}(N_d, \mu_d))\}}{\lambda} + \frac{\gamma(\eta)}{1 - e^{-\lambda\tau}} \leq \underline{\alpha}(\varepsilon), \tag{11}$$

systems $S_\tau(\Sigma)$ and $S_{\mathbb{Q}}(\Sigma)$ are alternating ε–approximately bisimilar.

Proof: Consider the relation $\mathcal{R} \subseteq X \times X_{\mathbb{Q}}$ defined by $(x, y) \in \mathcal{R}$ if and only if $V(x, y) \leq \underline{\alpha}(\varepsilon)$. Condition (i) in Definition 2.5 is satisfied by the definition of $\mathcal{R}$ and condition (i) in Definition 2.2. Let us now show that condition (ii) in Definition 2.5 holds. Consider any $(x, y) \in \mathcal{R}$. For any $u_1 \in \mathcal{U}_\tau$ there exists $u_2 \in 2\mu_u\mathbb{Z}^m \cap U$ such that:

$$\|u_2 - u_1\|_\infty \leq \mu_u. \tag{12}$$

[3]Note that since V is smooth, if the state space X is bounded, which is the case in many concrete applications, one can always choose $\gamma(\|w - z\|) = \left(\max_{x,y \in X} \|\frac{\partial V}{\partial y}(x, y)\|\right) \|w - z\|$.

Set $d_1 = d_2 \in \mathcal{D}_\tau$ and $z = \xi_{yu_2d_2}(\tau)$. There exists $v \in X_{\mathbb{Q}}$ so that:

$$\|z - v\| \leq \eta. \tag{13}$$

Hence, by definition of $S_{\mathbb{Q}}(\Sigma)$, the transition $y \xrightarrow[\mathbb{Q}]{u_2,d_2} v$ is in $S_{\mathbb{Q}}(\Sigma)$. Consider now the transition $x \xrightarrow[\tau]{u_1,d_1} w$ in $S_\tau(\Sigma)$. By Assumption (A1), condition (ii) in Definition 2.2 and the inequality in (12), one gets:

$$\begin{aligned}
&\tfrac{\partial V}{\partial w} f(w, u_1, d_2) + \tfrac{\partial V}{\partial z} f(z, u_2, d_2) \\
&\leq -\lambda V(w, z) + \sigma_u(\|u_1 - u_2\|) + \sigma_d(\|d_1 - d_2\|) \\
&\leq -\lambda V(w, z) + \sigma_u(\mu_u),
\end{aligned}$$

which, by Assumption (A2), the definition of $\mathcal{R}$ and the inequality in (13), implies:

$$\begin{aligned}
V(w, v) &\leq V(w, z) + \gamma(\|z - v\|) \\
&\leq V(w, z) + \gamma(\eta) \\
&\leq e^{-\lambda\tau} V(x, y) + (1 - e^{-\lambda\tau}) \frac{\sigma_u(\mu_u)}{\lambda} + \gamma(\eta) \\
&\leq e^{-\lambda\tau} \underline{\alpha}(\varepsilon) + (1 - e^{-\lambda\tau}) \frac{\sigma_u(\mu_u)}{\lambda} + \gamma(\eta).
\end{aligned} \tag{14}$$

Hence, by the inequality in (11), $V(w, v) \leq \underline{\alpha}(\varepsilon)$, from which $(w, v) \in \mathcal{R}$ and condition (ii) in Definition 2.5 is proved. By using similar arguments, it is possible to show that condition (iii) in Definition 2.5 is true, as well.
Finally, by definition of $\mathcal{R}$, it is easy to see that $\mathcal{R}(X) = X_{\mathbb{Q}}$ and $\mathcal{R}^{-1}(X_{\mathbb{Q}}) = X$. ■

V. An Illustrative Example

Consider a nonlinear control system Σ described by the following differential equation:

$$\dot{x} = f(x, u, d) = \begin{bmatrix} -4y + z^2 - u \\ 2y - 7\sin z + d \end{bmatrix}, \tag{15}$$

where $x = (y, z) \in X = [-1, 1[\times [-1, 1[$, $u \in U = [-1, 1]$ and $d \in D = [-0.005, 0.005]$. We first note that $M_d = 0.005$. We now construct a symbolic model for Σ. To this aim, we apply Theorem 4.2. Consider the following quadratic function:

$$V(x_1, x_2) = 0.5\|x_1 - x_2\|_2^2.$$

It is readily seen that V satisfies condition (i) of Definition 2.2 with $\underline{\alpha}(r) = 0.5\, r^2$ and $\overline{\alpha}(r) = r^2$, $r \in \mathbb{R}_0^+$. Moreover:

$$\begin{aligned}
&\frac{\partial V}{\partial x_1} f(x_1, u_1, d_1) + \frac{\partial V}{\partial x_2} f(x_2, u_2, d_2) = \\
&= (x_1 - x_2)' [f(x_1, u_1, d_1) - f(x_2, u_2, d_2)] = \\
&\leq -5V(x_1, x_2) + 2\|u_1 - u_2\| + 4\|d_1 - d_2\|.
\end{aligned}$$

Condition (ii) of Definition 2.2 is therefore fulfilled with $\lambda = 5$, $\sigma_u(r) = 2r$, $\sigma_d(r) = 4r$, $r \in \mathbb{R}_0^+$. Hence, by Theorem 2.3, system (15) is δ–ISS. In this example, we consider disturbance inputs with Lipschitz constant $\kappa_d = 2 \cdot 10^{-3}$. For a chosen precision $\varepsilon = 0.25$, the inequality in (11) is satisfied with the following choice of parameters:

$$\tau = 2 \quad \eta = 0.005 \quad \mu_u = 0.005 \quad \mu_d = 0.001 \quad N_d = 1.$$

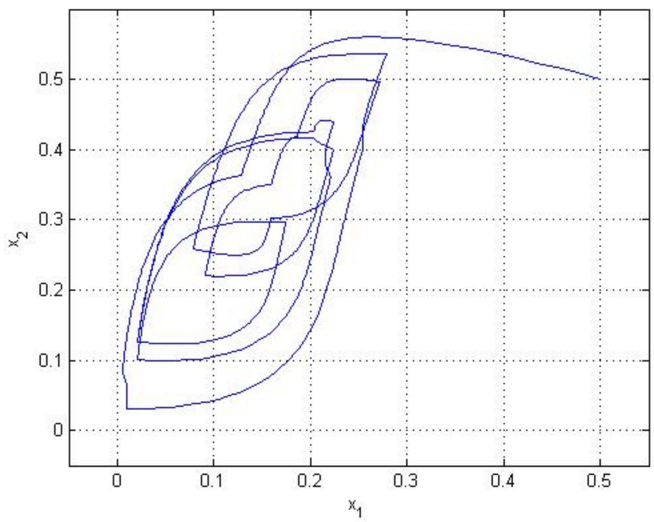

Fig. 2. State trajectory of control system Σ.

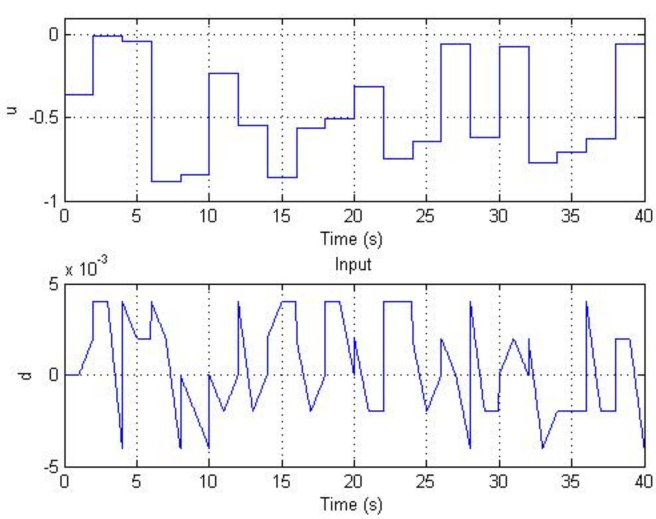

Fig. 3. Control input and disturbance input realization.

We have now all the ingredients to construct a symbolic model that is alternating ε–approximately bisimilar to the system in (15). The resulting number of states is 40401, the number of control inputs is 201 and the number of disturbance inputs is 66. Due to the large size of the symbolic model obtained, further details are not included here. Instead, we use the obtained symbolic model to solve the control design problem of enforcing a trajectory starting from the initial state $x_0 = (0.5, 0.5)$ to definitively remain in the positive orthant, independently from the disturbance signal realization. By using standard fixed–point algorithms, we designed the symbolic controller enforcing the prescribed specification. Figure 2 shows that the specification is indeed satisfied for the disturbance signal illustrated in Figure 3.

VI. Conclusion

In this paper we proposed symbolic models that approximate nonlinear control systems affected by disturbances, in the sense of alternating approximate bisimulation. The results presented in this paper provide an important improvement upon the results reported in [14] in that they propose symbolic models that can be effectively computed, through spline analysis.

Acknowledgment. The authors are grateful to Paulo Tabuada and Antoine Girard for fruitful discussions on the topic of the present paper.

References

[1] M. Egerstedt, E. Frazzoli, and G. J. Pappas, Eds., *Special issue on symbolic methods for complex control systems*. IEEE Transactions on Automatic Control, July 2006, vol. 51, no. 6.

[2] P. Tabuada, *Verification and Control of Hybrid Systems: A Symbolic Approach*. Springer, 2009.

[3] R. Alur, T. A. Henzinger, G. Lafferriere, and G. J. Pappas, "Discrete abstractions of hybrid systems," *Proceedings of the IEEE*, vol. 88, pp. 971–984, 2000.

[4] P. Tabuada and G. Pappas, "Linear Time Logic control of discrete-time linear systems," *IEEE Transactions on Automatic Control*, vol. 51, no. 12, pp. 1862–1877, 2006.

[5] L. Habets, P. Collins, and J. V. Schuppen, "Reachability and control synthesis for piecewise-affine hybrid systems on simplices," *IEEE Transaction on Automatic Control*, vol. 51, no. 6, pp. 938–948, 2006.

[6] C. Belta and L. Habets, "Controlling a class of nonlinear systems on rectangles," *IEEE Transactions of Automatic Control*, vol. 51, no. 11, pp. 1749–1759, 2006.

[7] R. Milner, *Communication and Concurrency*. Prentice Hall, 1989.

[8] D. Park, "Concurrency and automata on infinite sequences," ser. Lecture Notes in Computer Science, Springer-Verlag, Ed., vol. 104, 1981, pp. 167–183.

[9] A. Girard and G. Pappas, "Approximation metrics for discrete and continuous systems," *IEEE Transactions on Automatic Control*, vol. 52, no. 5, pp. 782–798, 2007.

[10] G. Pola, A. Girard, and P. Tabuada, "Approximately bisimilar symbolic models for nonlinear control systems," *Automatica*, vol. 44, pp. 2508–2516, October 2008.

[11] A. Girard, G. Pola, and P. Tabuada, "Approximately bisimilar symbolic models for incrementally stable switched systems," *IEEE Transactions of Automatic Control*, vol. 55, no. 1, pp. 116–126, January 2010.

[12] G. Pola, P. Pepe, M. Di Benedetto, and P. Tabuada, "Symbolic models for nonlinear time-delay systems using approximate bisimulations," *Systems and Control Letters*, vol. 59, pp. 365–373, 2010.

[13] G. Pola, P. Pepe, and M. Di Benedetto, "Alternating approximately bisimilar symbolic models for nonlinear control systems with unknown time-varying delays," in *49th IEEE Conference on Decision and Control (CDC)*, dec. 2010, pp. 7649 –7654.

[14] G. Pola and P. Tabuada, "Symbolic models for nonlinear control systems: Alternating approximate bisimulations," *SIAM Journal on Control and Optimization*, vol. 48, no. 2, pp. 719–733, 2009.

[15] R. Alur, T. Henzinger, O. Kupferman, and M. Vardi, "Alternating refinement relations," in *Proceedings of the 8th International Conference on Concurrence Theory*, ser. Lecture Notes in Computer Science, no. 1466. Springer, 1998, pp. 163–178.

[16] M. H. Schultz, *Spline Analysis*. Prentice Hall, 1973.

[17] E. Sontag, *Mathematical Control Theory*, 2nd ed., ser. Texts in Applied Mathematics. New-York: Springer-Verlag, 1998, vol. 6.

[18] D. Angeli and E. Sontag, "Forward completeness, unboundedness observability, and their Lyapunov characterizations," *Systems and Control Letters*, vol. 38, pp. 209–217, 1999.

[19] D. Angeli, "A Lyapunov approach to incremental stability properties," *IEEE Transactions on Automatic Control*, vol. 47, no. 3, pp. 410–421, 2002.