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Optimal Encoder and Control Strategies in Stochastic Control Subject to Rate Constraints for Channels with Memory and Feedback

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Abstract— **This paper is concerned with the joint design of encoder/control strategies which minimize a control pay-off subject to a rate constraint on the feedback information provided by the system. The control system is described by a conditional distribution, while the rate is described by the directed information of a noisy communication channel. By generalizing the Posterior Matching Scheme (PMS), a scheme that achieves operational capacity, to channels with memory and feedback, the joint optimality of encoder/control strategies is analyzed. A principle of optimality is derived and a dynamic programming equation is obtained, which together describe how the optimal encoder/control strategies should be selected. It is shown that joint optimality separates into an optimal encoder strategy based on the PMS, and a control strategy obtained via the dynamic programming equation.**

I. Introduction

The problems in information transmission as put forward by Shannon can be classified into 1) analysis, and 2) design or synthesis. Analysis deals with questions of coding theorems (direct and converse part) providing a mathematical formulae for the information capacity to be operational. Design deals with questions of encoder and decoder synthesis which achieve the operational capacity. Although, sufficient progress have been made over the last 50 years in analyzing and synthesizing for the Discrete Memoryless Channel (DMC), for channels with memory and feedback little progress has been reported [1].

Coding theorems for DMC and for channels with memory and feedback are derived in many places; Shannon's initial work under ergodicity, Dubrushins information density, Verdu-Han information spectrum, and their generalization to feedback channels in [2], [3].

For channels with memory and feedback the measure of information often employed is the directed information, which accounts for causality and the direction of information flow introduced in [4] and subsequently applied in [5]. A capacity achieving encoder design for DMC is introduced recently in [6], [7], [8], via the so-called Posterior Matching Scheme (PMS), a recursive encoding scheme which achieves the capacity of DMC with feedback.

This paper is concerned with analysis and synthesis questions, related to stochastic optimal control over finite rate noisy channels, under general conditions on the channel and control system conditional distributions. The main problem investigated is to find optimal encoders and controllers which minimize a control pay-off, subject to a rate constraint. The rate constraint is the directed information between the control system output and the channel output.

The main results include 1) properties of optimal encoders/controllers, 2) generalization of PMS to channels with memory and feedback, and 3) joint optimality of encoders/controllers, which under certain conditions separate into PMS encoder and controller design via dynamic programming.

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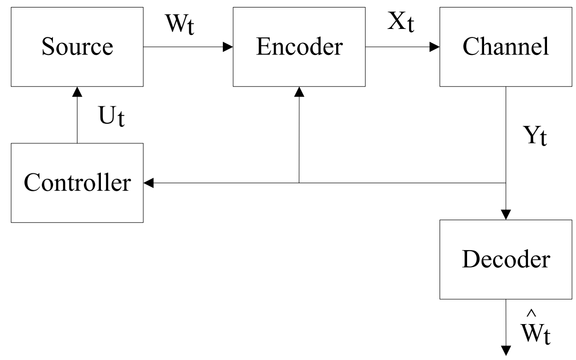

Fig. II.1. Control Communication System with Feedback

II. Problem Formulation

Define $\mathbb{Z}_+^n \stackrel{\triangle}{=} \{0,1,\ldots,n\}$, $n \in \mathbb{Z}_+ \stackrel{\triangle}{=} \{0,1,\ldots\}$, and assume all processes (introduced below) are defined on a complete probability space $(\Omega, \mathbb{F}(\Omega), \mathbb{P})$ with filtration $\{\mathbb{F}_t : t \in \mathbb{Z}_+^n\}$. The basic model is shown in Fig.II.1. The alphabets of the source output, channel input, channel output, decoder output, and controller output are sequences of Polish spaces, $\{\mathscr{W}_t : t \in \mathbb{Z}_+^n\}$, $\{\mathscr{X}_t : t \in \mathbb{Z}_+^n\}$, $\{\mathscr{Y}_t : t \in \mathbb{Z}_+^n\}$, $\{\hat{\mathscr{W}}_t : t \in \mathbb{Z}_+^n\}$, and $\{\mathscr{U}_t : t \in \mathbb{Z}_+^n\}$, respectively. The abstract alphabets are associated with their corresponding measurable spaces $(\mathscr{W}_t, \mathscr{B}(\mathscr{W}_t))$, $(\mathscr{X}_t, \mathscr{B}(\mathscr{X}_t))$, $(\mathscr{Y}_t, \mathscr{B}(\mathscr{Y}_t))$, $(\hat{\mathscr{W}}_t, \mathscr{B}(\hat{\mathscr{W}}_t))$, and $(\mathscr{U}_t, \mathscr{B}(\mathscr{U}_t))$. Thus, sequences are identified with the product measurable spaces, $(\mathscr{W}_{0,n}, \mathscr{B}(\mathscr{W}_{0,n}) \stackrel{\triangle}{=} \times_{k=0}^n (\mathscr{W}_k, \mathscr{B}(\mathscr{W}_k))$, and similarly for the rest. The source output, channel input, channel output, decoder output, and controller output are processes denoted by $W^n \stackrel{\triangle}{=} \{W_t : t \in \mathbb{Z}_+^n\}, W_t : \Omega \mapsto \mathscr{W}_t$, and similarly for the rest. Probability measures on any measurable space $(\mathscr{Z}, \mathscr{B}(\mathscr{Z}))$ are denoted by $\mathscr{M}_1(\mathscr{Z})$.

A. Definition of Sub-Systems

Information Source: is a sequence of stochastic Kernels (conditional distributions)

$$\{P_j(dw_j|w^{j-1},u^{j-1},y^{j-1},x^{j-1}) : j=0,1,...n\}$$

Depending on the measurability properties of U^n and X^n the source may be simplified.

Channel Encoder: is a sequence of stochastic Kernels

$$\{P_j(dx_j|x^{j-1},w^j,u^j,y^{j-1}) : j=0,1,...,n\}$$

Definition 2.1: (Encoder Strategies)
a) The set of deterministic encoder strategies denoted by $\mathscr{E}_{ad}^{nm}[0,n]$ is a sequence of functions $\{e_j : \mathscr{X}_{0,j-1} \times \mathscr{W}_{0,j} \times \mathscr{U}_{0,j} \times \mathscr{Y}_{0,j-1} \rightarrow \mathscr{X}_j : x_j = e_j(x^{j-1},w^j,u^j,y^{j-1}), e_j(\cdot)$ is measurable $j \in \mathbb{Z}_+^n\}$.
b) Markov with respect to the source strategies denoted by $\mathscr{E}_{ad}[0,n] \subseteq \mathscr{E}_{ad}^{nm}[0,n]$ are functions $\{e_j(w_j,u^j,y^{j-1}) : j \in \mathbb{Z}_+^n\}$.

Definition 2.2: (Control Strategies)
The set of admissible deterministic control strategies denoted by $\mathscr{C}_{ad}[0,n]$ is a sequence of functions $\{c_j : \mathscr{Y}_{0,j-1} \times \mathscr{U}_{0,j-1} \rightarrow \mathscr{U}_j : u_j = c_j(y^{j-1},u^{j-1}), c_j(\cdot)$ is measurable $j \in \mathbb{Z}_+^n\}$.

Communication Channel: is a sequence of stochastic Kernels

$$\{P_j(dy_j|y^{j-1},x^j,w^j,u^j) : j=0,1,...n\}$$

If $P_j(dy_j|y^{j-1},x^j,w^j,u^j) = P(dy_j|x_j)$, *a.s.*, $j=0,1,...,n$, the channel is called a Discrete Memoryless Channel (DMC).

Channel Decoder: is a sequence of stochastic Kernels

$$\{P_j(d\hat{w}_j|\hat{w}^{j-1},y^j,u^j) : j=0,1,...,n\}$$

A deterministic decoder is a sequence of delta measures identified by a sequence of functions $\{d_j : \hat{\mathscr{W}}_{0,j-1} \times \mathscr{Y}_{0,j} \times \mathscr{U}_{0,j} \rightarrow \hat{\mathscr{W}}_j : \hat{w}_j = d_j(\hat{w}^{j-1},y^j,u^j),\ d_j(.)$ is measurable $j \in \mathbb{Z}_+^n\}$

The $\varepsilon-$achievable rate for a channel is defined below.

Definition 2.3: a) An $(n,\mathscr{M}_n,\varepsilon_n)$ code for the channel consists of the following:

1) A set of messages $\mathscr{M}_n \triangleq \{1,2,\ldots,M_n\}$ and a class of encoders (deterministic or random) measurable mappings $\{\varphi_i : \mathscr{M}_n \times \mathscr{Y}_{i-1} \rightarrow \mathscr{X}_i : i \in \mathbb{Z}_+^n\}$ that transforms each message $W \in \mathscr{M}_n$ into a channel input $X^{n-1} \in \mathscr{X}_{0,n-1}$.
2) A class of decoder measurable mappings $d_n : \mathscr{Y}_{0,n-1} \rightarrow \mathscr{M}_n$, such that the average probability of decoding error satisfies

$$P_e^{(n)} \triangleq \frac{1}{M_n} \sum_{w \in \mathscr{M}_n} Prob(\hat{W} \neq w | W=w) = \varepsilon_n, \quad \hat{W} = d_n(Y^{n-1})$$

b) R is an ε-achievable rate if there exists an (n,M_n,ε_n) code satisfying $\limsup_{n\rightarrow\infty} \varepsilon_n \leq \varepsilon$ and $\liminf_{n\rightarrow\infty} \frac{1}{n}\log M_n \geq R$. The supremum of all ε achievable rates R for all $0 \leq \varepsilon < 1$ is the channel capacity.

B. Controlled Pay-Off and Directed Information

In this section we recall some basic results found in [10] and establish generalizations. The joint probability measure induced on the overall control/communication system is

$$\begin{aligned} P_{0,n}(dw^n,dx^n,dy^n,d\hat{w}^n,du^n) &= \otimes_{i=0}^n P_i(d\hat{w}_i|\hat{w}^{i-1},y^i,u^i) \\ &\otimes P_i(dy_i|y^{i-1},x^i,w^i,u^i) \otimes P_i(dx_i|y^{i-1},x^{i-1},w^i,u^i) \\ &\otimes P_i(dw_i|w^{i-1},u^i,y^{i-1},x^{i-1}) \otimes P_i(du_i|u^{i-1},y^{i-1}) \end{aligned}$$

Depending on the measurability properties of the sub-systems, the joint probability is simplified. Given encoder and control strategies, the average information flow from any source sequence W^n to a corresponding output sequence Y^n is defined via directed information [4]:

$$\begin{aligned} I(W^n \rightarrow Y^n) &\triangleq \sum_{i=0}^n I(W^i;Y_i|Y^{i-1}) \\ &= \sum_{i=0}^n \int \log \frac{P_i(dy_i|y^{i-1},w^i)}{P_i(dy_i|y^{i-1})} P_{0,i}(dy^i,dw^i) \end{aligned} \quad \text{(II.1)}$$

The pay-off for controlling the system is defined by

$$\begin{aligned} &J_{0,n}(\{c_j\}_{j=0}^n,\{e_j\}_{j=0}^n) \\ &\triangleq E\Big\{\sum_{i=0}^{n-1} \ell(W_i,e_i(X^{i-1},W^i,U^i,Y^{i-1}),c_i(Y^{i-1})) + \kappa(W_n)\Big\} \end{aligned}$$

where $\ell(.),\kappa(.)$ are assumed bounded and continues.

C. Optimization Problems

The list of problems of interest arising is capacity computation, encoder design which achieves the capacity, and joint encoder/control strategies which minimize the control pay-off subject to rate constraint.

Problem 2.4: (Information Capacity)
Given an admissible set of source and channel inputs $\mathscr{A}_{ad}[0,n]$, the information capacity is defined by

$$C_{0,n} \triangleq \sup_{(W^n,X^n)\in\mathscr{A}_{ad}[0,n]} \frac{1}{n+1} I(W^n \rightarrow Y^n) \quad \text{(II.2)}$$

The operational meaning of (II.2) assumes conditions for existence of $C_{0,\infty} \triangleq \liminf_{n\rightarrow\infty} C_{0,n}$ and direct and converse coding theorems [1], [2], [3]. Given $C_{0,\infty}$ exists, one can maximize $\frac{1}{n+1}I(W^n \rightarrow Y^n)$ over admissible encoders for a fixed control strategy, hence the next problem.

Problem 2.5: (Maximizing Directed Information) Given an admissible control class $\mathscr{C}_{ad}[0,n]$ find an admissible encoder, $\{e_j^*(x^{j-1},w^j,u^j,y^{j-1}) :\ j \in \mathbb{Z}_+^n\} \in \mathscr{E}_{ad}^{nm}[0,n]$ which maximizes

$$J_{0,n}^1(\{c_j\}_{j=0}^n,\{e_j^*\}_{j=0}^n) \triangleq \max_{\{e_j\}_{j=0}^n \in \mathscr{E}_{ad}^{nm}[0,n]} \frac{1}{n+1} I(W^n \rightarrow Y^n)$$

The joint encoder/control problem of interest is to fix the rate and minimize a control pay-off, over admissible encoders and control strategies.

Problem 2.6: (Minimizing Control Pay-Off Subject to Rate Constraint)

Given $R_n \in (0, C_{0,n}]$, find $\{e_j^* : j \in \mathbb{Z}_+^n\} \in \mathscr{E}_{ad}^{nm}[0,n]$ and $\{c_j^* :\in \mathbb{Z}_+^n\} \in \mathscr{C}_{ad}[0,n]$ which minimize

$$J_{0,n}^2(\{c_j^*\}_{j=0}^n, \{e_j^*\}_{j=0}^n) \stackrel{\triangle}{=} \inf_{\{c_j\}_{j=0}^n \in \mathscr{C}_{ad}[0,n]} \inf_{\{e_j\}_{j=0}^n \in \mathscr{E}_{ad}^{nm}[0,n]: \frac{1}{n+1} I(W^n \to Y^n) \leq R_n} \frac{1}{n+1} J_{0,n}(\{c_j\}_{j=0}^n, \{e_j\}_{j=0}^n)$$

Remark 2.7:
1) Problem 2.4 with $C_{0,\infty} \stackrel{\triangle}{=} \liminf_{n\to\infty} C_{0,n}$, together with coding theorems, has operational meaning, hence it corresponds to channel capacity.
2) Problem 2.5 with $J_{0,\infty}^1 \stackrel{\triangle}{=} \liminf_{n\to\infty} J_{0,n}^1$ is an infinite horizon encoder design.
3) Problem 2.6 with $J_{0,\infty}^2 \stackrel{\triangle}{=} \liminf_{n\to\infty} J_{0,n}^2$ is an infinite horizon joint encoder/control design.
4) Problem 2.6 is very general since it accounts for rate constraints, for any channel and control system. The authors believe that a full analysis of the finite horizon case will provide insight how to address the infinite horizon version of Problems 2.5, 2.6, which is a subject of future research. For the DMC and source which satisfies $P_j(dw_i|w^{i-1}, u^{i-1}, y^{i-1}, x_{i-1}) = P_j(dw_i|w_{i-1}, u_{i-1})$ several results currently in the literature can be used to address the infinite horizon case.

III. Markovian Encoders and Separated Control Strategies

The first goal is to identify general conditions so that maximizing $I(W^n \to Y^n)$ over an encoder with information structure $\{(x^{j-1}, w^j, u^j, y^{j-1}) : j \in \mathbb{Z}_n^+\}$ is equivalent to maximizing $I(W^n \to Y^n)$ over an encoder with information structure $\{(w_j, u^j, y^{j-1}) : j \in \mathbb{Z}_n^+\}$. Similarly for the joint encoder/controller minimization Problem 2.6.

The following conditions are sufficient to establish encoder laws which are Markov with respect to the source.

Assumptions 3.1: The information source satisfies

$$\begin{aligned} &P_j(dw_j|w^{j-1}, u^{j-1}, y^{j-1}, x^{j-1}) \\ &= P_j(dw_j|w_{j-1}, u^{j-1}, y^{j-1}, x_{j-1}), a.s., j \in \mathbb{Z}_+^n \end{aligned}$$

Assumptions 3.2: The communication channel satisfies

$$\begin{aligned} &P_j(dy_j|y^{j-1}, x^j, w^j, u^j) = P_j(dy_j|y^{j-1}, x_j, w_j, u^j), \\ &a.s., j \in \mathbb{Z}_+^n \end{aligned}$$

These assumptions are general and include many interesting control models, including non-linear models.

Theorem 3.3: Under Assumptions 3.1 and 3.2 and a fixed control class $\mathscr{C}_{ad}[0,n]$ the sequence of optimal encoder strategies for Problems 2.5 and 2.6 over $\mathscr{E}_{ad}^{nm}[0,n]$ have the form

$$e_j^*(x^{j-1}, w^j, u^j, y^{j-1}) = g_j(w_j, u^j, y^{j-1}), j \in \mathbb{Z}_+^n$$

and

$$\begin{aligned} &J_{0,n}^1(\{c_j\}_{j=0}^n, \{e_j^*\}_{j=0}^n) \stackrel{\triangle}{=} \\ &= \sup_{\{g_j\}_{j=0}^n \in \mathscr{E}_{ad}[0,n]} \frac{1}{n+1} I(W^n \to Y^n) \end{aligned}$$

$$J_{0,n}^2(\{c_j^*\}_{j=0}^n, \{e_j^*\}_{j=0}^n) \stackrel{\triangle}{=} \inf_{\{c_j\}_{j=0}^n \in \mathscr{C}_{ad}[0,n]} \inf_{\{e_j\}_{j=0}^n \in \mathscr{E}_{ad}[0,n]: \frac{1}{n+1} I(W^n \to Y^n) \leq R_n} \frac{1}{n+1} J_{0,n}(\{c_j\}_{j=0}^n, \{e_j\}_{j=0}^n)$$

The previous theorem generalizes [10].

The point to be made in Theorem 3.3 is that, under Assumptions 3.1 and 3.2, encoder strategies optimizing Problems 2.5 and 2.6 are equivalent to Markov (with respect to the source) strategies, hence only current source symbols are encoded and transmitted over each time the channel is used. The next definition of separated encoder strategies is often employed in stochastic control systems with partial information.

Definition 3.4: (Separated Encoder and Control Strategies) Given specific encoder and control strategies, define the conditional distribution $\Pi^{x,u}(dw_j|y^{j-1}) \stackrel{\triangle}{=} Prob(W_j \in dw_j|Y^{j-1} = y^{j-1})$, $j \in \mathbb{Z}_+^n$.

A deterministic encoder $\{g_j\}_{j=0}^n \in \mathscr{E}_{ad}[0,n]$ is called separated if $x_j = g_j(w_j, y^{j-1}, u^j)$ depends on $Y^{j-1} = y^{j-1}$ only through the conditional distribution $\Pi^{x,u}(dw_j|y^{j-1})$, $j \in \mathbb{Z}_+^n$. The set of separated deterministic encoder strategies is denoted by $\mathscr{E}_{ad}^{sep}[0,n]$. Separated control strategies are defined similarly and they are denoted by $\mathscr{C}_{ad}^{sep}[0,n]$.

IV. Generalized PMS for Channels with Memory and Feedback

Solutions to Problems 2.5 and 2.6 are obtained by first analyzing the optimality of the encoder. The results of this section are valid under the following assumptions.

Assumptions 4.1: Assumptions 3.1, 3.2 hold and in addition

$$P_j(dy_j|y^{j-1}, x_j, w_j) = P_j(dy_j|y^{j-1}, x_j), a.s., j \in \mathbb{N}_+^n$$

Assumptions 4.1 are sufficient to define capacity via the channel input and output.

The next statement is easily shown.

$$\text{Assumptions 4.1} \Rightarrow I(W^n \to Y^n) = \sum_{i=0}^n I(X_i; Y_i|Y^{i-1})$$

Hence, the following definition of information capacity.

Definition 4.2: Suppose Assumptions 4.1 hold. Let $\mathscr{Q}_{ad}^{pc}$ denote the set of channel input distributions which satisfy power constraints. The finite time capacity is defined by

$$C_{0,n}^1 \stackrel{\triangle}{=} \sup_{\{P_i(dx_i; y^{i-1})\}_{i=0}^n \in \mathscr{Q}_{ad}^{pc}[0,n]} \frac{1}{n+1} \sum_{i=0}^n I(X_i; Y_i|Y^{i-1})$$

The infinite horizon information capacity is defined by

$$C^1_{0,\infty} = \liminf_{n\to\infty} \sup_{\{P_i(dx_i;y^{i-1})\}_{i=0}^n \in \mathscr{Q}^{pc}_{ad}[0,n]} \frac{1}{n+1} \sum_{i=0}^n I(X_i;Y_i|Y^{i-1})$$

Theorem 4.3: Suppose Assumptions 4.1 hold and $\{X_i, Y_i : i \in \mathbb{Z}_+\}$ are stationary ergodic.

a) Any achievable rate R satisfies

$$R \leq \liminf_{n\to\infty} \frac{1}{n} \log M_n \leq \liminf_{n\to\infty} \sup_{\{P_i(dx_i;y^{i-1})\}_{i=0}^{n-1} \in \mathscr{Q}^{pc}_{ad}[0,n-1]} \frac{1}{n} \sum_{i=0}^{n-1} I(X_i;Y_i|Y^{i-1})$$

b) The channel capacity is $C^1_{0,\infty}$

Proof. Follows from stationarity and ergodicity.

The operational meaning of $C^1_{0,\infty}$ can be generalized to information stable processes (Dubrushin's definition).

Here it is shown how to design an encoder so that the directed information including the encoder, $I(W^n \to Y^n)$ is precisely equal to the supremum in $C^1_{0,n}$.

Let $\{P^*_i(dx_i|y^{i-1}) : i \in \mathbb{Z}^n_+\} \in \mathscr{Q}^{pc}_{0,n}$ be the sequence of stochastic kernels which achieves the supremum of $C^1_{0,n}$. Let $F^*_{X_i|Y^{i-1}}(x_i)$ be its corresponding conditional distribution.

Consider an encoder of the form

$$\Big\{x^*_i = g^*_i(w_i, y^{i-1}) = g^{*,s}_i(w_i, P_i(dw_i|y^{i-1})) : i = 0,1,\ldots,n\Big\} \in \mathscr{E}^{sep}_{ad}[0,n]$$

where $P_i(dw_i|y^{i-1})$ is a stochastic kernel, and denote by $F_{W_i|Y^{i-1}}(w_i)$ its corresponding conditional distribution function.

Define the Posterior Matching Scheme

$$\Big\{X^*_i = g^{*,s}_i(W_i, F_{W_i|Y^{i-1}}(W_i)) = F^{*,-1}_{X_i|Y^{i-1}} \circ F_{W_i|Y^{i-1}}(W_i) : i = 0,1,\ldots,n\Big\} \tag{IV.3}$$

This scheme corresponds to an encoder transmitting at each $i \in \mathbb{N}_+$ the symbol X^*_i via the mapping $g^{*,s}_i(\cdot,\cdot)$. The following hold at each $i \in \mathbb{N}_+$.

1) For a fixed $Y^{i-1} = y^{i-1}$, $F_{W_i|Y^{i-1}}(w_i)$ is a random variable uniformly distributed on the interval $[0,1)$. Hence, it is independent of y^{i-1}.
2) For a fixed $Y^{i-1} = y^{i-1}$, $F^{*,-1}_{X_i|Y^{i-1}}(.)$ is the inverse of a distribution function, applied to a uniformly distributed random variable. Hence, it transforms the uniform random variable $U_i = F_{W_i|Y^{i-1}}(w_i)$ into a R.V. X^*_i having the finite capacity achieving distribution $F^*_{X_i|Y^{i-1}}(x_i)$. That is, $F^{*,-1}_{X_i|Y^{i-1}} \circ F_{W_i|Y^{i-1}}(w_i)$ for a fixed $Y^{i-1} = y^{i-1}$ transforms X^*_i into a RV distributed according to $F^*_{X_i|Y^{i-1}}$.

The above PMS yields the following [10].

$$I(W^n \to Y^n) = \sum_{i=0}^n \int \log \frac{P_i(dy_i|y^{i-1}, x^*_i)}{P_i(dy_i|y^{i-1})} P_i(dy_i|y^{i-1}, x^*_i) \times P_i(dx^*_i|y^{i-1}) P_{0,i-1}(dy^{i-1}) = C^1_{0,n}$$

With respect to Problem 2.5, the PMS maximizes $I(W^n \to Y^n)$ over the encoders for a fixed control law. With respect to Problem 2.6, for any $R_n \in (0, C^1_{0,n}]$, one can design an encoder which satisfies the rate constraint inequality with equality. Hence, one can fix the encoder and minimize the pay-off in Problem 2.6 subject to an equality constraint.

V. Dynamic Programming

Armed with the PMS, the joint design of encoder/controller for Problem 2.6 is analyzed. Let $(\Omega, \mathbb{F}(\Omega), \mathbb{P}^{x,u})$ be a complete probability space. Define the following complete σ algebras:

$$\mathscr{G}_{0,n} \triangleq \sigma\{W_0, W_1, \ldots, W_n, Y_0, Y_1, \ldots, Y_n\}$$
$$\mathscr{F}_{0,n} \triangleq \sigma\{W_0, W_1, \ldots, W_n, Y_0, Y_1, \ldots, Y_{n-1}\}$$
$$\mathscr{J}_{0,n} \triangleq \{Y_0, Y_1, \ldots, Y_n\}$$

A deterministic encoder e_j is $\mathscr{F}_{0,j}-$ measurable and a deterministic control c_j is $\mathscr{J}_{0,j-1}-$ measurable.

A. Generalized Dynamic Programming

Consider Problem 2.6. Define the conditional control pay-off on the interval $[k,n]$ by

$$\bar{J}_{k,n}(c^n_k, e^n_k, \mathscr{J}_{0,k}) \triangleq E\Big\{\sum_{i=k}^{n-1} \ell(W_i, e_i(W^i, U^i, Y^{i-1}), c_i(U^{i-1}, Y^{i-1})) + \kappa(W_n))|\mathscr{J}_{0,k-1}\Big\}$$

By the smoothing property of conditional expectation

$$J_{k,n}(c^n_k, e^n_k) \triangleq E\Big\{\bar{J}_{k,n}(c^n_k, e^n_k, \mathscr{J}_{0,k-1})\Big\}$$

Similarly, define the conditional rate over encoder class $\mathscr{E}^{nm}_{ad}[0,n]$, on the interval $[k,n]$ by

$$\bar{I}_{k,n}(c^n_k, e^n_k, \mathscr{J}_{0,k-1}) \triangleq E\Big\{\sum_{i=k}^n \log \Big(\frac{P_i(dy_i|y^{i-1}, e^i(w^i, u^i, y^{i-1}), w^i, c^i(u^{i-1}, y^{i-1}))}{P_i(dy_i|y^{i-1})}\Big)|\mathscr{J}_{0,k-1}\Big\}$$

By the smoothing property of conditional expectation,

$$I_{k,n}(c^n_k, e^n_k) = E\Big\{\bar{I}_{k,n}(c^n_k, e^n_k, \mathscr{J}_{0,k-1})\Big\} = \sum_{i=k}^n I(W^i; Y_i|Y^{i-1})$$

The unconstrained cost-to-go is

$$\inf_{c^n_k \in \mathscr{C}_{ad}[k,n]} \inf_{e^n_k \in \mathscr{E}^{nm}_{ad}[k,n]} \Big\{J_{k,n}(c^n_k, e^n_k) + s\Big(I_{k,n}(c^n_k, e^n_k) - R_n\Big)\Big\}$$
$$= E\Big\{\inf_{c^n_k \in \mathscr{C}_{ad}[k,n]} \inf_{e^n_k \in \mathscr{E}^{nm}_{ad}[k,n]} \Big\{\bar{J}_{k,n}(c^n_k, e^n_k, \mathscr{J}_{0,k-1}) + s\Big(\bar{I}_{k,n}(c^n_k, e^n_k, \mathscr{J}_{0,k-1}) - R_n\Big)\Big\}\Big\}$$

where $s \in \mathbb{R}$ is a Langrange multiplier.

Theorem 5.1: Suppose there exist strategies $\{c_j^* : j \in \mathbb{Z}_+^n\} \in \mathscr{C}_{ad}[0,n]$, $\{e_j^* : j \in \mathbb{Z}_+^n\} \in \mathscr{E}_{ad}^{nm}[0,n]$ and a function $V_k(\mathscr{I}_{0,k-1})$ which satisfies the dynamic programming recursion:

$$V_k(\mathscr{I}_{0,k-1}) = \inf_{c_k \in \mathscr{C}_{ad}[k,k]} \inf_{e_k \in \mathscr{E}_{ad}^{nm}[k,k]} E\Big\{\ell(W_k, e_k(W^k, U^k, Y^{k-1}), c_k(U^{k-1}, Y^{k-1})) + s\Big(\log \frac{P_k(dY_k|Y^{k-1}, e^i(W^k,U^k,Y^{k-1}), W^k, c^k(U^{k-1},Y^{k-1}))}{P_k(dY_k|Y^{k-1})} - R_n\Big) + V_{k+1}(\mathscr{I}_{0,k}) | \mathscr{I}_{0,k-1}\Big\} \tag{V.4}$$

Then $\{e_j^* : j \in \mathbb{Z}_+^n\} \in \mathscr{E}_{ad}^{nm}[0,n]$, $\{c_j^* : j \in \mathbb{Z}_+^n\} \in \mathscr{C}_{ad}[0,n]$ obtained from (V.4) are optimal strategies.

$$J_{0,n}^2(\{c_j^*\}_{j=0}^n, \{e_j^*\}_{j=0}^n) = E\Big\{V_0(\mathscr{I}_{0,0})\Big\} \tag{V.5}$$

Theorem 5.1 is very general, although not easy to apply.

B. Dynamic Programming via Information State

This section invokes change of measure techniques to derive dynamic programming using a sufficient statistic.

Reference Probability Measure. Suppose Assumptions 3.1 and 3.2 hold, and in addition that

$$P_j(dy_j|y^{j-1}, x_j, w_j, u^j) = \rho_j(y_j|y^{j-1}, x_j, w_j, u^j) dy_j, a.s.$$

for $j = 0, 1, \ldots, n$, where the density is strictly positive. Suppose $\rho_j(y_j)$ is the unconditional probability density of Y_j and $\rho_j(y) > 0, \forall y, j = 0, 1, \ldots, n$. Assume encoder strategies $\mathscr{E}_{ad}[0,n]$. Define the following ratio of densities:

$$\Lambda_{0,k}^{x,u} \triangleq \prod_{l=0}^{k} \frac{\rho_l(y_l)}{\rho_l(y_l|y^{l-1}, w_l, x_l, u^l)} \tag{V.6}$$

Using (V.6) define a reference measure via $d\bar{\mathbb{P}}^{x,u} \triangleq \Lambda_{0,k}^{x,u}|_{\mathscr{G}_{0,k}} d\mathbb{P}^{x,u}$. Then by the Radon-Nikodym Derivative theorem, $d\bar{\mathbb{P}}^{x,u}(\Omega) = \int_\Omega \Lambda_{0,k}^{x,u}|_{\mathscr{G}_{0,k}} d\mathbb{P}^{x,u}(\omega) = 1$, hence it is a probability measure. The following holds.

Lemma 5.2: Under measure $\bar{\mathbb{P}}^{x,u}$ the random process $\{Y_k : k \in \mathbb{Z}_+^n\}$ are independent with density $\{\rho_k(y_k) : k \in \mathbb{Z}_+^n\}$ and $\{P_k(dw_k|w_{k-1}, y^{k-1}, x_{k-1}, u^{k-1}) : k \in \mathbb{Z}_+^n\}$ remains the same as that under measure $\mathbb{P}^{x,u}$.

Unnormalized Conditional Distributions. Start with $(\Omega, \mathbb{F}(\Omega), \mathbb{P}^{x,u})$ such that under $\bar{\mathbb{P}}^{x,u}$, $\{W_k : k \in \mathbb{Z}_+^n\}$ is a process with kernel $\{P_k(dw_k|w_{k-1}, y^{k-1}, x_{k-1}, u^{k-1}) : k \in \mathbb{Z}_+^n\}$ and $\{Y_k : k \in \mathbb{Z}_+^n\}$ are independent with density $\{\rho_k(y_k) : k \in \mathbb{Z}_+^n\}$. Define

$$\bar{\Lambda}_{0,k}^{x,u} \triangleq \Lambda_{0,k}^{x,u,-1} = \prod_{l=0}^{k} \frac{\rho_l(y_l|y^{l-1}, w_l, x_l, u^l)}{\rho_l(y_l)}$$

Define the real measure by $d\mathbb{P}^{x,u} = \bar{\Lambda}_{0,k}^{x,u} d\bar{\mathbb{P}}^{x,u}$. Consider any Borel function $f : \mathscr{W}_k \to \Re$ with compact support.
Then the following holds in view of Bayes rule.

$$\Pi_{k|k}^{x,u}(f(W)) \triangleq E\{f(W_k)|\mathscr{I}_{0,k}\} = \int_{\mathscr{W}_k} f(w) \Pi_{k|k}^{x,u}(dw|y^k) = \frac{\bar{E}\{f(W_k)\bar{\Lambda}_{0,k}^{x,u}|\mathscr{I}_{0,k}\}}{\bar{E}\{\bar{\Lambda}_{0,k}^{x,u}|\mathscr{I}_{0,k}\}} = \frac{\int_{\mathscr{W}_k} f(w)\pi_{k|k}^{x,u}(dw|y^k)}{\int_{\mathscr{W}_k} \pi_{k|k}^{x,u}(dw|y^k)}$$

$\Pi_{k|k}^{x,u}(dw|y^k)$ is the normalized stochastic kernel corresponding to the conditional distribution of W_k given $\mathscr{I}_{0,k}$ and $\pi_{k|k}^{x,u}(dw|y^k)$ is its unnormalized version. Therefore,

$$\Pi_{k|k}^{x,u}(dw|y^k) = \frac{\pi_{k|k}^{x,u}(dw|y^k)}{\int_{\mathscr{W}_k} \pi_{k|k}^{x,u}(dw|y^k)} \tag{V.7}$$

Similarly, the one step prediction is

$$\Pi_{k|k-1}^{x,u}(f(W)) \triangleq E\{f(W_k)|\mathscr{I}_{0,k-1}\} = \int_{\mathscr{W}_k} f(w)\Pi_{k|k-1}^{x,u}(dw|y^{k-1}) = \frac{\bar{E}\{f(W_k)\bar{\Lambda}_{0,k}^{x,u}|\mathscr{I}_{0,k-1}\}}{\bar{E}\{\bar{\Lambda}_{0,k}^{x,u}|\mathscr{I}_{0,k-1}\}} = \frac{\int_{\mathscr{W}_k} f(w)\pi_{k|k-1}^{x,u}(dw|y^{k-1})}{\int_{\mathscr{W}_k} \pi_{k|k-1}^{x,u}(dw|y^{k-1})} \tag{V.8}$$

$$\Pi_{k|k-1}^{x,u}(dw|y^{k-1}) = \frac{\pi_{k|k-1}^{x,u}(dw|y^{k-1})}{\int_{\mathscr{W}_k} \pi_{k|k-1}^{x,u}(dw|y^{k-1})} \tag{V.9}$$

Theorem 5.3 (Recursive Updates): Consider admissible encoders and controllers $\mathscr{E}_{ad}[0,n]$ and $\mathscr{C}_{ad}[0,n]$.
The stochastic kernel $\{\pi_{k|k-1}^{x,u}(dw; y^{k-1}) : k = 0, 1, \ldots, n\}$ satisfies the following recursive equation.

$$\pi_{k+1|k}^{x,u}(dw|y^k) = \int_{\mathscr{W}_k} \rho_k(y_k|y^{k-1}, w_k, x_k, u^k) \times P_{k+1}(dw|w_k, y^k, x_k, u^k)\pi_{k|k-1}^{x,u}(dw_k|y^{k-1})$$

and

$$\pi_{k|k}^{x,u}(dw|y^k) = \rho_k(y_k|y^{k-1}, w, x_k, u^k)\pi_{k|k-1}^{x,u}(dw|y^{k-1})$$

$$\pi_{k+1|k}^{x,u}(dw, dy|y^k) = \rho_{k+1}(y_{k+1})\pi_{k+1|k+1}^{x,u}(dw|y^{k+1})$$

Note that the update recursions of the previous theorem are not Markovian.

Dynamic Programming Recursion. Suppose Assumptions 3.1 and 3.2 hold. Consider the total rate pay-off $I(W^n \to Y^n)$:

$$I_{0,n}(c,g) = E\Big\{\sum_{i=0}^{n} \log \frac{\rho_i(y_i|y^{i-1}, w_i, g_i(w_i, y^{i-1}), c^i(u^{i-1}, y^{i-1}))}{\rho_i(y_i|y^{i-1})}\Big\} = \bar{E}\Big\{\sum_{i=0}^{n} \int \log\Big(\frac{\rho_i(y_i|y^{i-1}, w_i, g_i(w_i, y^{i-1}), c^i(u^{i-1}, y^{i-1}))}{\rho_i(y_i|y^{i-1})}\Big) \times \pi_{i|i-1}^{g,c}(dw_i, dy_i|y^{i-1})\Big\} \tag{V.10}$$

Similarly consider the controlled pay-off $J_{0,n}(c,g)$:

$$J_{0,n}(c,g) = \bar{E}\Big\{ \sum_{i=0}^{n-1} \int \ell(w_i, g_i(w_i, y^{i-1}), c_i(u^{i-1}, y^{i-1}))$$
$$\pi^{g,c}_{i|i-1}(dw_i|y^{i-1}) + \int \kappa(w_n) \pi^{g,c}_{n|n-1}(dw_n|y^{n-1}) \Big\}$$

Note that standard dynamic programming does not apply when the distribution process $\{\pi_{k|k-1}(.|y^{i-1}) : k \in \mathbb{Z}^n_+\}$ is not Markov. *For the rest of this section suppose the total cost $I_{0,n}(c,g)$ in (V.10) (based on the assumptions on the channel and source) is such that*

$$I_{0,n}(c,g) = \bar{E}\Big\{ \sum_{i=0}^{n} \int \log\Big(\frac{\rho_i(y_i|, w_i, g_i, c_i)}{\rho_i(y_i|y^{i-1})} \Big) \times \pi^{g,c}_{i|i-1}(dw_i, dy_i|y^{i-1}) \Big\} \quad \text{(V.11)}$$

and $\{\pi_{k|k-1}(.|y^{i-1}) : k \in \mathbb{Z}^n_+\}$ is Markov (e.g., the channel is a DMC).

Consider Problem 2.6. Suppose π is an information state at time k, then the remaining expected pay-off during the interval $g \in \mathscr{E}_{ad}[k,n]$ is defined by

$$V_k(c,g,\pi) \stackrel{\triangle}{=} \bar{E}\Big\{ \sum_{i=k}^{n-1} \int_{\mathscr{W}_i} \ell(w_i, g_i(w_i, y^{i-1}), c_i(y^{i-1}))$$
$$\pi^{g,c}_{i|i-1}(dw_i|y^{i-1}) + \int_{\mathscr{W}_n} \kappa(w_n) \pi^{g,c}_{n|n-1}(dw_n|y^{n-1}) +$$
$$s\Big(\sum_{i=k}^{n} \int_{\mathscr{W}_i \times \mathscr{Y}_i} \log \frac{\rho_i(y_i|w_i, g_i(w_i, y^{i-1}), c_i(y^{i-1}))}{\rho_i(y_i|y^{i-1})}$$
$$\pi^{g,c}_{i|i-1}(dw_i, dy_i|y^{i-1}) - R_n \Big) | \pi^{g,c}_{k|k-1} = \pi \Big\}$$

For any $0 \leq k \leq n$ the minimum pay-off during the interval $g \in \mathscr{E}_{ad}[k,n]$ and $c \in \mathscr{C}_{ad}[k,n]$ is defined by

$$V_k(\pi) \stackrel{\triangle}{=} \inf_{c \in \mathscr{C}_{ad}[k,n], g \in \in \mathscr{E}_{ad}[k,n]} V_k(c,g,\pi) \quad \text{(V.12)}$$

Thus, the following dynamic programming equation.

$$V_k(\pi) = \inf_{c \in \mathscr{C}_{ad}[k,n], g_k(\cdot, y^{k-1}) : w_k \to \mathscr{X}_k} \bar{E}\Big\{$$
$$\int_{\mathscr{W}_k} \ell(w_k, g_k(w_k, y^{k-1}), c_k(y^{k-1})) \pi^{g,c}_{k|k-1}(dw_k|y^{k-1}) + s\Big(\int_{\mathscr{W}_k \times \mathscr{Y}_k}$$
$$\log\Big(\frac{\rho_k(y_k|w_k, g_k(w_k, y^{k-1}), c_k(y^{k-1}))}{\rho_k(y_k|y^{k-1})} \Big)$$
$$\times \pi^{g,c}(dw_k, dy_k|y^{k-1}) - R_n \Big) + V_{k+1}(\pi^{g,c}_{k+1|k}) | \pi^{g,c}_{k|k-1} = \pi \Big\} \quad \text{(V.13)}$$

$$V_n(\pi) = \inf_{g_n(\cdot, y^{n-1}) : w_n \to \mathscr{X}_n} \Big\{ \int_{\mathscr{W}_n} \kappa(w_n) \pi^{g,c}_{n|n-1}(dw_n|y^{n-1}) +$$
$$+ s\Big(\int_{\mathscr{W}_n \times \mathscr{Y}_n} \log \frac{\rho_n(y_n|w_n, g_n(w_n, y^{n-1}), c_n(y^{n-1}))}{\rho_n(y_n|y^{n-1})}$$
$$\times \pi^{g,c}_{n|n-1}(dw_n, dy_n|y^{n-1}) - R_n \Big) \Big\} \quad \text{(V.14)}$$

It is noted that the backward dynamic programming recursion (V.13) with terminal condition (V.14) should be solved to determine the control and encoder laws $\{u^*_k = c^*_k(y^{k-1}) : k \in \mathbb{Z}^n_+\}$ and $\{x^*_k = g^*_k(w_k, y^{k-1}) : k \in \mathbb{Z}^n_+\}$.

Moreover, by Section V and by considering strategies $c \in \mathscr{C}^{sep}_{ad}[k,n]$ and $\{g_i(w_i, y^{i-1}) = g^s_i(w_i, P_i(dw_i|y^{i-1})) : i \in \mathbb{Z}^n_+\} \in \mathscr{E}^{sep}_{ad}$, the PMS achieves the inequality constraint with equality. Thus, the dynamic programming is minimized over $c \in \mathscr{C}^{sep}_{ad}[k,n]$. The point to be made is that the posterior matching scheme achieves the inequality constraint with equality, hence it is optimal. Hence, the dynamic programming is solved recursively for a fixed posterior matching encoder, over control laws. Thus, the joint encoder/control optimality separates into a PMS encoder strategy and an optimal control strategy obtained from (V.13) and (V.14).

VI. Conclusions and Future Work

This paper discusses joint optimality of encoder/control strategies which minimize a control pay-off subject to an information rate constraint. Properties of the encoder are derived, and the joint optimality of encoder/control strategies is shown to separate into the optimality of the encoder strategy with respect to a PMS, and the control strategy with respect to a dynamic programming recursion. Future work should examine the following items:

1) Analyze the infinite horizon case.
2) Compute examples of optimal encoder/controller strategies.

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