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A New Finite-Time Convergent and Robust Direct Model Reference Adaptive Control for SISO Linear Time Invariant Systems

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***Abstract*—The objective of this paper is to propose a new Direct MRAC (Model Reference Adaptive Control), having finite-time convergence of the tracking error and the parameter estimation algorithm under appropriate Persistence of Excitation Conditions. Moreover, enhanced robustness properties are also achieved. This is obtained by adding some strong, i.e. non locally Lipschitz or discontinuous, nonlinearities to the controller and parameter estimation algorithm of the classical MRAC. A Lyapunov-based approach is used to prove these properties. Some simulations illustrate how the proposed algorithm provides the MRAC with a much faster convergence of the tracking and parameter errors. Moreover, the improved convergence is obtained with less control action and with enhanced robustness properties of the control loop.**

I. INTRODUCTION

The direct Model Reference Adaptive Control (MRAC) is a well-known approach for adaptive control of linear and some nonlinear systems [13], [14], [2], [3]. When the Linear Time-Invariant (LTI) plant has relative degree $n^* = 1$ and the reference model is Strictly Positive Real (SPR), the controller is particularly simple to implement, and to design. In this case the convergence analysis becomes very simple by using the passivity properties of the error equation. Fig 1 gives the basic structure of the MRAC. In the Direct MRAC the tracking error dynamics, i.e. the difference between the desired behavior of the output of the plant, given by the behavior of the output of the reference model, and its actual value, is represented as a non-linear dynamical system, depending affinely on the Controller parameters. The Adjustment Mechanism tunes these controller parameters in such a way, that convergence of reference model's output y_m and plant's output y_p is achieved. Under appropriate persistence of excitation conditions asymptotic convergence of the parameters to their nominal values can be also achieved [13], [14], [2], [3].

In [8] a new recursive algorithm for parameter estimation in finite time, and with improved robustness properties, has been proposed. Its structure resembles the classical parameter estimation algorithms, but extra strong (not locally Lipschitz or discontinuous) nonlinear terms are added, so that the convergence and robustness properties of the basic linear algorithm are enhanced. These nonlinear terms are borrowed from the Super-Twisting Algorithm (STA), a second-order sliding mode algorithm proposed for the first time by [4]. Due to its strong convergence and robustness properties, this algorithm has proved to be useful in several applications as, for example, exact differentiators [5], [6], output feedback controllers [6], and observers [1]. A Lyapunov function for this algorithm has been presented in [10] (see also [9]), and the algorithm has been generalized in [11], [12]. The parameter estimation algorithm proposed in [8] inherits some of the properties of the Generalized Super-Twisting Algorithm.

The objective of this paper is to modify the Adjustment Mechanism of the classical Direct MRAC by adding the Super-Twisting-Like nonlinearities, so as to achieve the finite-time convergence and robustness properties. A Lyapunov-based approach is used to prove these properties. It is shown, also by means of simulations, that the proposed algorithm provides the MRAC a much faster convergence of the tracking error, and (when possible) of the parameter errors. More importantly, this improved convergence is obtained with less control action and with improved robustness properties of the control loop.

II. MRAC WITH RELATIVE DEGREE $n^* = 1$

A. Classical MRAC

The basic structure of a MRAC scheme is shown in Fig. 1. As it was mentioned the reference model is chosen to generate the desired trajectory, y_m, that the plant y_p output has to follow. The control $C(\theta)$ in the direct MRAC has an structure that depends on unknown constant parameters which are updated by an adaptive law.

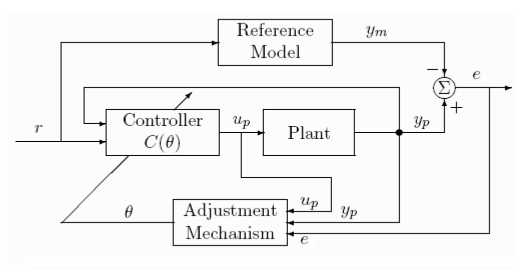

Fig. 1. General structure of MRAC scheme [2].

For the classical Direct MRAC [13], [2], [3] one considers the SISO, LTI plant

$$\begin{aligned}\dot{x}_p &= A_p x_p + B_p u_p \\ y_p &= C_p^T x_p\end{aligned} \tag{II.1}$$

where $x_p \in \mathbb{R}^n$; y_p, $u_p \in \mathbb{R}^1$ and A_p, B_p, C_p have the appropriate dimensions, that can also be described in the input/output form

$$y_p = G_p(s)u_p = k_p \frac{Z_p(s)}{R_p(s)} u_p \tag{II.2}$$

where Z_p, R_p are monic polynomials and k_p is a constant referred to as the "high frequency gain". The reference model, selected by the designer, is described by

$$\begin{aligned} \dot{x}_m &= A_m x_m + B_m r, \qquad x_m(0) = x_{m0} \\ y_m &= C_m^T x_m \end{aligned} \tag{II.3}$$

where $x_m \in \mathbb{R}^{p_m}$ for some integer p_m; y_m, $r \in \mathbb{R}^1$ and r is the reference input, which is assumed to be uniformly bounded and piecewise continuous function of time. The transfer function of the reference model, given by

$$y_m = W_m(s) r = k_m \frac{Z_m(s)}{R_m(s)} r \tag{II.4}$$

where $Z_m(s)$, $R_m(s)$ are monic polynomials, k_m is a constant. When the parameters of the plant are known, the Model Reference Control (MRC) problem consists in determining the plant input u_p so that all signals are bounded and the plant output y_p tracks the reference model output y_m for the class of reference input $r(t)$ mentioned. In order to meet the MRC objective it is assumed that both $G_p(s)$ and $W_m(s)$ satisfy the following assumptions:

- A1 An upper bound n of the degree n_p of $R_p(s)$ is known.
- A2 The relative degree $n^* = n_p - m_p$ of $G_p(s)$ is one, i.e. $n^* = 1$.
- A3 $Z_p(s)$ is a monic Hurwitz polynomial of degree $m_p = n_p - 1$.
- A4 The *sign* of the high frequency gain k_p is known.
- B1 $Z_m(s)$, $R_m(s)$ are monic Hurwitz polynomials of degree q_m, p_m, respectively, where $p_m \leq n$.
- B2 The relative degree $n_m^* = p_m - q_m$ of W_m is the same as that of $G_p(s)$, i.e., $n_m^* = n^* = 1$.
- B3 $W_m(s)$ is designed to be Strictly Positive Real (SPR).

Under these conditions, if the parameters of the plant (II.1) are known, the MRC problem can be solved by the control law given by [13], [2]

$$\dot{w}_1 = F w_1 + g u_p, \qquad w_1(0) = 0 \tag{II.5a}$$

$$\dot{w}_2 = F w_2 + g y_p, \qquad w_2(0) = 0 \tag{II.5b}$$

$$u_p = \theta^{*T} w \tag{II.5c}$$

where $w_1, w_2 \in \mathbb{R}^{n-1}$,

$$w = \begin{bmatrix} w_1^T & w_2^T & y_p & r \end{bmatrix}^T, \theta^* = \begin{bmatrix} \theta_1^{*T} & \theta_2^{*T} & \theta_3^{*T} & c_0^* \end{bmatrix}^T \tag{II.6}$$

$$F = \begin{bmatrix} -\lambda_{n-2} & -\lambda_{n-3} & \cdots & -\lambda_0 \\ 1 & 0 & \cdots & 0 \\ \vdots & \vdots & \ddots & \vdots \\ 0 & \cdots & 1 & 0 \end{bmatrix}, g = \begin{bmatrix} 1 \\ 0 \\ \vdots \\ 0 \end{bmatrix}, \tag{II.7}$$

λ_i are the coefficients of

$$\Lambda(s) = s^{n-1} + \lambda_{n-2} s^{n-2} + \cdots + \lambda_1 s + \lambda_0 = \det(sI - F),$$

that is an arbitrary monic Hurwitz polynomial of degree $n-1$ that contains $Z_m(s)$ as a factor, i.e., $\Lambda(s) = \Lambda_0(s) Z_m(s)$, being $\Lambda_0(s)$ monic, Hurwitz and of degree $n_0 = n - 1 - q_m$. θ^* is the vector of nominal values of the controller coefficients.

When the parameters of the plant are unknown, the objectives of the Model Reference Adaptive Control (MRAC) is also to design the control variable u_p so that the same MRC objectives are met. It is well-known [2], [14], [13]that this objective is attained if the control law (II.5c) is replaced by

$$u_p = \theta^T(t) w \tag{II.8}$$

where $\theta(t)$ is the estimate of θ^*, generated by

$$\dot{\theta}(t) = -\Gamma e_1 w \,\mathrm{sign}(\rho^*), \tag{II.9}$$

where

$$e_1 = y_p - y_m, \,\mathrm{sign}(\rho^*) = \mathrm{sign}\left(\frac{k_p}{k_m}\right), \Gamma = \Gamma^T > 0. \tag{II.10}$$

This is a classical Adaptive Control result [2], [13]

Theorem 1. *Under the stated assumptions the MRAC scheme given by (II.8-II.10) guarantees that:*

1) *All signals in the closed-loop plant are bounded and the tracking error e_1 converges to zero asymptotically with time for any reference input $r \in \mathscr{L}_\infty$.*
2) *If r is sufficiently rich of order $2n$, $\dot{r} \in \mathscr{L}_\infty$ and $Z_p(s)$, $R_p(s)$ are relatively coprime, then the parameter error $|\tilde{\theta}| = |\theta - \theta^*|$ and the tracking error e_1 converge to zero exponentially fast.*

B. Finite-time convergent MRAC

In order to enhance the convergence properties of the usual MRAC, some nonlinear correction terms will be added to the classical adaptive control law. These are motivated by the higher order sliding modes and, in particular, by the finite-time convergent parameter estimation algorithm introduced in [8]. The modified controller and adaption laws have the form

$$u_p = \theta^T(t) w - k_1 \phi_1(e_1) \,\mathrm{sign}(\rho^*) \tag{II.11}$$

$$\dot{\theta}(t) = -\Gamma \phi_2(e_1) w \,\mathrm{sign}(\rho^*), \tag{II.12}$$

where

$$\begin{aligned} \phi_1(e_1) &= \mu_1 |e_1|^{1/2} \mathrm{sign}(e_1) + \mu_2 e_1 \\ \phi_2(e_1) &= \frac{\mu_1^2}{2} \mathrm{sign}(e_1) + \frac{3}{2}\mu_1\mu_2 |e_1|^{1/2} \mathrm{sign}(e_1) + \mu_2^2 e_1 \end{aligned} \tag{II.13}$$

and $\mu_1 > 0$, $\mu_2 > 0$ are arbitrary positive constants. If $\mu_1 = 0$, $\mu_2 = 1$, $k_1 = 0$ and $\Gamma = 1$ the classical MRAC (II.8-II.9) is recovered. The extra non Lipschitz continuous term $|e_1|^{1/2} \mathrm{sign}(e_1)$ in ϕ_1 and the discontinuous term $\mathrm{sign}(e_1)$ in ϕ_2 are able to accelerate the tracking error e_1 and the parameter error $\tilde{\theta}$ so that, under appropriate conditions, they will converge in finite time. The main contribution of this paper is the following Theorem (cfr. with Theorem 1).

Theorem 2. *Assume that $Z_m(s)=1$, i.e. it is a polynomial of degree zero. Under the stated assumptions the MRAC scheme given by (II.11-II.12) guarantees that:*

1) *All signals in the closed-loop plant are bounded and the tracking error e_1 converges to zero asymptotically with time for any reference input $r \in \mathscr{L}_\infty$.*
2) *If r is sufficiently rich of order $2n$, $\dot{r} \in \mathscr{L}_\infty$ and $Z_p(s)$, $R_p(s)$ are relatively coprime, then the parameter error $|\tilde{\theta}| = |\theta - \theta^*|$ and the tracking error e_1 converge to zero in finite time.*

It is also shown in the proof of the Theorem, that the control loop is robust against additive perturbations in the control channel and in the parameters, i.e. bounded perturbations entering in the control channel, and/or bounded perturbations of the parameters' derivatives, cause a bounded control error.

III. Proof of Theorem 2

The proof will be divided in several parts.

A. The error dynamics

Plant (II.1) and controller (II.11) can be represented as

$$\begin{aligned} \dot{X}_c &= A_0 X_c + B_c u_p, \qquad X_c(0) = X_0 \\ y_p &= C_c^T X_c \\ u_p &= \theta^T w - k_1 \phi_1(e_1)\,\mathrm{sign}(\rho^*) \end{aligned} \tag{III.1}$$

where $X_c = \begin{bmatrix} x_p^T & w_1^T & w_2^T \end{bmatrix}^T$,

$$\begin{aligned} A_0 &= \begin{bmatrix} A_p & 0 & 0 \\ 0 & F & 0 \\ gC_p^T & 0 & F \end{bmatrix}, \qquad B_c = \begin{bmatrix} B_p \\ g \\ 0 \end{bmatrix} \\ C_c^T &= \begin{bmatrix} C_p^T & 0 & 0 \end{bmatrix} \end{aligned} \tag{III.2}$$

and then add and subtract the desired input $B_c \theta^{*T} w$ to obtain

$$\dot{X}_c = A_0 X_c + B_c \theta^{*T} w + B_c (u_p - \theta^{*T} w). \tag{III.3}$$

If we absorb the term $B_c \theta^{*T} w$ into the homogeneous part of (III.3), we end up with the representation

$$\begin{aligned} \dot{X}_c &= A_c X_c + B_c c_0^* r + B_c \left(u_p - \theta^{*T} w\right), \quad X_c(0) = X_0 \\ y_p &= C_c^T X_c \end{aligned} \tag{III.4}$$

where $A_c = \begin{bmatrix} A_p + B_p \theta_3^* C_p^T & B_p \theta^{*T} & B_p \theta^{*T} \\ g\theta^* C_p^T & F + g\theta_1^{*T} & g\theta_2^{*T} \\ gC_p^T & 0 & F \end{bmatrix}$ and $u_p = \theta(t)^T w - k_1 \phi_1(e_1)\,\mathrm{sign}(\rho^*)$.

If the parameters of the plant were known, then the nominal control law $u_p = \theta^{*T} w$ would produce in equation (III.4) the closed loop in the nominal case

$$\begin{aligned} \dot{X}_m &= A_c X_m + B_c c_0^* r, \quad X_m(0) = X_{m0} \\ y_m &= C_c^T X_m \end{aligned} \tag{III.5}$$

that corresponds exactly with the nominal model, i.e. $W_m(s) = C_c^T (sI - A_c)^{-1} B_c c_0^*$. Note that (III.5) is a nonminimal representation of the reference model. This shows that equation (III.4) is the same as the closed-loop system in the known parameter case except for the additional input term $B_c\left(u_p - \theta^{*T} w\right)$ that depends on the choice of the input u_p. Let $e = X_c - X_m$ and $e_1 = y_p - y_m$, where X_m is the state of the reference model (III.5), we have the error equation

$$\begin{aligned} \dot{e} &= A_c e + B_c\left(u_p - \theta^{*T} w\right), \quad e(0) = e_0 \\ e_1 &= C_c^T e\,. \end{aligned} \tag{III.6}$$

Substituting for the control law we obtain the error equation

$$\begin{aligned} \dot{e} &= A_c e + \bar{B}_c \rho^* \left(\tilde{\theta}^T w - k_1 \phi_1(e_1)\,\mathrm{sign}(\rho^*)\right), \quad e(0) = e_0 \\ e_1 &= C_c^T e\,, \end{aligned} \tag{III.7}$$

where $\bar{B}_c = B_c c_0^*$ and $\rho^* = 1/c_0^*$. Because $W_m(s) = C_c^T (sI - A_c)^{-1} B_c c_0^*$ is SPR and A_c is Hurwitz, equation (III.7) can be transformed to the normal form

$$\begin{aligned} \dot{e}_1 &= A_{11} e_1 + A_{12} e_2 + \rho^* \left(\tilde{\theta}^T w - k_1 \phi_1(e_1)\,\mathrm{sign}(\rho^*)\right) \\ \dot{e}_2 &= A_{21} e_1 + A_{22} e_2. \end{aligned} \tag{III.8}$$

with $e_1(0) = e_{10}$, $e_2(0) = e_{20}$ and $\dot{e}_2 = A_{22} e_2$ is the zero dynamics (see for example [15]). Recall

Lemma 3 (Meyer-Kalman-Yakubovich (MKY) [13], [2])**.** *Given a stable matrix A, vectors B, C and a scalar $d \geq 0$, we have the following: If $G(s) = C^T (sI - A)^{-1} B + d$ is SPR, then for any given $L = L^T > 0$, there exists a scalar $\nu > 0$, a vector q and a $P = P^T$ such that*

$$\begin{aligned} A^T P + PA &= -qq^T - \nu L \\ PB - C &= \pm q\sqrt{2d}\,. \end{aligned} \tag{III.9}$$

Applying this MKY-Lemma to the model (III.8) of the SPR transfer function $W_m(s)$ one concludes that for every $l_{11} > 0$ and $L_{22} = L_{22}^T > 0$ there exists a scalar q_1 , a vector q_2, a scalar $\nu > 0$ and a matrix $P_{22} = P_{22}^T > 0$ so that

$$\begin{aligned} &A_{22}^T P_{22} + P_{22} A_{22} = -q_2 q_2^T - \nu L_{22} \\ &A_{21}^T P_{22} + A_{12} = -q_1 q_2^T\,, 2A_{11} = -q_1^2 - \nu l_{11}\,. \end{aligned} \tag{III.10}$$

B. Uniform Stability

To prove the first item in the Theorem we propose a Lyapunov-like function

$$V\left(e, \tilde{\theta}\right) = \frac{1}{2}\phi_1^2(e_1) + e_2^T P_{22} e_2 + \frac{1}{2}|\rho^*|\,\tilde{\theta}^T \Gamma^{-1} \tilde{\theta}\,, \tag{III.11}$$

where $\Gamma = \Gamma^T > 0$ and $P_{22} = P_{22}^T$ satisfies (III.10). The time derivative of V is given by

$$\begin{aligned} \dot{V} = {} & \phi_1(e_1)\,\phi_1'(e_1)\left(A_{11} e_1 + A_{12} e_2 + \rho^* \tilde{\theta}^T w - k_1 |\rho^*|\,\phi_1(e_1)\right) + \\ & + 2 e_1 A_{21}^T P_{22} e_2 + e_2^T \left(A_{22}^T P_{22} + P_{22} A_{22}\right) e_2 - \rho^* \tilde{\theta}^T \phi_2(e_1)\, w\,. \end{aligned}$$

Using $\phi_2(e_1) = \phi_1(e_1)\,\phi_1'(e_1)$ and (III.10) it turns out that

$$\begin{aligned} \dot{V} = {} & -k_1 |\rho^*|\,\phi_1(e_1)\,\phi_2(e_1) + \phi_2(e_1)\left(A_{11} e_1 + A_{12} e_2\right) + \\ & + 2 e_1 \left(-A_{12} - q_1 q_2^T\right) e_2 - e_2^T q_2 q_2^T e_2 - \nu e_2^T L_{22} e_2\,. \end{aligned}$$

Using (from (III.10)) $q_1^2 = -2A_{11} - \nu l_{11}$ and adding, substracting the term $2e_1\left(A_{11} + \frac{1}{2}\nu l_{11}\right) e_1$ and factorizing becomes

$$\begin{aligned} \dot{V} = {} & -k_1 |\rho^*|\,\phi_1(e_1)\,\phi_2(e_1) + \phi_2(e_1)\left(A_{11} e_1 + A_{12} e_2\right) - \nu e_2^T L_{22} e_2 \\ & + 2 e_1 \left(-A_{12} e_2 - \left(A_{11} + \frac{1}{2}\nu l_{11}\right) e_1\right) - \psi^T(e)\,\psi(e)\,, \end{aligned}$$

where $\psi(e) = q_2^T e_2 + \sqrt{-2\left(A_{11} + \frac{1}{2}\nu l_{11}\right)} e_1$. When $Z_m(s) = 1$ the reference model has no zero dynamics, that is all zeros of the system (III.8) are unobservable from the output e_1, what implies that $A_{12} = 0$. Using this fact one finally obtains

$$\begin{aligned}\dot{V} \le & -k_1 |\rho^*| \phi_1(e_1) \phi_2(e_1) + \phi_2(e_1) A_{11} e_1 + \\ & - \nu e_2^T L_{22} e_2 + e_1 q_1^2 e_1 .\end{aligned} \tag{III.12}$$

It is easily seen that the rhs of the previous inequality is negative definite in e_1 and e_2 if k_1 is large enough, or $\mu_2 \ge 1$. (III.11) and (III.12) imply that V and, therefore, $e, \tilde{\theta} \in \mathscr{L}_\infty$. Because $e = X_c - X_m$ and $X_m \in \mathscr{L}_\infty$, we have $X_c \in \mathscr{L}_\infty$, which implies that $y_p, w_1, w_2 \in \mathscr{L}_\infty$. Because $u_p = \theta^T(t) w - k_1 \phi_1(e_1) \operatorname{sign}(\rho^*)$ and $\theta, w, e_1 \in \mathscr{L}_\infty$ we also have $u_p \in \mathscr{L}_\infty$. Therefore all the signals in the closed loop plant are bounded.

We finally show that the tracking error e_1 converges to zero, i.e. $\lim_{t\to\infty} e_1 = \lim_{t\to\infty}(y_p - y_m) = 0$. From (III.11) and (III.12) we establish that e and therefore $e_1 \in \mathscr{L}_2$. Furthermore, using $\theta, w, e_1 \in \mathscr{L}_\infty$ in (III.8) we have that $\dot{e}, \dot{e}_1 \in \mathscr{L}_\infty$. Hence, $e_1, \dot{e}_1 \in \mathscr{L}_\infty$ and $e_1 \in \mathscr{L}_2$, which, by Barbalat's Lemma, imply that $\lim_{t\to\infty} e_1(t) = 0$.

C. Convergence under Persistence of Excitation Conditions

From [13], [2], [3] (see also the pertinent observations in [7]), it follows that for the auxiliary nonlinear system

$$\Sigma_{L1} : \begin{cases} \dot{e}_1 = & -k_1 |\rho^*| e_1 + \rho^* w^T(t, e, \tilde{\theta}) \tilde{\theta} \\ \dot{\tilde{\theta}} = & -\Gamma e_1 w(t, e, \tilde{\theta}) \operatorname{sign}(\rho^*) \end{cases} \tag{III.13a}$$

$$\Sigma_2 : \{ \; \dot{e}_2 = A_{22} e_2 + A_{21} e_1 \tag{III.13b}$$

under the conditions of item 2 of Theorem 1, the equilibrium point $\left(e_1, e_2^T, \tilde{\theta}^T\right) = 0$ is exponentially stable, when $Z_m = 1$ (and therefore $A_{12} = 0$), and the persistence of excitation conditions for the bounded part of $w(t, e, \tilde{\theta})$. In [8] it was shown that under the same conditions and with $w(t)$ holding the persistence excitation condition for the system

$$\Sigma_{NL1} : \begin{cases} \dot{e}_1 = & -k_1 |\rho^*| \phi_1(e_1) + \rho^* w^T(t) \tilde{\theta} + \delta_1 \\ \dot{\tilde{\theta}} = & -\Gamma \phi_2(e_1) w(t) \operatorname{sign}(\rho^*) + \delta_2 \end{cases}$$

e_1 and $\tilde{\theta}$ converge to zero in finite time, when $\delta_1 = 0$ and $\delta_2 = 0$. Moreover, for bounded signals (normally seen as perturbations) δ_1 and δ_2 the variables e_1 and $\tilde{\theta}$ remain bounded, i.e. the system is ISS stable with input δ_1, δ_2. Since system Σ_{NL1} is strictly passive from the input δ_1 to the output $\phi_2(e_1)$, with storage function $V\left(e, \tilde{\theta}\right) = \frac{1}{2}\phi_1^2(e_1) + \frac{1}{2}|\rho^*| \tilde{\theta}^T \Gamma^{-1} \tilde{\theta}$, it follows that the trayectories of the error equation

$$\Sigma_1 : \begin{cases} \dot{e}_1 = & -k_1 |\rho^*| \phi_1(e_1) + \rho^* w^T(t, e, \tilde{\theta}) \tilde{\theta} + A_{11} e_1 + \delta_1 \\ \dot{\tilde{\theta}} = & -\Gamma \phi_2(e_1) w(t, e, \tilde{\theta}) \operatorname{sign}(\rho^*) + \delta_2 \end{cases} \tag{III.14}$$

$$\Sigma_2 : \{ \; \dot{e}_2 = A_{22} e_2 + A_{21} e_1 \tag{III.15}$$

converge asymptotically to the equilibrium point $\left(e_1, e_2^T, \tilde{\theta}^T\right) = 0$, with e_1 and $\tilde{\theta}$ converging in finite time and e_2 converging exponentially, when $\delta_1 = 0$ and $\delta_2 = 0$. Furthermore, for bounded δ_1 and δ_2 the variables e_1, e_2 and $\tilde{\theta}$ remain bounded, i.e. the cascade system $\Sigma_1 - \Sigma_2$ is ISS stable with input δ_1, δ_2. □

IV. Example

In order to compare the controllers with only linear terms and with the extra nonlinear terms proposed in this paper, we consider the second order plant (used as example in [13])

$$y_p = \frac{k_p(s + b_0)}{s^2 + a_1 s + a_0} u_p . \tag{IV.1}$$

The values for the unknown constants used in the simulation are $k_p = 1$, $b_0 = 1$, $a_1 = -3$ and $a_0 = 2$, and the sign of $k_p > 0$ is known. The reference model is given by

$$y_m = \frac{1}{s+1} r . \tag{IV.2}$$

A nominal controller is given by [13], [2]

$$\begin{aligned} \dot{w}_1 &= -2w_1 + u_p, \quad w_1(0) = 0 \\ \dot{w}_2 &= -2w_2 + y_p, \quad w_2(0) = 0 \\ u_p &= \theta_1 w_1 + \theta_2 w_2 + \theta_3 y_p + c_0 r , \end{aligned} \tag{IV.3}$$

where $\theta = [\theta_1, \theta_2, \theta_3, c_0]^T$, $w = [w_1, w_2, y_p, r]^T$, and the nominal parameter values are $\theta^* = [\theta_1^*, \theta_2^*, \theta_3^*, c_0^*]^T = [0, 6, -5, 1]^T$. Three simulation scenarios will be presented.

A. Persistence of Excitation conditions, without perturbations

For the simulations the reference signal r has been selected as $r = 5\cos(t) + 10\cos(5t)$, that is sufficiently rich for the system, so that the regressor w is persistently exciting. The classical (linear) MRAC is given by (II.8-II.10), where Γ has been set as the identity matrix of dimension 4. The proposed (nonlinear) MRAC is given by (II.11-II.13), where the gains have been set to $k_1 = 10$, $\mu_1 = 1$ and $\mu_2 = 1$. Fig. 2 shows the output reference given by the model y_m (continuous line) and the plant's output y_p (dotted line) with the classical MRAC scheme, whereas Fig. 3 the same results for the proposed (nonlinear) MRAC scheme. The corresponding control variable u_p is shown in Fig. 4. From these figures it is clear that the proposed (nonlinear) controller is much faster than the classical one, with a smaller control effort! Furthermore, the parameter convergence, shown in Fig. 5, is also reached in a finite time and much faster than the classical algorithm.

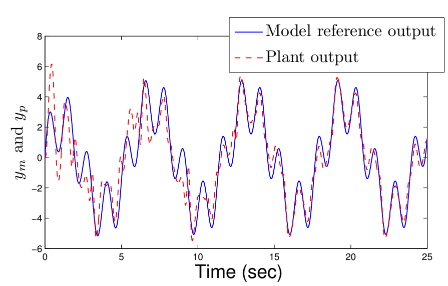

Fig. 2. Model y_m (continuous line) and the Plant's output y_p (dotted line) with the classical MRAC scheme with reference signal $r = 5\cos(t) + 10\cos(5t)$.

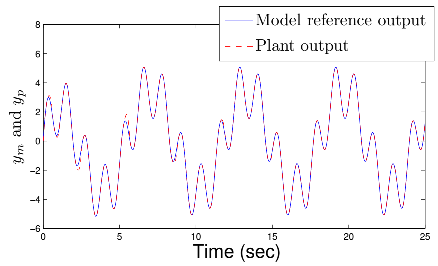

Fig. 3. Model y_m (continuous line) and the Plant's output y_p (dotted line) with the nonlinear MRAC scheme with reference signal $r = 5\cos(t) + 10\cos(5t)$.

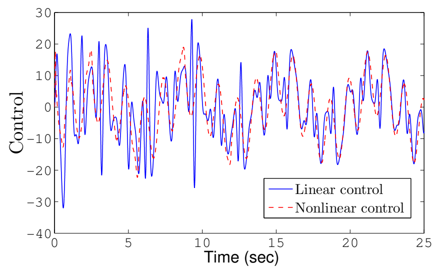

Fig. 4. Control variable u_p for the classical MRAC (continuous line) and the proposed NL MRAC (dotted line) with reference $r = 5\cos(t) + 10\cos(5t)$.

B. Persistence of Excitation conditions, with perturbations

It was shown in the proof of the Theorem, that the proposed MRAC (nonlinear) algorithm (II.11-II.13) is robust against additive perturbations in the control input and in the parameters, due for example to slowly time varying parameters. In order to illustrate these features a simulation with a perturbation $p(t) = 5\sin(6t)$, entering at the control input of the plant, has been done. Fig. 6 shows the tracking error $e_1 = y_p - y_m$ for the classical (continuous line) and the proposed (dotted line) MRAC schemes. It is clear that the additional nonlinear terms of the proposed algorithm lead to a much smaller tracking error. Moreover, the parameter estimation (shown in Fig. 7) is faster and better for the proposed algorithm than for the classical one. It is interesting to note that for the nonlinear algorithm the parameter estimation oscillates around the true value, while the linear estimation error has a bias.

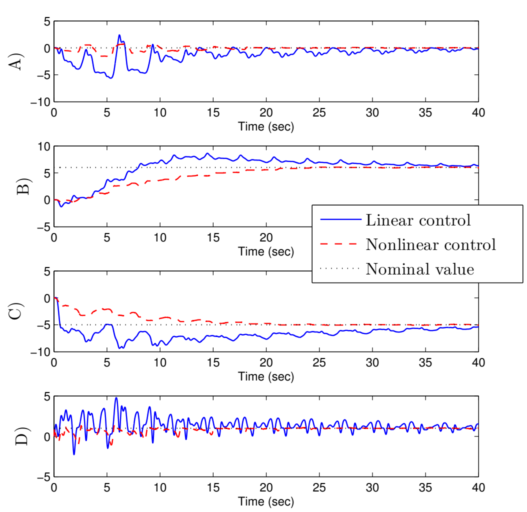

Fig. 5. Parameter convergence to the real values with reference signal $r = 5\sin(t) + 10\sin(5t)$. A) $\theta_1^* = 0$, B) $\theta_2^* = 6$, C) $\theta_3^* = -5$ and D) $c_0^* = 1$

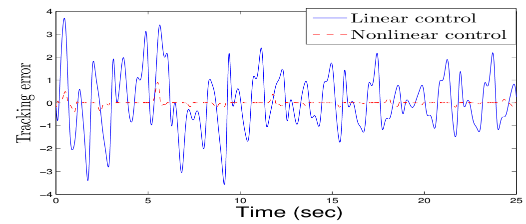

Fig. 6. Tracking error $e_1 = y_p - y_m$ for the classical (continuous line) and the proposed (dotted line) MRAC schemes with reference signal $r = 5\cos(t) + 10\cos(5t)$, with perturbation $p(t) = 5\sin(6t)$.

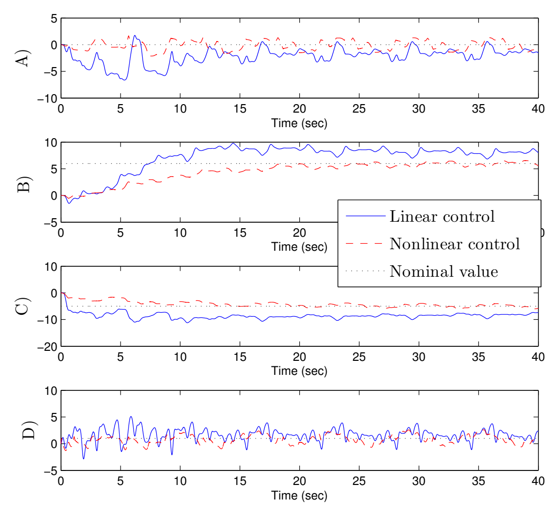

Fig. 7. Parameter convergence to the real values with reference signal $r = 5\sin(t) + 10\sin(5t)$, and perturbation $p(t) = 5\sin(6t)$. A) $\theta_1^* = 0$, B) $\theta_2^* = 6$, C) $\theta_3^* = -5$ and D) $c_0^* = 1$.

Plant output with noise: Fig. 8 shows the tracking error when a measurement noise of approx. 5% of plant's output is added. Note that the the nonlinear MRAC still converges faster than the linear one, and the noise effect in steady state is similar for both controllers.

C. Lack of Persistence of Excitation conditions

It is well-known that the properties of the MRAC are weaker when the Persistence of Excitation conditions are not satisfied. For a constant reference signal $r(t) = 6$, Fig. 9 shows the output reference given by the model y_m (continuous line) and the plant's output y_p (dotted line) with the classical MRAC scheme, while Fig. 10 shows the same results for the proposed (nonlinear) MRAC scheme. The corresponding

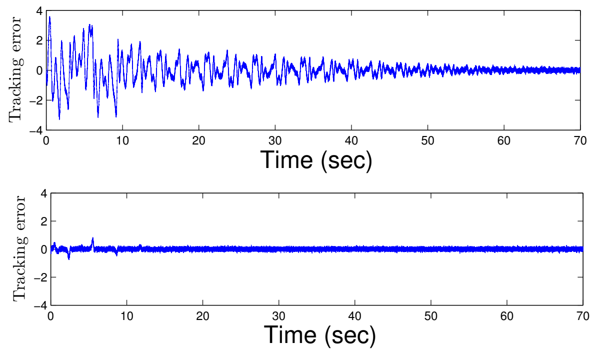

Fig. 8. Tracking error $e_1 = y_p - y_m$ for the classical (up) and the proposed (down) MRAC schemes with a maximum amplitude noise of 0.25 added to the plant's output.

control variable u_p is shown in Fig. 11. It is remarkable that again the convergence of the proposed nonlinear MRAC is much faster in this case, with a much smaller control effort.

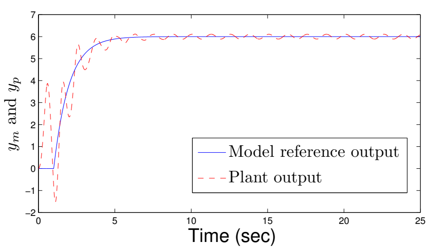

Fig. 9. Model y_m (continuous line) and the Plant's output y_p (dotted line) with the classical MRAC scheme with constant reference signal $r = 6$.

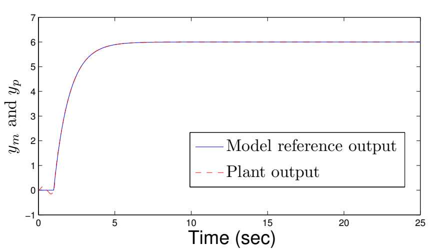

Fig. 10. Model y_m (continuous line) and the Plant's output y_p (dotted line) with the nonlinear MRAC scheme with reference signal $r = 6$.

V. Conclusions

It has been shown that a modification of the Adjustment Mechanism of the classical Direct MRAC, by adding Super-Twisting-Like nonlinearities, is able to achieve finite-time convergence and improved robustness properties. A Lyapunov-based approach was used to prove these properties. Some simulations have shown that the proposed algorithm provides the MRAC a much faster convergence of the tracking error, and (when possible) of the parameter errors. More importantly, this improved convergence is obtained with less control action and with improved robustness properties of the control loop.

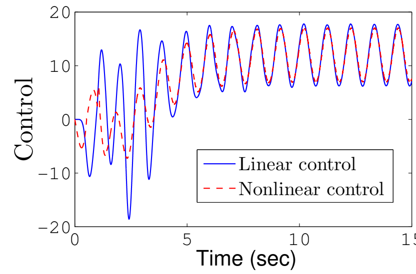

Fig. 11. Control variable u_p for the classical MRAC (continuous line) and the proposed nonlinear MRAC (dotted line) with reference signal $r = 6$.

For simplicity of the presentation, in this paper we have restricted ourselves to a very simple reference model without zeros. It is possible however, to extend the analysis to the more general case of an SPR reference model with zero dynamics. A promising research direction is to extend the presented idea to linear systems with arbitrary relative degree, and for nonlinear systems.

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References

[1] J. Davila, L. Fridman, and A. Poznyak, "Observation and identification of mechanical systems via second order sliding modes," *Int. J. of Control.*, vol. 79, no. 10, pp. 1251–1262, 2006.
[2] P. A. Ioannou and J. Sun, *Robust Adaptive Control*. Upper Saddle River, NJ, 1996.
[3] P. A. Ioannou and B. Findan, *Adaptive Control Tutorial*. Philadelphia, SIAM, 2006.
[4] A. Levant, "Sliding order and sliding accuracy in sliding mode control," *Int. J. of Control*, vol. 58, no. 6, pp. 1247–1263, 1993.
[5] ——, "Robust exact differentiation via sliding mode technique," *Automatica*, vol. 34, no. 3, pp. 379–384, 1998.
[6] ——, "Higher-order sliding modes, differentiation and output-feedback control," *Int. J. of Control*, vol. 76, no. 9/10, pp. 924–941, 2003.
[7] A. Loría and E. Panteley, "Uniform exponential stability of linear time-varying systems: revisited," *Sys. & Cont. Lett.*, vol. 47, pp. 13–24, 2002.
[8] J.A. Moreno and E. Guzmán, "A New Recursive Finite-Time Convergent Parameter Estimation Algorithm," *18th IFAC World Congress*, Milan, Italy, August 28- September 2, 2011. p. 3439 – 3444.
[9] Moreno, J.A.; Osorio, M. Strict Lyapunov Functions for the Super-Twisting Algorithm. IEEE Tran. on Aut. Control. To appear.
[10] Moreno, J.A., Osorio, M. (2008) A Lyapunov approach to second-order sliding mode controllers and observers. *47th IEEE Conference on Decision and Control*. CDC 2008, pp.2856-2861.
[11] J. A. Moreno, "A Linear Framework for the Robust Stability Analysis of a Generalized Super-Twisting Algorithm." *Electrical Engineering, Computing Science and Automatic Control (CCE 2009) 6th International Conference*, pp. 12–17, Nov. 2009. [Online]. Available: http://ieeexplore.ieee.org/lpdocs/epic03/wrapper.htm?arnumber=5393477
[12] Moreno, J.A. (2011). Lyapunov Approach for Analysis and Design of Second Order Sliding Mode Algorithms. In: Fridman, L., Moreno, J.A.; Iriarte, R. (Ed.) Sliding Modes after the first decade of the 21st Century. Springer-Verlag, LNCIS 412. pp. 113 – 150.
[13] K. S. Narendra and A. Annaswamy, *Stable Adaptive Systems*. Dover Publications, Inc. Mineola, New York 2005.
[14] S. Sastry and M. Bodson, *Adaptive Control Stability. Convergence and Robustness*. Prentice Hall, Englewood Cliffs, NJ, 1989.
[15] R. Sepulchre, M. Jankovic and P.V. Kokotovic, *Constructive Nonlinear Control*. Springer, London, 1997.