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Dissipativity Based $\mathcal{H}_\infty$ Control for Nonlinear Stochastic Systems with Time-Varying Delays

Huiping Li and Yang Shi

Abstract— **This paper is concerned with the $\mathcal{H}_\infty$ state feedback control problem for a class of nonlinear stochastic systems with time-varying state delays. Firstly, the stochastic dissipativity of the time-delay nonlinear system is established, based on which the $\mathcal{H}_\infty$ state feedback controller is synthesized. We show that the closed-loop system rendered by the designed controller can achieve the L_2-gain performance and exponential stability in the sense of mean square. Finally, these sufficient conditions are further specialized as solving a linear matrix inequality (LMI) and a bilinear matrix inequality (BMI) for a class of special stochastic nonlinear delayed systems.**

I. INTRODUCTION

In recent years, the research on the $\mathcal{H}_\infty$ control for stochastic nonlinear systems has received increasing attention. The $\mathcal{H}_\infty$ stochastic control and estimation problem was treated as nonlinear stochastic minimax dynamic games in [1], and the equivalent results to deterministic nonlinear systems were also derived. The $\mathcal{H}_\infty$ state feedback controller design for discrete-time stochastic nonlinear systems was studied in [2] and the corresponding continuous-time case was investigated in [3] and [4]. [3] analyzed the time-invariant case, and [4] tackled the time-variant case. Furthermore, the extended work of [2] and [4] was presented in [5], where some sufficient conditions for the design of an output feedback controller were demonstrated, while the related $\mathcal{H}_\infty$ filtering problem for stochastic continuous-time systems was addressed in [6].

On the other hand, time delays arise quite commonly in many engineering systems such as chemical processes, electrical networks, biological systems, and so on. It has been shown that the existence of delays may be a main source of instability, oscillation and erratic system behaviors, see, e.g. [7], [8] and the references therein. The time delay makes the system stabilization, control and estimation problems more difficult. Over the past few years, a great deal of effort has been devoted to the $\mathcal{H}_\infty$ control problem for linear systems with time delays, see, e.g. [9]–[12] and the references therein.

It is worth noting that the nonlinear systems with time delays are more challenging compared to linear systems, and only few results have been contributed in the literature. For instance, approximation approaches such as the linearisation algorithm, the Taylor expansion and the first-order approximation were used to tackle nonlinear systems with delays [13]–[15]; besides, some works were devoted to dealing with special nonlinear systems with time delays. For example, the stabilization problem for a class of triangular structural time-delay nonlinear systems was investigated in [16]. Robust stability analysis was reported for a class of nonlinear systems with constant delays in [17], where nonlinearities was involved in a given polytopic region. In spite of these developments, to the best of the authors' knowledge, the $\mathcal{H}_\infty$ control problem for stochastic nonlinear systems with time-varying state delays has not been fully solved yet, which is the focus of this paper.

In this paper, we address the $\mathcal{H}_\infty$ state feedback controller design problem for a class of stochastic nonlinear systems with time-varying state delays. The stochastic nonlinear system is described as a delayed Itô-type stochastic differential equation with state and disturbance-dependent noise. The purpose of this paper is to synthesize an $\mathcal{H}_\infty$ states feedback controller for the studied nonlinear stochastic systems such that the L_2-gain performance is achieved and the closed-loop system is exponentially stable in the sense of mean square.

The rest of the paper is organized as follows. In Section II, the problem formulation and preliminaries are presented. The sufficient condition for guaranteeing the stochastic dissipative property for the investigated time-delay systems is firstly established, based on which the $\mathcal{H}_\infty$ state feedback controller are synthesized In Section IV, and the simplified sufficient conditions for a special type of nonlinear system is developed in the same section. The conclusion remarks are presented in Section IV.

The following notations are used in the sequel for the rest of the paper. The superscripts T and -1 stand for the matrix transposition and the matrix inverse, respectively. $\mathbb{R}^n$ denotes the n-dimensional Euclidean space. $\mathbb{R}^{n\times m}$ is the set of $n \times m$ real matrices. The notation $P > 0$ ($P \geq 0$) means that P is real symmetric and positive definite (positive semi-definite). Tr means the trace of a matrix, and $\otimes$ denote the Kronecker product. $\|\cdot\|$ refers to the Euclidean norm for vectors and induced 2-norm for matrices. $\mathbb{E}$ stands for the mathematical expectation operator. Let $(\Omega, \mathcal{F}, \{\mathcal{F}_t\}_{t\geq 0}, \mathcal{P})$ be a complete probability space with a filtration $\{\mathcal{F}_t\}_{t\geq 0}$ containing all $\mathcal{P}$-null sets and being right continuous. $\mathcal{L}^2([0,\infty],\mathbb{R}^n)$ is the space of the nonanticipative stochastic processes $y(t)$ with respect to filtration $\mathcal{F}_t$ satisfying $\|y(t)\|^2 := \mathbb{E}\int_0^\infty \|y(t)\|^2 dt < \infty$. $\mathcal{C}^{2,1}(\mathbb{R}^n \times \mathbb{R}; \mathbb{R})$ is a class of functions $V(x,t)$ second-order differentiable with respect to x and first-order differentiable

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Huiping Li is with Department of Mechanical Engineering, University of Victoria, Victoria,B.C., Canada. Email: huiping@uvic.ca

Yang Shi is with the Department of Mechanical Engineering, University of Victoria, Victoria, B.C., Canada. Email: yshi@uvic.ca

with respect to t. The space of continuously differentiable functions $\phi : [-T, 0] \rightarrow \mathbb{R}^n$ with finite norm $\|\phi\|_T = \sup_{-T\leq t\leq 0} \|\phi(t)\|$ is denoted by $\mathcal{D}$. $x_t \in \mathcal{D}$ is a segment of the function $x(\cdot)$ given by $x_t(\theta) = x(t+\theta), \forall\theta \in [-T, 0]$.

II. Problem statement

Consider the following time-delay nonlinear stochastic system:

$$\begin{cases} dx(t) = [f(x(t), x(t-\tau(t)))x(t) \\ +f_1(x(t), x(t-\tau(t)))x(t-\tau(t)) \\ +g(x(t), x(t-\tau(t)))u(t) + g_1(x(t), x(t-\tau(t)))v(t)]dt \\ +s(x(t), x(t-\tau(t))x(t)d\omega(t) \\ +s_1(x(t), x(t-\tau(t))v(t)d\omega_1(t) \\ z(t) = \begin{bmatrix} h(x(t), x(t-\tau(t)))x(t) \\ u(t) \end{bmatrix} \\ x(t) = \phi(t), \quad \forall t \in [-T, 0], \end{cases} \tag{1}$$

where $x(t) \in \mathbb{R}^n$ is called the system state; $z(t) \in \mathcal{L}^2([0,\infty], \mathbb{R}^{n_v})$ is the regulated output; $v(t) \in \mathbb{R}^{n_v}$ is the exogenous disturbance which belongs to $\mathcal{L}^2([0,\infty]; \mathbb{R}^{n_v})$. $W(t) = [\omega^{\mathrm{T}}(t), \omega_1^{\mathrm{T}}(t)]^{\mathrm{T}}$ is the standard Wiener process (Brownian motion) defined on the complete probability space $(\Omega, \mathcal{F}, \{\mathcal{F}_t\}_{t\geq 0}, \mathcal{P})$ with the natural filter $\mathcal{F}_t$ generated by $W(\cdot)$ up to time t, where $\omega(t)$ and $\omega_1(t)$ are independent $\mathbb{R}^1$-valued and $\mathbb{R}^1$-valued Wiener processes, respectively. $u(t) \in \mathcal{L}^2([0,\infty]; \mathbb{R}^{n_u})$ is the control input, which is an adapted process with respect to $\{\mathcal{F}_t\}_{t\geq 0}$. Furthermore, $\tau(t)$ is the time-varying delays satisfying $0 \leq \tau(t) \leq T < \infty, \forall t > 0, \tau(0) = 0$ and $0 \leq \dot{\tau}(t) \leq \mu < 1$; $\phi(.) \in \mathcal{D}$, is the system initial function. Besides, the following assumptions are made. All the functions in (1) are assumed to be Borel measurable relative to the appropriate spaces, i.e. $f : \mathbb{R}^n \times \mathbb{R}^n \rightarrow \mathbb{R}^n, f_1 : \mathbb{R}^n \times \mathbb{R}^n \rightarrow \mathbb{R}^n$ $g : \mathbb{R}^n \times \mathbb{R}^n \rightarrow \mathbb{R}^{n\times n_u}$, $g_1 : \mathbb{R}^n \times \mathbb{R}^n \rightarrow \mathbb{R}^{n\times n_v}$, $s : \mathbb{R}^n \times \mathbb{R}^n \rightarrow \mathbb{R}^{n\times n}$, $s_1 : \mathbb{R}^n \times \mathbb{R}^n \rightarrow \mathbb{R}^{n\times n_v}$ and $h : \mathbb{R}^n \times \mathbb{R}^n \rightarrow \mathbb{R}^{n\times n_z}$. In addition, it is assumed that for all admissible control inputs $u(t) \in \mathcal{L}^2([0,\infty]; \mathbb{R}^{n_u})$ and disturbances $v(t) \in \mathcal{L}^2([0,\infty]; \mathbb{R}^{n_v})$ such that system (1) has a unique strong solution; for more details, the reader is referred to [18], [19].

In what follows, we will introduce the dissipativity concept for the stochastic time-delay systems (1) which is the basis on which the $\mathcal{H}_\infty$ controller can be synthesized.

Let $S(v, z)$ be a real-value function $S : \mathbb{R}^{n_v} \times \mathbb{R}^{n_z+n_u} \rightarrow \mathbb{R}$, and assume that for an admissible $\{v(t)\}_{t\geq 0}$ and the solution $\{x(t)\}_{t\geq 0}$ to (1), $\mathbb{E}\int_0^t |S(v(s), z(s))|ds < \infty$ for all $t \geq 0$, then $S(v, z)$ is said to be a supply rate for the system in (1). In general, the supply rate can be taken as many forms [4]. Here, we take the supply rate $S(v, z) = \gamma^2\|v\|^2 - \|z\|^2$ for this study.

By using the notation of the supply rate, the stochastic dissipative concept for delay-free system is introduced in [20]. In this paper, we extend the stochastic dissipative concept for the time-delay system in (1) as follows.

Definition 1: For the system in (1) and the supply rate defined above, if there exists a function $V : \mathbb{R}^n \times [0,\infty] \rightarrow \mathbb{R}_{\geq 0}$, for all $x_0 \in \mathbb{R}^n$, such that $\mathbb{E}\{V(x_0, 0)\} < \infty$, and

$$\mathbb{E}\{V(x(t_2), t_2)\} - \mathbb{E}\{V(x(t_1), t_1)\} \leq \mathbb{E}\left\{\int_{t_1}^{t_2} S(v(s), z(s))ds\right\}, \tag{2}$$

for all $t_2 \geq t_1 \geq 0$ and all the admissible $\{v(t)\}_{t\geq 0}$ and $\{u(t)\}_{t\geq 0}$, then the system in (1) is said to be dissipative with respect to the supply rate S, and V is called a storage function of the system in (1).

To precisely characterize the performance of the $\mathcal{H}_\infty$ state feedback controller to be synthesized, the design criteria of the controller for system (1) is presented in definition 2.

Definition 2: For the system in (1), and a prescribed attenuation $\gamma > 0$, a control input $u^*(t) \in \mathcal{L}^2([0,\infty]; \mathbb{R}^{n_u})$ is said to be an $\mathcal{H}_\infty$ state feedback controller for system (1) if for $\forall$ $v(t) \in \mathcal{L}^2([0,\infty]; \mathbb{R}^{n_v})$ such that the following two conditions hold:

H1) The closed-loop system (1) with $u(t) = u^*(t)$, and $v(t) \equiv 0$ is exponentially stable in the sense of mean square.

H2) The property of L_2-gain less than or equal to γ is satisfied as

$$\mathbb{E}\left\{\int_0^T \|z(t)\|^2 dt\right\} \leq \gamma^2\mathbb{E}\left\{\beta(x_0, 0) + \int_0^T \|v(t)\|^2 dt\right\}, \tag{3}$$

where $\beta(x_0, 0) \geq 0$.

III. Main results

In this section, we firstly propose sufficient conditions of guaranteeing the dissipativity property for the system in (1), under which the $\mathcal{H}_\infty$ state feedback controller is synthesized for the system in (1). Then the proposed method is further specialized for a special type of nonlinear systems.

A. State feedback control design

Before deriving the dissipative property for the system in (1), we state a fact for the dissipativity as follows.

Lemma 1: If there exist a controller $u(t) = u(x(t), t)$ such that the system in (1) is dissipative with respect to the supply rate $S(v, z) = \gamma^2\|v\|^2 - \|z\|^2$ for an associated storage function satisfying $\mathbb{E}\{V(x_0, 0)\} \leq \gamma^2\mathbb{E}\beta(x_0, 0)$, then the closed-loop system in (1) possesses the property of L_2-gain less than or equal to γ.

In the sequel, the sufficient conditions of guaranteeing the dissipativity for the time-delay system in (1) are given in Theorem 1. For simplicity, we write $x(t-\tau(t))$ as x_τ and omit the arguments of those functions without causing any confusions.

Theorem 1: For the system in (1) and a prescribed attenuation γ, suppose the control input is $u(t) = u^*(t) := -g^{\mathrm{T}}(x, x_\tau)\tilde{Q}(x, x_\tau)x$; if there exist two symmetric positive-definite matrix functions $Q(x, x_\tau) \in \mathcal{C}^{2,1}(\mathbb{R}^{n\times n}; \mathbb{R}^{n\times n})$, and $P(x_t(\theta)) \in \mathcal{C}^{2,1}(\mathcal{C}([-T,0];\mathbb{R}^n);\mathbb{R}^{n\times n})$, such that the following inequalities hold:

$$\Gamma(x,x_\tau) := \gamma^2 I - s_1^{\mathrm{T}}(x,x_\tau)\bar{Q}(x,x_\tau)s_1(x,x_\tau) > 0 \quad (4)$$

$$\left[\begin{array}{ccc} \Xi_1(x,x_\tau) & \tilde{Q}(x,x_\tau)f_1(x,x_\tau) & \tilde{Q}(x,x_\tau)g_1(x,x_\tau) \\ f_1^{\mathrm{T}}(x,x_\tau)\tilde{Q}(x,x_\tau) & -(1-\mu)P(x_\tau) & 0 \\ g_1^{\mathrm{T}}(x,x_\tau)\tilde{Q}(x,x_\tau) & 0 & -\gamma^2 I + s_1^{\mathrm{T}}(x,x_\tau)\bar{Q}(x,x_\tau)s_1(x,x_\tau) \end{array}\right] \le 0, \quad (5)$$

where

$$\begin{aligned} \Xi_1(x,x_\tau) :=& 2\tilde{Q}(x,x_\tau)f_1(x,x_\tau) + h^{\mathrm{T}}(x,x_\tau)h(x,x_\tau) \\ &+ 2s_1^{\mathrm{T}}(x,x_\tau)\bar{Q}(x,x_\tau)s_1(x,x_\tau) + P(x) \\ &- \tilde{Q}(x,x_\tau)g(x,x_\tau)g^{\mathrm{T}}(x,x_\tau)\tilde{Q}(x,x_\tau), \quad (6) \\ \tilde{Q}(x,x_\tau) :=& Q(x,x_\tau) + \frac{1}{2}\left(I_n \otimes x^{\mathrm{T}}\right)\frac{\partial Q(x,x_\tau)}{\partial x}, \quad (7) \\ \bar{Q}(x,x_\tau) :=& Q(x,x_\tau) + \left(\frac{\partial Q(x,x_\tau)}{\partial x^{\mathrm{T}}}\right)(I_n \otimes x) \\ &+ \left(I_n \otimes x^{\mathrm{T}}\right)\left(\frac{\partial Q(x,x_\tau)}{\partial x^{\mathrm{T}}}\right)^{\mathrm{T}} + \frac{1}{2}\left(I_n \otimes x^{\mathrm{T}}\right) \\ &\times \frac{\partial^2 Q(x,x_\tau)}{\partial x \partial x^{\mathrm{T}}}(I_n \otimes x), \quad (8) \end{aligned}$$

then the system in (1) is dissipative with respect to the supply rate $S(v,z) = \gamma^2\|v\|^2 - \|z\|^2$, and the associated storage function is $V(x(t),t) = x^{\mathrm{T}}(t)Q(x,x_\tau)x(t) + \int_{-\tau(t)}^{0} x_t^{\mathrm{T}}(\theta)P(x_t(\theta))x_t(\theta)d\theta$.

Proof: Taking a positive $V(x(t),t)$ for the system in (1) as follows:

$$\begin{aligned} V(x(t),t) =& x^{\mathrm{T}}(t)Q(x,x_\tau)x(t) \\ &+ \int_{-\tau(t)}^{0} x_t^{\mathrm{T}}(\theta)P(x_t(\theta))x_t(\theta)d\theta. \quad (9) \end{aligned}$$

In light of the Itô-formula, we obtain

$$\begin{aligned} dV(x,t) =& \{x^{\mathrm{T}}P(x)x - (1-\dot{\tau}(t))x_\tau^{\mathrm{T}}P(x_\tau)x_\tau \\ &+ 2x^{\mathrm{T}}\tilde{Q}(x,x_\tau)(f(x,x_\tau)x + f_1(x,x_\tau)x_\tau \\ &+ g(x,x_\tau)u + g_1(x,x_\tau)v) \\ &+ \mathrm{Tr}\left(x^{\mathrm{T}}s^{\mathrm{T}}(x,x_\tau)\bar{Q}(x,x_\tau)s(x,x_\tau)x\right) \\ &+ \mathrm{Tr}\left(v^{\mathrm{T}}s_1^{\mathrm{T}}(x,x_\tau)\bar{Q}(x,x_\tau)s_1(x,x_\tau)v\right)\} dt \\ &+ 2x^{\mathrm{T}}\tilde{Q}(x,x_\tau)s(x,x_\tau)xd\omega(t) \\ &+ 2x^{\mathrm{T}}\tilde{Q}(x,x_\tau)s_1(x,x_\tau)vd\omega_1(t) \\ =& \mathcal{A}V(x,t) + 2x^{\mathrm{T}}\tilde{Q}(x,x_\tau)s(x,x_\tau)xd\omega(t) \\ &+ 2x^{\mathrm{T}}\tilde{Q}(x,x_\tau)s_1(x,x_\tau)vd\omega_1(t) \quad (10) \end{aligned}$$

In (10), the weak infinitesimal operator $\mathcal{A}V(x,t)$ can be further calculated as

$$\begin{aligned} \mathcal{A}V(x,t) =& \left\|u + g^{\mathrm{T}}(x,x_\tau)\tilde{Q}(x,x_\tau)x\right\|^2 \\ &- \left\|\left\{v - \Gamma^{-1}(x,x_\tau)g_1^{\mathrm{T}}(x,x_\tau)\tilde{Q}(x,x_\tau)x\right\}^T \right. \\ &\left. \times\, \Gamma^{\frac{1}{2}}(x,x_\tau)\right\|^2 \\ &- \|z\|^2 + \gamma^2\|v\|^2 + 2x^{\mathrm{T}}\tilde{Q}(x,x_\tau)(f(x,x_\tau)x \\ &+ f_1(x,x_\tau)x_\tau) + x^{\mathrm{T}}h^{\mathrm{T}}(x,x_\tau)h(x,x_\tau)x \\ &+ x^{\mathrm{T}}P(x)x - (1-\dot{\tau}(t))x_\tau^{\mathrm{T}}P(x_\tau)x_\tau \\ &- x^{\mathrm{T}}\tilde{Q}(x,x_\tau)g(x,x_\tau)g^{\mathrm{T}}(x,x_\tau)\tilde{Q}(x,x_\tau)x \\ &+ x^{\mathrm{T}}\tilde{Q}(x,x_\tau)g_1(x,x_\tau)\Gamma^{-1}(x,x_\tau)g_1^{\mathrm{T}}(x,x_\tau) \\ &\times \tilde{Q}(x,x_\tau)x + x^{\mathrm{T}}s^{\mathrm{T}}(x,x_\tau)\bar{Q}(x,x_\tau)s(x,x_\tau)x. \end{aligned} \quad (11)$$

Using $\dot{\tau}(t) \le \mu$ and substituting the notations of $u^*(t)$, $v^*(t)$ and $H(Q,P,x,x_\tau)$ into (11) result in

$$\begin{aligned} \mathcal{A}V(x,t) \le& \|u - u^*(t)\|^2 - \|z\|^2 + \gamma^2\|v\|^2 \\ &- \left\|(v - v^*(t))^{\mathrm{T}}\Gamma^{\frac{1}{2}}(x,x_\tau)\right\|^2 \\ &+ H(Q,P,x,x_\tau), \quad (12) \end{aligned}$$

where $\tilde{Q}(x,x_\tau)$ and $\bar{Q}(x,x_\tau)$ are defined in (7) and (8), respectively, $v^*(t) := \Gamma^{-1}(x,x_\tau)g_1(x,x_\tau)\tilde{Q}(x,x_\tau)$ and $H(Q,P,x,x_\tau)$ is defined as follows:

$$\begin{aligned} &H(Q,P,x,x_\tau) \\ :=& 2(\tilde{Q}(x,x_\tau)x)^{\mathrm{T}}(f(x,x_\tau)x + f_1(x,x_\tau)x_\tau) \\ &+ (h(x,x_\tau)x)^{\mathrm{T}}(h(x,x_\tau)x) + x^{\mathrm{T}}P(x)x \\ &- (1-\mu)x_\tau^{\mathrm{T}}P(x_\tau)x_\tau - (\tilde{Q}(x,x_\tau)x)^{\mathrm{T}} \\ &\times g(x,x_\tau)g^{\mathrm{T}}(x,x_\tau)\tilde{Q}(x,x_\tau)x \\ &+ (\tilde{Q}(x,x_\tau)x)^{\mathrm{T}}g_1(x,x_\tau)\Gamma^{-1}(x,x_\tau) \\ &\times g_1(x,x_\tau)^{\mathrm{T}}\tilde{Q}(x,x_\tau)x \\ &+ x^{\mathrm{T}}s^{\mathrm{T}}(x,x_\tau)\bar{Q}(x,x_\tau)s(x,x_\tau)x. \end{aligned}$$

It is observed that $H(Q,P,x,x_\tau)$ can be further written as

$$H(Q,P,x,t) = \left[\begin{array}{c} x \\ x_\tau \end{array}\right]^{\mathrm{T}}\left[\begin{array}{cc} \Omega_1 & \Omega_2 \\ \Omega_2^{\mathrm{T}} & \Omega_3 \end{array}\right]\left[\begin{array}{c} x \\ x_\tau \end{array}\right], \quad (13)$$

where

$$\begin{aligned} \Omega_1 =& 2\tilde{Q}(x,x_\tau)f(x,x_\tau) + h^{\mathrm{T}}(x,x_\tau)h(x,x_\tau) + P(x) \\ &- \tilde{Q}(x,x_\tau)g(x,x_\tau)g^{\mathrm{T}}(x,x_\tau)\tilde{Q}(x,x_\tau) + \tilde{Q}(x,x_\tau) \\ &\times g_1(x,x_\tau)\Gamma^{-1}(x,x_\tau)g_1^{\mathrm{T}}(x,x_\tau)\tilde{Q}(x,x_\tau) \\ &+ s^{\mathrm{T}}\bar{Q}(x,x_\tau)s(x,x_\tau), \\ \Omega_2 =& \tilde{Q}(x,x_\tau)f_1(x,x_\tau), \\ \Omega_3 =& -(1-\mu)P(x_\tau). \end{aligned}$$

By applying Schur's Complement to (4) and (5), it can be obtained that

$$\left[\begin{array}{cc} \Omega_1 & \Omega_2 \\ \Omega_2^{\mathrm{T}} & \Omega_3 \end{array}\right] \le 0,$$

that is, $H(Q,P,x,x_\tau)\le 0$. By the Itô formula, (10) may be written as

$$\begin{aligned}&V(x(t_2),t_2)\\ \le& V(x(t_1),t_1)+\int_{t_1}^{t_2}\Big[\gamma^2\|v\|^2-\|z\|^2+\|u-u^*(t)\|^2\\ &-\Big\|(v-v^*(t))^{\mathrm T}\Gamma^{\frac12}(x,x_\tau)\Big\|^2\Big]dt\\ &+\int_{t_1}^{t_2}2x^{\mathrm T}\tilde Q(x,x_\tau)s(x,x_\tau)xd\omega(t)\\ &+\int_{t_1}^{t_2}2x^{\mathrm T}\tilde Q(x,x_\tau)s_1(x,x_\tau)vd\omega_1(t),\quad \mathcal P-a.e.\end{aligned}\tag{14}$$

where $t_2\ge t_1\ge 0$. By substituting $u(t)=u^*(t)$ into (14), and taking expectation of both sides, we get

$$\begin{aligned}\mathbb E V(x(t_2),t_2)\le&\mathbb E V(x(t_1),t_1)\\ &+\mathbb E\int_{t_1}^{t_2}\left[\gamma^2\|v(t)\|^2-\|z(t)\|^2\right]dt.\end{aligned}$$

Therefore, the system in (1) is dissipative with respect to the supply rate $S(v,z)$ and the associated storage function is $V(x,t)$ given by (9). The proof is completed. ■

Remark 1 It is noted that, the result of Theorem 1 can also be analyzed from the point view of system energy. In fact, if we define a system function $J(v,u,t_1,t_2):=\mathbb E\int_{t_1}^{t_2}-S(v,z)dt$, for all $t_2\ge t_1\ge 0$, which is the measurement of the system energy storage, and $J_0(v,u,t_1,t_2)$ as

$$\begin{aligned}J_0(v,u,t_1,t_2):=&\mathbb E V(x(t_1),t_1)-\mathbb E V(x(t_2),t_2)\\ &+\mathbb E\int_{t_1}^{t_2}\Big\{\|u-u^*(t)\|^2\\ &-\Big\|(v-v^*(t))^{\mathrm T}\Gamma^{\frac12}(x,x_\tau)\Big\|^2\Big\}dt,\end{aligned}$$

then according to (14), it can be derived $J(v,u,t_1,t_2)\le J_0(v,u,t_1,t_2)$. Furthermore, it is observed that $J_0(v,u^*(t),t_1,t_2)\le J_0(v,u,t_1,t_2)$ and $J_0(v^*(t),u^*(t),t_1,t_2)\le J_0(v,u^*(t),t_1,t_2)$. That is to say, on the one hand, the system energy storage is minimized by $u(t)=u^*(t)$, which makes the system in (1) easier to be stabilized; on the other hand, the worst external disturbance $v(t)=v^*(t)$ is chosen such that the dissipative property is guaranteed, i.e., the system is robust against for all $v(t)\in\mathcal L^2([0,\ \infty],\mathbb R^{n_v})$.

Remark 2 It should be pointed out that the dissipative conditions for the system in (1) in Theorem 1 are different from those developed in [3], [4] for delay-free nonlinear systems. In particular, the explicit time-delay storage functional are constructed. Moreover, the dissipative conditions developed can be used to cover both the time-delay nonlinear systems as well as delay-free systems. Thus, the results developed in Theorem 1 may include the results in [3], [4] as special cases.

Based on Theorem 1, the controller can be synthesized in the following theorem.

Theorem 2: For the system in (1), suppose that 1) the conditions of Theorem 1 hold; 2) $h(x(t),x_\tau)$ is lower norm-bounded by a positive number, i.e., $\inf_{t\ge -T}\|h(x(t),x_\tau)\|\ge\alpha>0$ for all $x(t)$ and x_τ; 3) the eigenvalues of $Q(x,x_\tau)$ and $P(x_t(\theta))$ are bounded for all $x(t)$ and x_τ, i.e., $\sup_{t\ge -T}\lambda[Q(x,x_\tau)]\le\lambda_Q<\infty$, and $\sup_{t\ge -T}\lambda[P(x_\tau)]\le\lambda_P<\infty$, then the control input $u(t)=u^*(t)$ is the $\mathcal H_\infty$ state feedback controller for the system in (1).

Proof: Since 1) holds, according to Theorem 1, the system in (1) with control input $u(t)=u^*(t)$ is dissipative with respect to the supply rate $S(v,z)$. Then in terms of Lemma 1, the system in (1) has the property of L_2-gain less than or equal to γ, i.e., H2) is satisfied.

Next we show that the system in (1) with $v(t)\equiv 0$ and $u(t)=u^*(t)$ is exponentially stable in mean square. Here, we choose the Lyapunov function as $V(x,t)$ in (9) for the system in (1) with zero external disturbance. Then, the weak infinitesimal operator $\mathcal A V(x,t)$ associated with system (1) is evaluated as

$$\begin{aligned}\mathcal A V(x,t)=&x^{\mathrm T}P(x)x-(1-\dot\tau(t))x_\tau^{\mathrm T}P(x_\tau)x_\tau\\ &+2x^{\mathrm T}\tilde Q(x,x_\tau)\\ &\times\left(f(x,x_\tau)x+f_1(x,x_\tau)x_\tau+g(x,x_\tau)u\right)\\ &+\mathrm{Tr}\left(xs^{\mathrm T}(x,x_\tau)\bar Q(x,x_\tau)s(x,x_\tau)x\right).\end{aligned}\tag{15}$$

In (15), by considering $\dot\tau(t)\le\mu$ and some manipulations, we have

$$\begin{aligned}\mathcal A V(x,t)\le&\Big\|u+g^{\mathrm T}(x,x_\tau)\tilde Q(x,x_\tau)x\Big\|^2-\|z\|^2\\ &+2x^{\mathrm T}\tilde Q(x,x_\tau)(f(x,x_\tau)x+f_1(x,x_\tau)x_\tau)\\ &+x^{\mathrm T}h^{\mathrm T}(x,x_\tau)h(x,x_\tau)x\\ &+x^{\mathrm T}P(x)x-(1-\dot\tau(t))x_\tau^{\mathrm T}P(x_\tau)x_\tau\\ &-x^{\mathrm T}\tilde Q(x,x_\tau)g(x,x_\tau)g^{\mathrm T}(x,x_\tau)\\ &\times\tilde Q(x,x_\tau)x\\ &+x^{\mathrm T}s^{\mathrm T}(x,x_\tau)\bar Q(x,x_\tau)s(x,x_\tau)x.\end{aligned}\tag{16}$$

By rearranging the right-hand side of (16), the following can be derived:

$$\begin{aligned}\mathcal A V(x,t)\le&\|u-u^*(t)\|^2-\|z\|^2+H(Q,P,x,t)\\ &-x^{\mathrm T}\tilde Q(x,x_\tau)g_1(x,x_\tau)\\ &\times\Gamma^{-1}(x,x_\tau)g_1^{\mathrm T}(x,x_\tau)\tilde Q(x,x_\tau)x.\end{aligned}\tag{17}$$

In (17), by using $H(Q,P,x,x_\tau)\le 0$ and taking $u(t)=u^*(t)$, we have

$$\begin{aligned}\mathcal A V(x,t)\le&-\|z\|^2\\ =&-\|u^*(t)\|^2-\|h(x,x_\tau)x\|^2\\ \le&-\|h(x,x_\tau)\|^2\|x(t)\|^2\\ \le&-\alpha^2\|x(t)\|^2.\end{aligned}\tag{18}$$

Here, the last inequality follows because the norm of $h(x,x_\tau)$ is lower bounded. Furthermore, for the eigenvalues of $Q(x,x_\tau)$ and $P(x(s))$ are bounded $\forall x(s)\in\mathcal{D}$, we obtain

$$V(x(t),t)\geq x^{\mathrm{T}}(t)Q(x,x_\tau)x(t) \geq \inf_{t\geq -T}\lambda[Q(x,x_\tau)]\|x(t)\|^2, \quad (19)$$

and

$$V(x(t),t)\leq x^{\mathrm{T}}(t)Q(x,x_\tau)x(t) + \tau(t)\sup_{t-T\leq s\leq t}\lambda[P(x(s))]\sup_{t-T\leq s\leq t}\|x(s)\|^2 \leq (\lambda_Q+T\lambda_P)\sup_{t-T\leq s\leq t}\|x(s)\|^2. \quad (20)$$

By combining (18), (19) and (20), we conclude that the system in (1) with $v\equiv 0$ and $u(t)=u^*(t)$ is exponentially stable in mean square. The proof is completed. ■

Remark 3 Note that the conditions 2) and 3) in Theorem 2 are reasonable. In fact, those conditions are generalized from linear systems, for which they are satisfied naturally. Furthermore, the controller design method proposed in Theorem 2 is based on the dissipative theory, which are quite different from those developed for time-delay systems in [10], [13]. The controller $u(t)=u^*(t)$ can minimize the system energy storage as well as satisfying the criteria H1) and H2).

B. Extension to a special type of nonlinear stochastic systems

Now we are in a position to specialize the derived results for a special type of nonlinear stochastic systems, that is, the nonlinearities are described as norm-bounded parameter uncertainties:

$$\begin{aligned} dx(t) &= [(A+E_1F(x)H)x(t)+(A_1+E_2F(x)H)x_\tau \\ &\quad +Gu(t)+G_1v(t)]\,dt+(S+E_3F(x)H)x(t)d\omega(t) \\ &\quad +S_1v(t)d\omega_1(t), \\ z(t) &= \begin{bmatrix} H_1x(t) \\ u(t)\end{bmatrix} \\ x(t) &= \phi(t), \quad \forall t\in[-T,0], \end{aligned} \quad (21)$$

where $A\in\mathbb{R}^{n\times n}$, $A_1\in\mathbb{R}^{n\times n}$, $E_1,E_2,E_3\in\mathbb{R}^{n\times m}$, $H\in\mathbb{R}^{m\times n}$, $G\in\mathbb{R}^{n\times n_u}$, $G_1\in\mathbb{R}^{n\times n_v}$, $S_1\in\mathbb{R}^{n\times n_v}$, $S\in\mathbb{R}^{n\times n}$ are constant matrices. $F(x)\in\mathbb{R}^{m\times m}$ is the norm-bounded uncertainty, satisfying $F^{\mathrm{T}}(x)F(x)\leq I$. To solve the $\mathcal{H}_\infty$ control problem and guarantee that the system (21) has the disturbance attenuation property and is exponential stable in mean square, we have the following corollary. Before establishing the corollary, we need the following lemma, which will play an instrumental role in the proof.

Lemma 2: [21] For constant matrices E and H, and a matrix $F(x)$ with $F^{\mathrm{T}}(x)F(x)\leq I$, then there exists a positive scalar ε such that

$$HF(x)E+E^{\mathrm{T}}F^{\mathrm{T}}(x)H^{\mathrm{T}}\leq\varepsilon HH^{\mathrm{T}}+\varepsilon^{-1}E^{\mathrm{T}}E \quad (22)$$

Corollary 1: For system (21), if there exist four positive scalars α, β_1, β_2 and β_3, and two positive-definite matrices $Q=Q^{\mathrm{T}}$ and $P=P^{\mathrm{T}}$, and a prescribed scalar $\gamma>0$, such that the following matrix inequalities hold:

$$\gamma^2I-S_1^{\mathrm{T}}QS_1\geq\alpha I, \quad (23)$$

$$\begin{bmatrix} \Phi_1 & \Phi_2 & S^{\mathrm{T}}Q & \beta_1H^{\mathrm{T}} & QE_1 & 0 & QE_2 & QG_1 \\ * & \Phi_3 & 0 & 0 & 0 & \beta_3H^{\mathrm{T}} & 0 & 0 \\ * & * & -Q & 0 & QE_3 & 0 & 0 & 0 \\ * & * & * & -\beta_1I & 0 & 0 & 0 & 0 \\ * & * & * & * & -\beta_1I & 0 & 0 & 0 \\ * & * & * & * & * & -\beta_2I & 0 & 0 \\ * & * & * & * & * & * & -\beta_2I & 0 \\ * & * & * & * & * & * & * & -\alpha I \end{bmatrix}\leq 0, \quad (24)$$

where

$$\begin{aligned} \Phi_1 &= QA+A^{\mathrm{T}}Q+H_1^{\mathrm{T}}H_1+P-QGG^{\mathrm{T}}Q, \\ \Phi_2 &= QA_1, \\ \Phi_3 &= -(1-\mu)P, \end{aligned}$$

then the $\mathcal{H}_\infty$ state feedback controller is given by $u^*(t)=-GQx(t)$.

Proof: Choose the following explicit storage function:

$$V(x(t),t)=x^{\mathrm{T}}(t)Qx(t)+\int_{-\tau(t)}^{0}x_t^{\mathrm{T}}(\theta)Px_t(\theta)d\theta.$$

According to Theorem 2 and its proof, the following inequality holds:

$$\begin{aligned} H_s(Q,P,x,x_\tau) :=& 2x^{\mathrm{T}}Q\left[(A+E_1F(x)H)x\right. \\ & \left.+(A_1+E_2F(x)H)x_\tau\right]+x^{\mathrm{T}}H_1^{\mathrm{T}}H_1x \\ & +x^{\mathrm{T}}Px-(1-\mu)x_\tau^{\mathrm{T}}Px_\tau \\ & -x^{\mathrm{T}}QGG^{\mathrm{T}}Qx+x^{\mathrm{T}}QG_1 \\ & \times(\gamma^2I-S_1^{\mathrm{T}}QS_1)^{-1}G_1^{\mathrm{T}}Qx \\ & +x^{\mathrm{T}}(S+E_3F(x)H)^{\mathrm{T}} \\ & \times Q(S+E_3F(x)H)x\leq 0, \end{aligned} \quad (25)$$

and

$$\gamma^2I-S_1^{\mathrm{T}}QS_1>0. \quad (26)$$

then the inequality (26) holds iff the there exists positive scalar α, such that $\gamma^2I-S_1^{\mathrm{T}}QS_1\geq\alpha I$, i.e., inequality (23) holds. Furthermore, note that the above inequality (25) can be written as

$$\begin{bmatrix} x \\ x_\tau\end{bmatrix}^{\mathrm{T}}\begin{bmatrix}\bar\Phi_1 & \bar\Phi_2 \\ \bar\Phi_2^{\mathrm{T}} & \bar\Phi_3\end{bmatrix}\begin{bmatrix} x \\ x_\tau\end{bmatrix}\leq 0, \quad (27)$$

where

$$\begin{aligned} \bar\Phi_1 =& Q(A+E_1F(x)H)+(A+E_1F(x)H)^{\mathrm{T}}Q+H_1^{\mathrm{T}}H_1 \\ & +QG_1(\gamma^2I-S_1^{\mathrm{T}}QS_1)^{-1}G_1^{\mathrm{T}}Q+P-QGG^{\mathrm{T}}Q \\ & +(S+E_3F(x)H)^{\mathrm{T}}Q(S+E_3F(x)H), \\ \bar\Phi_2 =& Q(A_1+E_2F(x)H), \\ \bar\Phi_3 =& -P(1-\mu). \end{aligned}$$

By Schur's Complement and the condition (23), we have the following matrix inequality:

$$\begin{bmatrix} \bar{\Phi}_4 & \bar{\Phi}_2 & (S+E_3F(x)H)^{\mathrm{T}} \\ * & \bar{\Phi}_3 & 0 \\ * & * & -Q^{-1} \end{bmatrix} \leq 0, \qquad (28)$$

where

$$\begin{aligned}\bar{\Phi}_4 =& Q(A+E_1F(x)H)+(A+E_1F(x)H)^{\mathrm{T}}Q \\ &+H_1^{\mathrm{T}}H_1+QG_1\alpha^{-1}G_1^{\mathrm{T}}Q+P-QGG^{\mathrm{T}}Q.\end{aligned}$$

Furthermore, the following inequality holds as a result of some algebraic operations for inequality (28),

$$\begin{aligned}&\Xi_3+N_1^{\mathrm{T}}F^{\mathrm{T}}(x)M_1^{\mathrm{T}} \\ &+M_1F(x)N_1+N_2^{\mathrm{T}}F^{\mathrm{T}}(x)M_2^{\mathrm{T}}+M_2F(x)N_2 \leq 0, \quad (29)\end{aligned}$$

where

$$\Xi_3 = \begin{bmatrix} \Phi_1 & \Phi_2 & S^{\mathrm{T}} \\ * & \Phi_3 & 0 \\ * & * & -Q^{-1} \end{bmatrix},$$

$N_1 = [\ H \quad 0 \quad 0\ ]$, $N_2 = [\ 0 \quad H \quad 0\ ]$, $M_1^{\mathrm{T}} = [\ E_1^{\mathrm{T}}Q \quad 0 \quad E_3^{\mathrm{T}}\ ]$, and $M_2^{\mathrm{T}} = [\ E_2^{\mathrm{T}}Q \quad 0 \quad 0\ ]$.

By applying Lemma 2 to inequality (29), then there exist positive scalars, β_1, β_2 and β_3 such that inequality (29) is equivalent to the following one:

$$\Xi_3+\sum_{i=1}^{2}(\beta_iN_i^{\mathrm{T}}N_i+\beta_i^{-1}M_iM_i^{\mathrm{T}}) \leq 0. \qquad (30)$$

By pre-multiplying Λ and post-multiplying Λ^{T} to the result that is obtained by applying Schur's Complement to inequality (30), then inequality (24) follows. Here, Λ is given by

$$\Lambda = \mathrm{diag}\left(\left[\ I \quad I \quad Q \quad \beta_1 I \quad I \quad \beta_2 I \quad I \quad I\ \right]\right).$$

Moreover, according to Theorem 2, the control input is synthesized as $u^*(t) = -GQx(t)$. On the other hand, the exponential stability in mean square for the closed-loop system with $v(t) \equiv 0$ can be derived by following the similar line. This completes the proof. ■

Remark 4 It is observed that the sufficient conditions in Theorem 2 are expressed in the form of matrix inequalities. Further these conditions are converted into a LMI (23) and a BMI (24) in Corollary 1. LMI (23) is easy to be solved, while to solve BMI (24), firstly, the LMI based lower bound and upper bound is determined using the same techniques in [22]. Then BMI (24) can be solved by applying the brand and bound algorithm [22].

IV. Conclusion

In this paper, we have investigated the $\mathcal{H}_\infty$ state feedback control problem for a class of nonlinear stochastic systems with time-varying state delays. The stochastic dissipativity for the time delay nonlinear systems has been established. Based on the dissipation, two matrix inequalities have been developed as the sufficient conditions to synthesize the $\mathcal{H}_\infty$ state feedback controller, under which the closed-loop system achieves the disturbance attenuation level γ and exponential stability in the sense of mean square. The developed results have been further specialized for a type of special nonlinear systems, and solvable conditions have been presented.

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