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Stochastic Stability of Semi-Markov Jump Linear Systems: An LMI Approach

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***Abstract*— The semi-Markov jump linear system is more general than the classic Markov jump linear system. In the semi-Markov jump linear systems, the governing stochastic process is not a Markov process, but a semi-Markov process. Instead of the exponential distribution for the sojourn-time in each mode in the jump linear system, the Weibull distribution is considered in this paper. By deriving the infinitesimal generator for the Lyapunov function of the semi-Markov jump linear system, the numerically testable sufficient conditions for stochastic stability of semi-Markov jump linear systems are obtained. Numerical examples are provided to validate the proposed sufficient stochastic stability conditions.**

I. INTRODUCTION

The Markov jump linear systems (MJLSs) have received considerable research attention in recent years. These types of systems are modeled by a set of linear systems with the transitions among the linear systems governed by Markov chain. The MJLS can describe different types of systems subject to abrupt changes. Hence, it finds many applications in control systems, such as target tracking systems, fault tolerant systems, manufactory processes, networked systems; see, e.g., [1]–[4]. Many important results have been reported in the literature. For instance, the stability analysis and control design are investigated in [5]–[7]. The stability analysis of MJLS with partially known transition rates is studied in [8]. From the mathematic point of view, the jump linear systems belong to the stochastic systems.

In jump linear systems, the **sojourn-time** is the time duration between the two jumps. The sojourn-time h is a random variable following continuous probability distribution F in continuous-time jump linear systems. For example, in the MJLS, F is an exponential distribution. Closely related to the probability distribution, the transition rate $\lambda_{ij}(h)$ is the speed/frequency that the system jumps from mode i to mode j. The transition rate is also referred as failure rate or hazard rate [9]. $\lambda_{ij}(h)$ is determined by F. For example, if F is an exponential distribution, then $\lambda_{ij}(h) \equiv \lambda_{ij}$ is a constant. This time-invariant property of the transition rate can also be derived from the memoryless property of the exponential distribution. Because the memoryless property indicates that the jump speed of the stochastic process is independent of the past/history. In fact, only the exponential distribution among the continuous-time probability distributions pertains the memoryless property [9]. For this reason, using MJLS to model the stochastic system requires that the transition rate of the system is independent of past. This requirement, however, is too restrictive, because the transition rates for many practical systems are not constants. For example, in the fault tolerant control systems, the bathtub curve is widely used to describe a particular form of the transition rate function which comprises three parts: a) decreasing, b) constant (roughly), c) increasing [10]. A typical bathtub curve is reported in [11] shown in Fig. 1 where the system jumps to mode i at $h = 0$. As a result, such a process cannot be modeled as Markov process, so the MJLS has some limitations in applications.

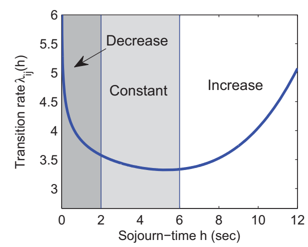

Fig. 1. Transition rates in bathtub shape.

So what if the governing stochastic process of a stochastic system is not a Markov process? To investigate the more general stochastic systems with non-Markov jumps, it is natural to relax the assumptions introduced by Markov process. Focusing on the continuous-time system in this paper and considering the time-varying transition rate described above, we relax the probability distribution of the sojourn-time h from exponential distribution to a more general probability distribution. As a result, the transition rate λ_{ij} is relaxed from a constant to a time-varying variable $\lambda_{ij}(h)$. The continuous stochastic process whose sojourn-time is non-exponentially distribution is referred to as continuous semi-Markov process. Accordingly, the jump linear system whose parameter switches according to semi-Markov process is referred as a semi-Markov jump linear system (S-MJLS) [12].

The necessary and sufficient conditions for optimal control for S-MJLS were discussed in [13]. Nevertheless, no systematic algorithms or numerical algorithms were provided to test if an S-MJLS is stable or not. A stability condition for the S-MJLS controller design was obtained in [12] where the MJLS stability condition was adopted to design the controller. Although the condition was verified on a bunch-train cavity interaction system, the sojourn-time distribution was "nearly exponential" which indicated the S-MJLS was nearly MJLS and the time-varying information of the transition rate was not considered in the controller design. Hou *et*

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al. [14] addressed the stochastic stability for the linear system with semi-Markov jump parameters and similar results have been obtained as in the Markov jump systems. In [14], due to the density property of phase-type (PH) distributions of all probability distributions on $[0, +\infty)$, the PH semi-Markov process was firstly defined and the stability of simple linear systems with PH semi-Markov jump parameters was addressed. The stabilization of the MJLS with time-varying transition rates which were described by polytopic sets were studied in [15], where a conservative result was reported. It is noticed that, although the stability and control design problems for S-MJLS have been a research focus for several years, little attention has been paid to develop numerically testable stochastic stability conditions.

Due to the practical importance of developing numerically testable stability conditions for the S-MJLS and based on the fact that the S-MJLS is a generalization of MJLS, we generalize the stochastic stability condition for MJLS to S-MJLS in terms of linear matrix inequalities (LMIs) which can be solved by the standard software package.

The remainder of this paper is organized as follows. The formulation of the S-MJLS and the objectives are given in Section II. In Section III, the sufficient conditions for stochastic stability of S-MJLS are obtained. To validate the proposed theorems in Section III, simulation examples are provided in Section IV. Finally, the concluding remarks are addressed in Section V.

II. Problem Statement

Considering a dynamical system defined in a probability space $(\Omega, \mathcal{F}, \mathbb{P})$ where its state equation is as follows:

$$\begin{aligned} \dot{x}(t) &= A(r(t))x(t), \\ x(0) &= x_0, \quad r(0) = r_0, \end{aligned} \tag{1}$$

where $\{r(t), t \geq 0\}$ is a continuous-time semi-Markov process taking values in a finite space $\mathcal{S} = \{1, 2, \cdots, N\}$, $x(t) \in \mathbb{R}^n$ is the state vector, $x_0 \in \mathbb{R}^n$ is the initial state at $t = 0$, r_0 is the initial mode in the semi-Markov process at $t = 0$, $A(r(t))$, $r(t) = i \in \mathcal{S}$ are system matrices with compatible dimensions which depend on $r(t)$. For simplicity, $A(i)$ is denoted by A_i.

The evolution of the semi-Markov process $\{r(t), t \geq 0\}$ is governed by the following probability transitions:

$$\begin{aligned} &\Pr\{r(t+h) = j | r(t) = i\} \\ &= \begin{cases} \lambda_{ij}(h)h + o(h), & r(t) \text{ jumps from } i \text{ to } j, \\ 1 + \lambda_{ii}(h)h + o(h), & \text{otherwise}, \end{cases} \end{aligned} \tag{2}$$

where $\lambda_{ij}(h)$ is the transition rate from mode i to mode j at t when $i \neq j$ and $\lambda_{ii}(h) = -\sum_{j=1, j\neq i}^{N} \lambda_{ij}(h)$ and $o(h)$ is little-o notation defined by $\lim_{h\to 0} \frac{o(h)}{h} = 0$.

The objective of this paper is: Propose the numerically testable stochastic stability conditions of the S-MJLS in (1). For the stochastic stability, we adopt the definition in [16].

Definition 1: System (1) with all modes and all $t \geq 0$ is said to be stochastically stable if there exists a finite positive constant $T(x_0, r_0)$ such that the following holds for any initial condition (x_0, r_0):

$$\mathbb{E}\left[\int_0^\infty \|x(t)\|^2 dt | x_0, r_0\right] \leq T(x_0, r_0). \tag{3}$$

To address the stability condition of the S-MJLS, it is necessary to revisit the stochastic stability of the MJLS.

A. The MJLS Revisit

If the stochastic process $r(t)$ involved in (1) is a standard Markov process, the system in (1) is referred as MJLS. Due to the exponential distribution for the sojourn-time, the transition rate from mode i to mode j is constant, i.e.

$$\lambda_{ij}(t) = \frac{f_{ij}(t)}{1 - F_{ij}(t)} = \frac{\lambda_{ij} e^{-\lambda_{ij} t}}{1 - (1 - e^{-\lambda_{ij} t})} = \lambda_{ij},$$

where λ_{ij} is independent of t. The Lyapunov function of system (1) can be chosen as the following quadratic form

$$V(x(t), r(t)) = x^{\mathrm{T}}(t) P(r(t)) x(t),$$

where $P(r(t))$ is a set of symmetric positive definite matrices. At time t, $x(t) = x$ and $r(t) = i$, $\tilde{A}V(x(t), r(t))$ emanating from the point (t, i) is given by:

$$\tilde{A}V(x(t), r(t)) = x^{\mathrm{T}}(t) Q(i) x(t),$$

where

$$Q(i) = A_i^{\mathrm{T}} P(i) + P(i) A_i + \sum_{j\in N} \lambda_{ij} P(j). \tag{4}$$

According to the Dynkin's formula [17]

$$\begin{aligned} &\mathbb{E}\left[V(x(t+\Delta), r(t+\Delta)) | x(t), r(t)\right] - V(x(t), r(t)) \\ &= \int_t^{t+\Delta} \tilde{A}V(x(s), r(s)) ds. \end{aligned} \tag{5}$$

Noting here, the Dynkin's formula can be regarded as a stochastic generalization of the Newton-Leibniz formula. If $\tilde{A}V(\cdot) < 0$, the system is stochastically stable [16]. Therefore, the stability of MJLS can be verified by the existence of the symmetric and positive-definite matrices $\mathcal{P} = (P(1), P(2), \cdots, P(N))$.

III. Main Results

To investigate the stochastic stability of S-MJLSs, analog to MJLS, we firstly construct a stochastic Lyapunov function V and then derive the infinitesimal generator of V. The infinitesimal generator $\tilde{A}$ can be considered as the derivative of the function $V(\cdot)$ [18].

A. From MJLS to S-MJLS Using Weibull Distribution

From MJLS to S-MJLS, we relax the probability distribution of sojourn-time from exponential distribution to Weibull distribution so the transition rate in S-MJLS will be time-varying instead of constant in MJLS. The Weibull distribution is a continuous probability distribution with shape parameter $\beta > 0$ and scale parameter $\alpha > 0$. A random variable H is said to follow a two-parameter Weibull distribution if its probability distribution function (PDF) is

$$f(h) = \begin{cases} \frac{\beta}{\alpha^\beta} h^{\beta-1} \exp\left[-\left(\frac{h}{\alpha}\right)^\beta\right], & h \geq 0, \\ 0, & h < 0. \end{cases}$$

The cumulative distribution function (CDF) of H is

$$F(h) = \begin{cases} 1 - \exp\left[-\left(\frac{h}{\alpha}\right)^\beta\right], & h \geq 0, \\ 0, & h < 0. \end{cases}$$

The transition rate function $\lambda(h)$ is

$$\lambda(h) = \frac{f(h)}{1 - F(h)} = \frac{\beta}{\alpha^\beta} h^{\beta-1}.$$

Remark 1 It is worth mentioning that for the Weibull distribution, if $\beta = 1$, it reduces to an exponential distribution. In such case, the semi-Markov process reduces to a standard Markov process where the sojourn-time at each mode is exponentially distributed. In other words, the transition rate $\lambda(h) \equiv 1/\alpha$ is a constant value (the straight black line in Fig 2). Determining the shape of the PDF and transition rate function $\lambda(h)$, β is therefore known as the shape parameter. For example, $\lambda(h)$ monotonically increases along t if $\beta > 1$ and monotonically decreases if $\beta < 1$ (see Fig. 2).

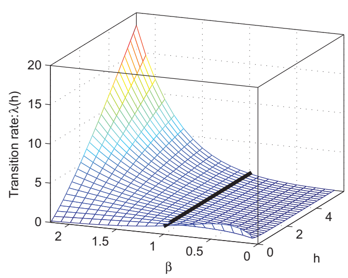

Fig. 2. Transition rates with different shape parameters β ($\alpha = 1$).

As a result of the relaxation, the memoryless property in the MJLS does not pertain in the S-MJLS, so the transition rate is not a constant in the S-MJLS. This poses the main technical difficulty for the stochastic stability analysis for the S-MJLS. The following theorem provides the sufficient condition of the S-MJLS is stochastically stable and it will play an instrumental role in the testable stochastic stability analysis.

Theorem 1: For the semi-Markov jump linear system in (1), where $\lambda_{ij}(h)$ depends on sojourn-time h where h is set to 0 when system jumps. The system is stochastically stable if there exist $P(i) > 0$, $i \in \mathcal{S}$, such that for all $i \in \mathcal{S}$

$$A_i^{\mathrm{T}} P(i) + P(i) A_i + \sum_{j=1}^{N} \lambda_{ij}(h) P(j) < 0. \tag{6}$$

Proof: Consider the following Lyapunov function:

$$V(x(t), r(t)) = x^{\mathrm{T}}(t) P(i) x(t),$$

where $P(i) > 0$ is a symmetric and positive definite matrix for every $i \in \mathcal{S}$.

$$\begin{aligned}
&\tilde{A} V(x(t), r(t)) \\
&= \lim_{\Delta \to 0} \frac{\mathbb{E}\left[V(x(t+\Delta), r(t+\Delta)) | x(t), r(t)\right] - V(x(t), r(t))}{\Delta} \\
&= \lim_{\Delta \to 0} \frac{1}{\Delta} \left[\sum_{j=1, j\neq i}^{N} \Pr\{r(t+\Delta) = j | r(t) = i\} \right. \\
&\quad x^{\mathrm{T}}(t+\Delta) P(j) x(t+\Delta) - x^{\mathrm{T}}(t) P(i) x(t) \\
&\quad \left. + \Pr\{r(t+\Delta) = i | r(t) = i\} x^{\mathrm{T}}(t+\Delta) P(i) x(t+\Delta) \right].
\end{aligned}$$

Noting that for a general distribution of the sojourn-time in mode i, without memoryless property, $\Pr\{r(t+\Delta) = j | r(t) = i\} \neq \Pr\{r(\Delta) = j | r(0) = i\}$. Using the conditional probability formula,

$$\begin{aligned}
\tilde{A} V(x(t), r(t)) = \lim_{\Delta \to 0} \frac{1}{\Delta} &\left[\sum_{j=1, j\neq i}^{N} \frac{q_{ij}(F_i(t+\Delta) - F_i(t))}{1 - F_i(t)} \right. \\
&x^{\mathrm{T}}(t+\Delta) P(j) x(t+\Delta) - x^{\mathrm{T}}(t) P(i) x(t) \\
&\left. + \frac{1 - F_i(t+\Delta)}{1 - F_i(t)} x^{\mathrm{T}}(t+\Delta) P(i) x(t+\Delta) \right],
\end{aligned}$$

where $F_i(t)$ is the CDF of the sojourn-time when system remains in mode i and q_{ij} is the probability intensity from mode i to mode j. Given Δ is small, $x(t+\Delta) = (A_i \Delta + I) x(t)$ and $\lim_{\Delta \to 0} \Delta^2 / \Delta = 0$. So

$$\begin{aligned}
&\tilde{A} V(x(t), r(t)) \\
&= \lim_{\Delta \to 0} \frac{1}{\Delta} \left[\sum_{j=1, j\neq i}^{N} \frac{q_{ij}(F_i(t+\Delta) - F_i(t))}{1 - F_i(t)} x^{\mathrm{T}}(t) \right. \\
&[A_i^{\mathrm{T}} P(j) \Delta + P(j) A_i \Delta + P(j)] x(t) - x^{\mathrm{T}}(t) P(i) x(t) + \\
&\left. \frac{1 - F_i(t+\Delta)}{1 - F_i(t)} x^{\mathrm{T}}(t) [A_i^{\mathrm{T}} P(i) \Delta + P(i) A_i \Delta + P(i)] x(t) \right].
\end{aligned}$$

Using the condition

$$\lim_{\Delta \to 0} \frac{F_i(t+\Delta) - F_i(t)}{1 - F_i(t)} = 0,$$

the time derivative of $V(x(t), r(t))$ becomes

$$\tilde{A} V(x(t), r(t)) = x^{\mathrm{T}}(t) Q(t, i) x(t),$$

where

$$\begin{aligned}
Q(t, i) = &\sum_{j=1, j\neq i}^{N} q_{ij} P(j) \lim_{\Delta \to 0} \frac{F_i(t+\Delta) - F_i(t)}{(1 - F_i(t))\Delta} \\
&+ \left[A_i^{\mathrm{T}} P(i) + P(i) A_i\right] \lim_{\Delta \to 0} \frac{1 - F_i(t+\Delta)}{1 - F_i(t)} \\
&+ P(i) \lim_{\Delta \to 0} \frac{F_i(t) - F_i(t+\Delta)}{(1 - F_i(t))\Delta}. \\
= &\left[A_i^{\mathrm{T}} P(i) + P(i) A_i\right] \lim_{\Delta \to 0} \frac{1 - F_i(t+\Delta)}{1 - F_i(t)} + \\
&\left[\sum_{j=1, j\neq i}^{N} q_{ij} P(j) - P(i) \right] \lim_{\Delta \to 0} \frac{F_i(t+\Delta) - F_i(t)}{(1 - F_i(t))\Delta}.
\end{aligned}$$

To evaluate the limit, take the Taylor series with respect to Δ at 0 as follows:

$$F_i(t+\Delta) = F_i(t) + \Delta \frac{\partial F_i(t+\Delta)}{\partial \Delta}\bigg|_{\Delta=0} + o(\Delta).$$

For finite t, $\exists \epsilon > 0$ such that $1 - F_i(t) > \epsilon$, so

$$\lim_{\Delta \to 0} \frac{1 - F_i(t+\Delta)}{1 - F_i(t)} = 1.$$

Besides,

$$\begin{aligned}\lim_{\Delta\to 0} \frac{F_i(t+\Delta) - F_i(t)}{(1-F_i(t))\Delta} =& \frac{1}{1-F_i(t)} \lim_{\Delta \to 0} \frac{F_i(t+\Delta) - F_i(t)}{\Delta} \\ =& \lambda_i(t).\end{aligned}$$

Here $\lambda_i(t)$ is the transition rate of the system jumping from mode i. Define $\lambda_{ij}(t) = \lambda_i(t) q_{ij}$ for $j \neq i$ and $\lambda_{ii}(t) = -\sum_{j=1, j\neq i} \lambda_{ij}(t)$, so

$$Q(t,i) = A_i^{\mathrm{T}} P(i) + P(i) A_i + \sum_{j=1}^{N} \lambda_{ij}(t) P(j).$$

Thus

$$\begin{aligned}\tilde{A}V(x(t), r(t)) =& x^{\mathrm{T}}(t) Q(t,i) x(t) \\ \leq & \max_{i\in\mathcal{S},t} \{\lambda_{\max} Q(t,i)\} \, x^{\mathrm{T}}(t) x(t).\end{aligned}$$

Here, we show that $\max_{i\in\mathcal{S},t}\{\lambda_{\max} Q(t,i)\}$ exists. Denote

$$Q(t,i) = Q_1(i) + Q_2(t,i),$$

where $Q_1(i)$ and $Q_2(t,i)$ are given as follows

$$Q_1(i) = A_i^{\mathrm{T}} P(i) + P(i) A_i, \quad Q_2(t,i) = \sum_{j\in\mathcal{S}} P(j) \lambda_{ij}(t).$$

Obviously, $\max_{i\in\mathcal{S}}\{\lambda_{\max} Q_1(i)\}$ and $\lambda_{\max} P(j)$ exist. Since $\lambda_{ij}(t) \leq \bar{\lambda}_{ij}$, hence,

$$Q(t,i) - I \max_{i\in\mathcal{S}} \{\lambda_{\max} Q_1(i)\} - I \sum_{j\in\mathcal{S}} \lambda_{\max} P(j) \bar{\lambda}_{ij} \leq 0.$$

Therefore, $\max_{i\in\mathcal{S},t}\{\lambda_{\max} Q(t,i)\}$ always exists.

By the generalized Dynkin's formula [19],

$$\begin{aligned}& \mathbb{E}[V(x(t), i)] - V(x_0, r_0) \\ =& \mathbb{E}\left[\int_0^t \tilde{A}V(x(s), r(s)) ds \middle| (x_0, r_0)\right] \\ \leq & \max_{i\in\mathcal{S},t}\{\lambda_{\max} Q(t,i)\} \, \mathbb{E}\left[\int_0^t x^{\mathrm{T}}(s) x(s) ds \middle| (x_0, r_0)\right].\end{aligned}$$

This, in turn, implies

$$\begin{aligned}& -\max_{i\in\mathcal{S},t}\{\lambda_{\max} Q(t,i)\} \, \mathbb{E}\left[\int_0^t x^{\mathrm{T}}(s) x(s) ds \middle| (x_0, r_0)\right] \\ \leq & \mathbb{E}\left[V(x_0, r_0)\right] - \mathbb{E}\left[V(x(t), i)\right] \leq \mathbb{E}\left[V(x_0, r_0)\right].\end{aligned}$$

Furthermore, (6) indicates $\max_{i\in\mathcal{S},t}\{\lambda_{\max} Q(t,i)\} < 0$, so

$$\mathbb{E}\left[\int_0^t x^{\mathrm{T}}(s) x(s) ds \middle| (x_0, r_0)\right] \leq -\frac{\mathbb{E}\left[V(x_0, r_0)\right]}{\max_{i\in\mathcal{S},t}\{\lambda_{\max} Q(t,i)\}}$$

holds for any $t > 0$. Letting t go to infinity implies that

$$\mathbb{E}\left[\int_0^\infty x^{\mathrm{T}}(s) x(s) ds \middle| (x_0, r_0)\right]$$

is bounded by the constant

$$T(x_0, r_0) = -\frac{\mathbb{E}\left[V(x_0, r_0)\right]}{\max_{i\in\mathcal{S},t}\{\lambda_{\max} Q(t,i)\}} > 0.$$

According to Definition 1, the system in (1) is stochastically stable. This completes the proof. ■

Since $\lambda_{ij}(t)$ is continuously time-varying, verifying the condition in (6) involves testing the feasibility of infinity many matrix inequalities which is numerical impossible [20]. So we seek the similar form in (4) which is numerically testable by existing solvers.

B. A Conservative Condition

A conservative but intuitive approach is to find one set of $P(i)$, $i \in \mathcal{S}$ such that condition (6) holds uniformly for all possible $\lambda_{ij}(t)$, $t > 0$. The following theorem gives an sufficient stochastic stability condition.

Theorem 2: For S-MJLS in (1), if there exist $P(i) > 0$, $i \in \mathcal{S}$, such that for all $i \in \mathcal{S}$

$$A_i^{\mathrm{T}} P(i) + P(i) A_i + \sum_{j=1}^{N} \underline{\lambda}_{ij} P(j) < 0 \tag{7}$$

$$A_i^{\mathrm{T}} P(i) + P(i) A_i + \sum_{j=1}^{N} \bar{\lambda}_{ij} P(j) < 0, \tag{8}$$

where $\underline{\lambda}_{ij} < \bar{\lambda}_{ij}$ are the lower and upper bounds of the transition rate, respectively. Then the S-MJLS with transition rate $\lambda_{ij}(t) \in [\underline{\lambda}_{ij},\ \bar{\lambda}_{ij}]$ is stochastically stable.

Proof: According to Theorem 1, the S-MJLS is stochastically stable with transition rate $\lambda_{ij}(t)$ if there exist $P(i) > 0$, $i \in \mathcal{S}$ such that the condition in (6) holds. $\lambda_{ij}(t)$ can be written as the linear combination $\lambda_{ij}(t) = \varepsilon_1 \underline{\lambda}_{ij} + \varepsilon_2 \bar{\lambda}_{ij}$ where $\varepsilon_1 + \varepsilon_2 = 1$ and $\varepsilon_1, \varepsilon_2 > 0$. For $i \in \mathcal{S}$, multiplying (7) by ε_1 and (8) by ε_2, the summation yields

$$A_i^{\mathrm{T}} P(i) + P(i) A_i + \sum_{j=1}^{N} (\varepsilon_1 \underline{\lambda}_{ij} + \varepsilon_2 \bar{\lambda}_{ij}) P(j) < 0.$$

By tuning ε_1 and ε_2, all possible $\lambda_{ij}(t) \in [\underline{\lambda}_{ij},\ \bar{\lambda}_{ij}]$ can be achieved. Therefore the condition (6) holds uniformly which means that the system in (1) is stochastically stable. ■

Observing from Fig. 2, for the sojourn-time with a Weibull distribution with parameters $\alpha = 1$, $\beta = 2$, the transition rate increases monotonically along h, accordingly, $\underline{\lambda}_{ij} = \lambda_{ij}(0) = 0$ and $\bar{\lambda}_{ij} = \lambda_{ij}(\infty) = \infty$. For $\lambda_{ij}(\infty) = \infty$ and $P(i) > 0$ the condition in (8) does not hold. This imposes the significant difficulty on the stochastic stability analysis. Therefore, the following assumption is needed to tackle the problem. For S-MJLS where the transition rate λ varies from 0 to ∞, we constrain λ to $[\underline{\lambda},\ \bar{\lambda}]$. The choice of $\underline{\lambda}$ and $\bar{\lambda}$ can guarantee the switching happens between $\underline{t}$ and $\bar{t}$ at 99% confidence level, i.e. $\Pr\{$the S-MJLS jumps between $\underline{t}$ and $\bar{t}\} > 0.99$.

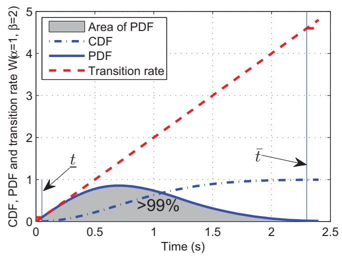

Fig. 3. Constrain the lower and upper bounds of the transition rates.

C. Reduce the Conservativeness

The condition in Theorem 2 is, however, conservative. To reduce the conservativeness of the condition, we partition the transition rate into M sections. The separating transition rates are $\lambda_{ij,1}, \lambda_{ij,2}, \cdots, \lambda_{ij,M-1}$, and $\underline{\lambda}_{ij}$, $\bar{\lambda}_{ij}$ are further denoted by $\lambda_{ij,0}$, $\lambda_{ij,M}$, respectively. So the minimum and maximum transition rate for section m are $(\lambda_{ij,m-1}\ \lambda_{ij,m})$. The following theorem shows the partition can reduce the conservativeness of the sufficient condition in Theorem 2.

Theorem 3: For the S-MJLS in (1), if there exist $P(i,m) > 0$, $i \in \mathcal{S}, m = 1, 2, \cdots, M$ that the following set of LMIs hold for every $i \in \mathcal{S}$ and $m = 1, 2, \cdots, M$

$$A_i^{\mathrm{T}} P(i,m) + P(i,m)A_i + \sum_{j=1}^{N} \lambda_{ij,m-1} P(j,m) < 0 \quad (9)$$

$$A_i^{\mathrm{T}} P(i,m) + P(i,m)A_i + \sum_{j=1}^{N} \lambda_{ij,m} P(j,m) < 0, \quad (10)$$

where $\lambda_{ij,m-1}$ and $\lambda_{ij,m}$ are the lower and upper bounds of transition rate in section m. Then the S-MJLS is stochastically stable.

Proof: Substituting $\underline{\lambda}_{ij}$ and $\bar{\lambda}_{ij}$ in Theorem 2 by $\lambda_{ij,m-1}$ and $\lambda_{ij,m}$, this theorem can be readily proved and hence omitted here. ■

Remark 2 The condition in Theorem 1 is not numerically testable by existing software package. Imposing constrains on the transition rates the testable sufficient condition is obtained in Theorem 2. To reduce the conservativeness, we partition the transition rates. Since $P(i,m_1)$ does not necessarily equal $P(i,m_2)$, so Theorem 3 allows more freedom in choosing feasible solutions which can be seen from the numerical Example 3.

IV. Numerical Examples

In this section, we present numerical simulation examples to illustrate the effectiveness of the developed sufficient stochastic stability conditions.

A. Example 1

Consider the S-MJLS with 2 modes

$$A_1 = \begin{bmatrix} -1.8588 & 0.4615 \\ 0.4969 & -0.1221 \end{bmatrix}, A_2 = \begin{bmatrix} 0.0048 & -0.0094 \\ -0.2932 & -0.7738 \end{bmatrix}.$$

The sojourn-time for each mode follows the same Weibull distribution with scale parameter $\alpha = 1$ and shape parameter $\beta = 2$. We constrain the transition rate bounded in $[0.1000\ \ 4.6000]$ which is valid at 99% confidence level. The eigenvalues of the two subsystems are -1.9821, 0.0012 and 0.0083, -0.7773, respectively. Both subsystems are unstable. Apply Theorem 2, solving (7) and (8) yields

$$P(1) = \begin{bmatrix} 6.4650 & 3.7088 \\ 3.7088 & 16.9694 \end{bmatrix}, P(2) = \begin{bmatrix} 12.5502 & 2.9348 \\ 2.9348 & 11.5181 \end{bmatrix}.$$

Therefore the S-MJLS is stochastically stable. With initial conditions $x_0 = [5\ \ -4]^{\mathrm{T}}$ and $r_0 = 1$, the state trajectories are shown in Fig. 4.

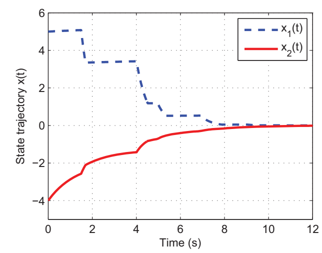

Fig. 4. State trajectories of Example 1.

B. Example 2

Consider the S-MJLS with 2 modes

$$A_1 = \begin{bmatrix} -0.4393 & 3.1629 \\ -2.4097 & 0.4621 \end{bmatrix}, A_2 = \begin{bmatrix} -0.8888 & -0.4272 \\ -0.0792 & -0.3879 \end{bmatrix}.$$

The eigenvalues of the two subsystems are $0.0114 + 2.7237i$, $0.0114 - 2.7237i$ and -0.9491, -0.3276, respectively. A_1 is unstable. For the purpose of comparison, the probability distributions and constraints for sojourn-time in each mode are the same as in the Example 1. The sufficient condition in Theorem 2 is infeasible, but the sufficient condition in Theorem 3 with $m = 2$ and separating transition rate $\lambda_{ij,1} = 0.8326$ is feasible. Solving (9) and (10) yields

$$P(1,1) = \begin{bmatrix} 1266.8 & -238.5 \\ -0238.5 & 1660.6 \end{bmatrix},$$

$$P(2,1) = \begin{bmatrix} 473.0 & -222.9 \\ -222.9 & 1050.9 \end{bmatrix},$$

$$P(1,2) = \begin{bmatrix} 1017.1 & -208.8 \\ -208.8 & 1310.4 \end{bmatrix},$$

$$P(2,2) = \begin{bmatrix} 676.7 & -237.0 \\ -237.0 & 1159.7 \end{bmatrix}.$$

Thus, the S-MJLS is stochastically stable. The state trajectories with the same initial conditions as in Example 1 are shown in Fig. 5.

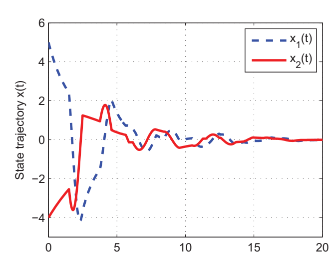

Fig. 5. State trajectories of Example 2.

C. Example 3

Consider the S-MJLS with 2 modes

$$A_1 = \begin{bmatrix} 0.0222 & -2.7965 \\ 1.7444 & -0.6698 \end{bmatrix}, A_2 = \begin{bmatrix} 0.0772 & 0.3068 \\ -0.1344 & -0.4202 \end{bmatrix}.$$

The eigenvalues of the two subsystems are $-0.3238 + 2.1814i, \ -0.3238 - 2.1814i$ and $-0.0279, \ -0.3151$, respectively. Both two subsystems are stable. However, the conditions in Theorem 2 and $m = 2$ in Theorem 3 are not feasible. Nevertheless, with $m = 4$ and the separating transition rates $\lambda_{ij,1} = 0.5364$, $\lambda_{ij,2} = 0.8326$, and $\lambda_{ij,3} = 1.1774$, the condition in Theorem 3 is feasible which yields

$$P(1,1) = \begin{bmatrix} 410.9 & -73.8 \\ -73.8 & 523.8 \end{bmatrix},$$
$$P(2,1) = \begin{bmatrix} 730.5 & 389.3 \\ 389.3 & 736.5 \end{bmatrix},$$
$$P(1,2) = \begin{bmatrix} 508.3 & -133.7 \\ -133.7 & 678.8 \end{bmatrix},$$
$$P(2,2) = \begin{bmatrix} 855.5 & 105.2 \\ 105.2 & 539.4 \end{bmatrix}.$$
$$P(1,3) = \begin{bmatrix} 515.7 & -145.0 \\ -145.0 & 688.3 \end{bmatrix},$$
$$P(2,3) = \begin{bmatrix} 786.5 & 16.6 \\ 16.6 & 537.5 \end{bmatrix},$$
$$P(1,4) = \begin{bmatrix} 508.0 & -150.3 \\ -150.3 & 679.5 \end{bmatrix},$$
$$P(2,4) = \begin{bmatrix} 670.7 & -64.4 \\ -64.4 & 565.6 \end{bmatrix}.$$

So the S-MJLS is stochastically stable. The state trajectories with the same initial conditions as in Example 1 are shown in Fig. 6.

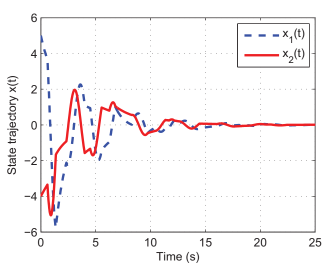

Fig. 6. State trajectories of Example 3.

Remark 3 Theorem 2 reduces to the conventional MJLS theory considering the time-varying transition rates described by polytopic uncertainties [15]. Shown by Example 2 and 3, the sufficient conditions of the conventional MJLS theory will not be satisfied, however, by separating the transition rates into several sections with Theorem 3, the S-MJLSs in Example 2 and 3 are stochastically stable. It demonstrates that the proposed condition is less conservative and more efficient on the stochastic stability analysis for the S-MJLS.

V. Conclusion

For the S-MJLS with the sojourn-time following Weibull distribution, the sufficient condition for stochastic stability of the S-MJLS is derived. Further constraining on the lower and upper bound of the transition rate, the testable sufficient conditions of the S-MJLS are derived in the form of a set of LMIs. To reduce the conservativeness of the sufficient condition, we separate the transition rates into small sections. The more sections are separated, the less conservative of the sufficient condition, however, computational complexity increased which cost higher computational burden. Simulation examples verify the proposed sufficient conditions. The examples also show that the instability of some or all of the modes in the S-MJLS does not necessarily mean the overall system is unstable.

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