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Novel results in averaging analysis of singularly perturbed hybrid systems

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Abstract— **We investigate stability of a class of singularly perturbed systems whose slow system is a set-valued average defined via an appropriate averaging procedure of the solutions of the continuous-time boundary layer system. An approximate hybrid system consisting of this average, the projection of the jump map in the direction of the slow states and flow and jump sets from the original dynamics is shown to approximate the actual singularly perturbed hybrid system. In particular, using forward pre-completeness of the average system we show that solutions of the actual and approximate systems are close in an appropriate sense on compact time intervals. It is also shown that global asymptotic stability of the average system implies semi-global practical asymptotic stability of the actual system. Several examples are presented to illustrate our results and relate them to previously published results in the literature.**

I. Introduction

Dynamical systems that exhibit two time scale behavior can be analyzed via singular perturbation techniques in which the actual dynamics are approximated with two auxiliary systems: a fast (or boundary layer) system and a slow (or reduced) system. The classical singular perturbation results assume that the solutions of the boundary layer system converge to an asymptotically stable manifold and that the vector fields are Lipschitz continuous [12], [16].

The classical results on singular perturbations of continuous-time systems can be used to conclude closeness of solutions between the actual system and reduced and boundary layer systems on compact time domains under the assumption that the reduced system is forward complete and the boundary layer system is exponentially stable, see [11]. Asymptotic or exponential stability of the actual system can be guaranteed if both reduced and boundary layer systems are asymptotically or exponentially stable [4], [16].

In some situations the boundary layer solutions do not converge to an equilibrium manifold but instead to a time-varying integral manifold on which the derivatives of the slow state variables can be averaged, see [4]; in those cases it is necessary to combine averaging and singular perturbation techniques. Moreover, combining averaging and singular perturbations may also be necessary when trajectories of the boundary layer system converge to a set instead of the equilibrium manifold, see [1]–[3], [7]–[9], [15].

Singular perturbation theory based on averaging leads to a reduced order system, where fast motions appear implicitly and only their average influence on slow motions is considered. In general, this approach requires the assumption that large time scale behavior of trajectories of the fast dynamics is in some sense independent of its initial values [10], or properties guaranteed by a unique invariant measure [1]–[3], [9] or some stability properties [6], [15], [17], [20].

In this paper we consider singular perturbations via averaging for a class of hybrid systems. Prior results on singular perturbations for hybrid systems can be found in [13], [14], [19]. In particular, results in [14] assume that the solutions of the boundary layer system converge to a quasi-steady state equilibrium manifold. Singular perturbations for hybrid control systems with fast actuators were considered in [13] where a set-valued mapping was used to approximate the limiting behavior of the boundary layer system. Continuous averages for the slow dynamics that are generated by solutions of the boundary layer system are considered in [19].

The main purpose of this paper is to extend the results in [19] (also presented in [18]) to cover a more general class of hybrid systems. Our results can be used as an analysis tool to design hybrid feedbacks for continuous-time plants implemented by fast but continuous actuators, see more details in [19, Example 5].

Assuming forward pre-completeness of the average system, we show that each solution of the slow dynamics of the singularly perturbed hybrid system can be made arbitrarily close on compact time domains to some solution of its average system by increasing the separation of time scales. We also show that a compact set is semi-globally practically asymptotically stable for the actual hybrid system if it is globally asymptotically stable for the average system. An example is used to illustrate that a continuous average defined in [19] for the slow dynamics may not exist and a set-valued average proposed in this paper may exist. Hence, our results can be applied to a larger class of systems than those considered in [19]. Moreover, using the same example, we show that our results give sharper conclusions in some cases than the results presented in [13].

The paper is organized as follows. We introduce a class of singularly perturbed hybrid systems in Section II. The main results and an example used to relate the main results to previously published results are given in Sections III and IV respectively. Section V contains conclusions. All proofs are omitted due to the space limitations.

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II. Singularly perturbed hybrid systems

The singularly perturbed hybrid systems that we consider are based on two time scales, (τ, j) and (t, j) with $\tau = \varepsilon t$ for a small parameter $\varepsilon > 0$, with the notations $x' = \frac{dx}{d\tau}$, $\dot{x} = \frac{dx}{dt}$. $\mathbb{Z}_{\geq 0} = \{0, 1, 2, \cdots\}$. $\mathbb{B}$ is the closed unit ball in an Euclidean space, the dimension of which should be clear from the context. A set-valued mapping $M : \mathbb{R}^n \rightrightarrows \mathbb{R}^n$ is outer semi-continuous at $x \in \mathbb{R}^n$ if for all sequences $x_i \to x$ and $y_i \in M(x_i)$ such that $y_i \to y$ we have $y \in M(x)$, and M is outer semi-continuous (OSC) if it is outer semi-continuous at each $x \in \mathbb{R}^n$. A set-valued mapping $M : \mathbb{R}^n \rightrightarrows \mathbb{R}^n$ is locally bounded if for any compact set $\mathcal{A} \subset \mathbb{R}^n$ there exists $r > 0$ such that $M(\mathcal{A}) := \bigcup_{x \in \mathcal{A}} M(x) \subset r\mathbb{B}$; if M is OSC and locally bounded, then $M(\mathcal{A})$ is compact for any compact set $\mathcal{A}$. A function $x : \mathbb{R}_{\geq 0} \to \mathbb{R}^n$ is locally absolutely continuous if its derivative is defined almost everywhere and we have $x(t) - x(t_0) = \int_{t_0}^{t} \dot{x}(s) ds$ for all $t \geq t_0 \geq 0$. Given a set S, $\overline{con} S$ denotes the closed convex hull of a set S. Given a compact set $\mathcal{A} \subset \mathbb{R}^n$ and a vector $x \in \mathbb{R}^n$, define $|x|_{\mathcal{A}} := \min_{y \in \mathcal{A}} |x - y|$. A continuous function $\sigma : \mathbb{R}_{\geq 0} \to \mathbb{R}_{\geq 0}$ is of class-$\mathcal{L}$ if it is non-increasing and converging to zero as its argument grows unbounded. A continuous function $\gamma : \mathbb{R}_{\geq 0} \to \mathbb{R}_{\geq 0}$ is of class-$\mathcal{G}$ if it is zero at zero and non-decreasing. It is of class-$\mathcal{K}$ if it is of class $\mathcal{G}$ and strictly increasing. A continuous function $\beta : \mathbb{R}_{\geq 0} \times \mathbb{R}_{\geq 0} \to \mathbb{R}_{\geq 0}$ is of class-$\mathcal{KL}$ if it is of class $\mathcal{K}$ in its first argument and class $\mathcal{L}$ in its second argument. A set $S \subset \mathbb{R}_{\geq 0} \times \mathbb{Z}_{\geq 0}$ is called a compact hybrid time domain if $S = \bigcup_{j=0}^{J-1} ([t_j, t_{j+1}], j)$ for some finite sequence of times $0 = t_0 \leq t_1 \leq t_2 \leq \cdots \leq t_J$. The set S is a hybrid time domain if for all $(T, J) \in S$, $S \cap ([0, T] \times \{0, 1, \cdots, J\})$ is a compact hybrid time domain.

Consider a class of singularly perturbed hybrid systems with the time variables (τ, j):

$$\mathcal{H}_\varepsilon \quad \left.\begin{aligned} x' &= f(x, z, \varepsilon) \\ z' &= \tfrac{1}{\varepsilon}\psi(x, z, \varepsilon) \end{aligned}\right\} \quad (x, z) \in C \times \Psi \qquad (1)$$
$$(x, z)^+ \in G(x, z) \qquad (x, z) \in D \times \Psi ,$$

where $x \in \mathbb{R}^n$, $z \in \mathbb{R}^m$, $C, D \subset \mathbb{R}^n$, $\Psi \subset \mathbb{R}^m$, $f : C \times \Psi \times \mathbb{R}_{\geq 0} \to \mathbb{R}^n$, $\psi : C \times \Psi \times \mathbb{R}_{\geq 0} \to \mathbb{R}^m$, $G : \mathbb{R}^n \times \mathbb{R}^m \rightrightarrows \mathbb{R}^n \times \mathbb{R}^m$, and $\varepsilon > 0$ is a small parameter reflecting that the flow dynamics of z are much faster than x. Let $f_0(x, z) := f(x, z, 0)$ and $\psi_0(x, z) := \psi(x, z, 0)$. We assume that system $\mathcal{H}_\varepsilon$ satisfies the following conditions.

Assumption 1: The sets C and D are closed and the set Ψ is compact. G is outer semi-continuous and locally bounded, and for each $(x, z) \in D \times \Psi$, $G(x, z)$ is nonempty. $f_0 : C \times \Psi \to \mathbb{R}^n$ and $\psi_0 : C \times \Psi \to \mathbb{R}^m$ are continuous, and for each $\delta > 0$ and compact $K \subset \mathbb{R}^n$ there exists $\varepsilon^* := \varepsilon^*(K, \delta) > 0$ such that

$$\left.\begin{aligned} |f(x, z, \varepsilon) - f_0(x, z)| &\leq \delta \\ |\psi(x, z, \varepsilon) - \psi_0(x, z)| &\leq \delta \end{aligned}\right\} \qquad (2)$$
$$\forall\ ((x, z), \varepsilon) \in ((C \cap K) \times \Psi) \times (0, \varepsilon^*] . \qquad \square$$

Note that in Assumption 1, the set Ψ is required to be compact as we wish to deal with compact attractors for the fast state z and without any assumption on the set-valued map G; if (1) admits solutions with a purely discrete-time domain then a jump rule like $z^+ = z$ will not allow z to converge to a compact set unless it is constrained to a compact set a priori.

We also express the system $\mathcal{H}_\varepsilon$ in (1) with the time variables (t, j) with $t := \tau/\varepsilon$:

$$\mathcal{H}_\varepsilon \quad \left.\begin{aligned} \dot{x} &= \varepsilon f(x, z, \varepsilon) \\ \dot{z} &= \psi(x, z, \varepsilon) \end{aligned}\right\} \quad (x, z) \in C \times \Psi \qquad (3)$$
$$(x, z)^+ \in G(x, z) \qquad (x, z) \in D \times \Psi ,$$

and define the boundary layer system of the system $\mathcal{H}_\varepsilon$ as

$$\mathcal{H}_{bl} \quad \left.\begin{aligned} \dot{x}_{bl} &= 0 \\ \dot{z}_{bl} &= \psi_0(x_{bl}, z_{bl}) \end{aligned}\right\} \quad (x_{bl}, z_{bl}) \in C \times \Psi , \qquad (4)$$

which is obtained by ignoring the jump mapping and setting $\varepsilon = 0$ in (3).

We next define the average of the function $f_0 : C \times \Psi \to \mathbb{R}^n$ with respect to $\psi_0 : C \times \Psi \to \mathbb{R}^m$, with which the flow dynamics of the system $\mathcal{H}_\varepsilon$ can be approximated by a differential inclusion.

Definition 1: For functions $f_0 : C \times \Psi \to \mathbb{R}^n$ and $\psi_0 : C \times \Psi \to \mathbb{R}^m$, the set-valued mapping $F_{av} : \mathbb{R}^n \rightrightarrows \mathbb{R}^n$ is said to be an average of f_0 with respect to ψ_0 on $C \times \Psi$ if for each compact set $K \subset \mathbb{R}^n$ there exists a class-$\mathcal{L}$ function σ_K such that, for each $L > 0$, $x \in C \cap K$ and each function $z_{bl} : [0, L] \mapsto \Psi$ satisfying $\dot{z}_{bl} = \psi_0(x, z_{bl})$ there exists a measurable function $f_{z_{bl}} : [0, L] \to \mathbb{R}^n$ such that $f_{z_{bl}}(s) \in F_{av}(x)$ for all $s \in [0, L]$ and the following holds:

$$\left| \frac{1}{L} \int_0^L f_0[(x, z_{bl}(s)) - f_{z_{bl}}(s)] ds \right| \leq \sigma_K(L). \qquad (5)$$

For the singularly perturbed system $\mathcal{H}_\varepsilon$ modeled in (1) or (3), its average system $\mathcal{H}_{av} := \{F_{av}, G_{av}, C, D\}$ is defined as:

$$\mathcal{H}_{av} \quad \begin{aligned} \xi' &\in F_{av}(\xi) & \xi &\in C \\ \xi^+ &\in G_{av}(\xi) & \xi &\in D , \end{aligned} \qquad (6)$$

where $\xi \in \mathbb{R}^n$, F_{av} comes from Def. 1 and G_{av} is the projection of $G(x, z)$ in the x direction:

$$G_{av}(x) := \{v_1 \in \mathbb{R}^n : (v_1, v_2) \in G(x, z), (z, v_2) \in \Psi \times \mathbb{R}^m\}. \qquad (7)$$

To illustrate how to get the jump mapping G_{av} of the averaged system from G of the actual hybrid system, a simple example is given.

Example 1: Consider the hybrid system $\mathcal{H}_\varepsilon$ with the data (f, ψ, G, C, D, Ψ) formed as (3). Let F_{av} be the average of f_0 with respect to ψ_0 on the set $C \times \Psi$. For some $\gamma > 0$, let G, D be defined as

$$G(x, z) := [-\gamma x + z_1^2, g(x, z)]^T \qquad (8)$$
$$D := \{x : x \leq 0\} ,$$

where $g(x,z)$ is an arbitrary function. Noting the definition of G_{av} in (7), we get the average of the hybrid system $\mathcal{H}_\varepsilon$ is

$$\begin{aligned} &\xi' \in F_{av}(\xi) && \xi \in C \\ &\xi^+ \in -\gamma\xi + [c_3, c_4] && \xi \in D\ , \end{aligned} \tag{9}$$

where the positive real numbers $c_3 := \min_{z\in\Psi}\{z_1^2\}$ and $c_4 := \max_{z\in\Psi}\{z_1^2\}$. □

To analyze the singularly perturbed system $\mathcal{H}_\varepsilon$ through its average system $\mathcal{H}_{av}$, we need the assumption that a well-defined average is admitted by $\mathcal{H}_\varepsilon$.

Assumption 2: The function $f_0 : C \times \Psi \to \mathbb{R}^n$ admits an outer semi-continuous, locally bounded and convex-valued average mapping $F_{av} : \mathbb{R}^n \rightrightarrows \mathbb{R}^n$ with respect to $\psi_0 : C \times \Psi \to \mathbb{R}^m$ on the set $C \times \Psi$. □

We provide a lemma on existence of averages for systems $\mathcal{H}_\varepsilon$ in (1) and the idea is implicit in the results of [15].

Assumption 3: For a given compact set $\Omega \subset C \times \Psi$, there exist an outer semi-continuous, locally bounded and convex-valued mapping $F_{av} : \mathbb{R}^n \rightrightarrows \mathbb{R}^n$ and a class-$\mathcal{L}$ function σ_Ω such that, for each $L > 0$ and function $z_{bl} : [0, L] \mapsto \Psi$ satisfying $(x, z_{bl}(0)) \in \Omega$ and $\dot{z}_{bl} = \psi_0(x, z_{bl})$ there exists a measurable function $f_{z_{bl}} : [0, L] \to \mathbb{R}^n$ such that $f_{z_{bl}}(s) \in F_{av}(x)$ for all $s \in [0, L]$ and the following holds:

$$\left| \frac{1}{L} \int_0^L [f_0(x, z_{bl}(s)) - f_{z_{bl}}(s)] ds \right| \le \sigma_\Omega(L).$$

Lemma 1: Suppose that the singularly perturbed system $\mathcal{H}_\varepsilon$ in (1) satisfies Assumptions 1. Assumption 2 holds if for each compact set $K \subset \mathbb{R}^n$ there exists a compact set $\Omega \subset (C \cap K) \times \Psi$ such that Assumption 3 holds and Ω is globally asymptotically stable for the boundary layer system in (4) with C replaced with $C \cap K$. □

Lemma 1 is helpful to show where Assumption 2 holds. For instance, it is showed that Assumption 2 holds in the case when the boundary layer system $\mathcal{H}_{bl}$ has a globally asymptotically stable quasi-steady state equilibrium manifold, see [19, Example 1], which is the essential assumption for classical singular perturbation theory. It also holds when solutions of the boundary layer system $\mathcal{H}_{bl}$ converge to a stable limit cycle [19, Example 2] and even when system $\mathcal{H}_{bl}$ contains equilibria that are neither stable nor attractive [19, Example 3].

Note that in [19], considering the same class of hybrid systems $\mathcal{H}_\varepsilon$ in (1), the average of the function f_0 with respect to ψ_0 is a continuous function, which can be taken as a special case comparing the set-valued mapping average in Def. 1 of the present paper. Through an example in Section IV, we show that Def. 1 pertains to a more general class of hybrid systems. This generalization is the main contribution of this paper.

III. Main results

Closeness of solutions between the singularly perturbed system and the solutions of its average system on compact time domains is considered as one of the main results under the assumption that the average hybrid system is forward pre-complete. Based on the hybrid time domain, the definitions of solutions and forward completeness for hybrid system $\mathcal{H}_{av}$ in (6) and closeness of hybrid signals are first reviewed, see more details in [5].

A hybrid signal is a function defined on a hybrid time domain. A hybrid signal $\xi :\ \mathrm{dom}\ \xi \mapsto \mathbb{R}^n$ is called a hybrid arc if $\xi(\cdot, j)$ is locally absolutely continuous for each j. A hybrid arc $\xi :\ \mathrm{dom}\ \xi \mapsto \mathbb{R}^n$ is a solution to the hybrid system $\mathcal{H}_{av}$ in (6) if $\xi(0,0) \in C \cup D$ and:

1) for all $j \in \mathbb{N}$ and almost all τ such that $(\tau, j) \in \mathrm{dom}\ \xi$, $\xi(\tau, j) \in C$ and $\xi'(\tau, j) \in F_{av}(\xi(\tau, j))$;
2) for all $(\tau, j) \in \mathrm{dom}\ \xi$ such that $(\tau, j+1) \in \mathrm{dom}\ \xi$, $\xi(\tau, j) \in D$ and $\xi(\tau, j+1) \in G_{av}(\xi(\tau, j))$.

A solution is maximal if it cannot be extended.

Definition 2: (Forward completeness) A hybrid solution is said to be forward complete if its domain is unbounded. A hybrid solution is said to be forward pre-complete if its domain is compact or unbounded. System $\mathcal{H}_{av}$ is said to be forward pre-complete from a compact set $K_0 \subset \mathbb{R}^n$ if all maximal solutions ξ with $\xi(0,0) \in K_0$ are forward pre-complete. □

Definition 3: (Closeness of hybrid signals) Two hybrid signals $\xi_1 :\ \mathrm{dom}\ \xi_1 \mapsto \mathbb{R}^n$ and $\xi_2 :\ \mathrm{dom}\ \xi_2 \mapsto \mathbb{R}^n$ are said to be (T, J, ρ)-close if:

1) for each $(t, j) \in \mathrm{dom}\ \xi_1$ with $t \le T$ and $j \le J$ there exists s such that $(s, j) \in \mathrm{dom}\ \xi_2$, with $|t - s| \le \rho$ and $|\xi_1(t, j) - \xi_2(s, j)| \le \rho$,
2) for each $(t, j) \in \mathrm{dom}\ \xi_2$ with $t \le T$ and $j \le J$ there exists s such that $(s, j) \in \mathrm{dom}\ \xi_1$, with $|t - s| \le \rho$ and $|\xi_2(t, j) - \xi_1(s, j)| \le \rho$. □

Now, we are ready to give results on closeness of the slow solutions x of the singularly perturbed system $\mathcal{H}_\varepsilon$ to the solutions of its average system $\mathcal{H}_{av}$ on compact time domains in Theorem 1.

Theorem 1: Suppose that the singularly perturbed system $\mathcal{H}_\varepsilon$ in (1) satisfies Assumptions 1 and 2 and that its average system $\mathcal{H}_{av}$ (6) is forward pre-complete from a compact set $K_0 \subset \mathbb{R}^n$. Then, for each $\rho > 0$ and any strictly positive real numbers T, J, there exists $\varepsilon^* > 0$ such that, for each $\varepsilon \in (0, \varepsilon^*]$ and each solution x to system $\mathcal{H}_\varepsilon$ with $x(0,0) \in K_0$, there exists some solution ξ to system $\mathcal{H}_{av}$ with $\xi(0,0) \in K_0$ such that x and ξ are (T, J, ρ)-close. □

We also consider the stability properties of the perturbed system under the assumption that the average system has a global asymptotic stability property. Global asymptotic stability for the system $\mathcal{H}_{av}$ in (6) and semi-global asymptotic stability for the system $\mathcal{H}_\varepsilon$ in (1) are defined as the follows.

Definition 4: For the hybrid system $\mathcal{H}_{av}$ in (6), the compact set $\mathcal{A}$ is said to be globally asymptotically stable with respect to $\beta \in \mathcal{KL}$ if

$$|\xi(\tau, j)|_{\mathcal{A}} \le \beta(|\xi(0,0)|_{\mathcal{A}}, \tau + j), \quad \forall\ (\tau, j) \in \mathrm{dom}\ \xi. \quad \square$$

Definition 5: For the hybrid system $\mathcal{H}_\varepsilon$ in (1), the compact set $\mathcal{A}$ is said to be semi-globally practically asymptotically stable (SGP-AS) with respect to $\beta \in \mathcal{KL}$ if, for each compact set K_0 and positive real number $\nu > 0$, there exists

$\varepsilon^* > 0$ such that, for each $\varepsilon \in (0, \varepsilon^*]$, each solution x with $x(0,0) \in K_0 + \varepsilon\mathbb{B}$ satisfies

$$|x(\tau, j)|_{\mathcal{A}} \leq \beta(|x(0,0)|_{\mathcal{A}}, \tau + j) + \nu, \qquad \forall\ (\tau, j) \in \mathrm{dom}\ x.$$

□

Then, the stability properties of system $\mathcal{H}_\varepsilon$ are considered in Theorem 2 under a global asymptotic stability assumption on the average system $\mathcal{H}_{av}$.

Theorem 2: Suppose that the singularly perturbed system $\mathcal{H}_\varepsilon$ in (1) satisfies Assumptions 1 and 2 and the compact set $\mathcal{A}$ is globally asymptotically stable for its average system $\mathcal{H}_{av}$ (6) with respect to $\beta \in \mathcal{KL}$. Then, the compact set $\mathcal{A} \times \Psi$ is SGP-AS for system $\mathcal{H}_\varepsilon$ with respect to β. □

Now we recall a comparison on our main results with classical singular perturbation theory from [19] and present them here for completeness. In classical singular perturbation theory, say [4], [11], [16], the boundary layer system $\mathcal{H}_{bl}$ is assumed to have a globally asymptotically stable equilibrium manifold. Such an assumption is formulated as follows.

Assumption 4: For the boundary layer system $\mathcal{H}_{bl}$ in (4), the function $h : C \to \Psi$ is continuous and for each compact set $K \subset \mathbb{R}^n$, the compact set $\mathcal{M}_K := \{(x, z_{bl}) : x \in C \cap K, z_{bl} = h(x)\}$ is globally asymptotically stable with respect to $\beta \in \mathcal{KL}$.

As shown in [19, Example 1] Assumption 4 is sufficient to guarantee Assumption 2 and the function $x \mapsto F_{av}(x) := f_0(x, h(x))$ is the average of f_0 with respect to ψ_0 for system $\mathcal{H}_\varepsilon$ in (1) based on Assumption 4. Then, its average system is $\mathcal{H}_{av}$ in (6) with

$$F_{av}(x) := f_0(x, h(x))\ \ \forall\ x \in C, \tag{10}$$

and the following two corollaries follow directly from our main results. Note that the results in the following corollaries are more general than [4], [11], [15], [16], where both the closeness of solutions between the actual continuous time system with its average system and the stability properties of the actual system are considered, since the assumption of Lipschitz continuity for the functions f_0 and ψ_0 in [4], [11], [15], [16] are not needed in the current paper.

Corollary 1: Suppose that the singularly perturbed system $\mathcal{H}_\varepsilon$ in (1) satisfies Assumptions 1 and 4 and its average system $\mathcal{H}_{av}$ defined in (6) and (10) is forward pre-complete from a compact set $K_0 \subset \mathbb{R}^n$. Then, for each $\rho > 0$ and any $T, J > 0$ there exists $\varepsilon^* > 0$ such that, for each $\varepsilon \in (0, \varepsilon^*]$ and each solution x to system $\mathcal{H}_\varepsilon$ with $x(0,0) \in K_0$ there exists some solution ξ to system $\mathcal{H}_{av}$ with $\xi(0,0) \in K_0$ such that x and ξ are (T, J, ρ)-close. □

Corollary 2: Suppose that the singularly perturbed system $\mathcal{H}_\varepsilon$ in (1) satisfies Assumption 1 and 4 and the compact set $\mathcal{A}$ is globally asymptotically stable for its average system $\mathcal{H}_{av}$ defined in (6) and (10) with respect to $\beta \in \mathcal{KL}$. Then, the compact set $\mathcal{A} \times \Psi$ is SGP-AS for system $\mathcal{H}_\varepsilon$ with respect to β. □

IV. An Example

In this section, we compare the main results given above with [19] and [13]. In particular, we show that our results apply to a larger class of systems than the singular perturbation results in [19]. Indeed, we show that the average in Def. 1 may exist even in cases when the average used in [19] for the slow dynamics may not exist. Using the same example, we also show that our results give sharper conclusions than [13] in some cases. We first recall the average definition in [19].

Definition 6: For functions $f_0 : C \times \Psi \to \mathbb{R}^n$ and $\psi_0 : C \times \Psi \to \mathbb{R}^m$, the function $f_{av} : C \to \mathbb{R}^n$ is said to be an average of f_0 with respect to ψ_0 on $C \times \Psi$ if for each compact set $K \subset \mathbb{R}^n$ there exists a class-$\mathcal{L}$ function σ_K such that, for each $L > 0$, $x \in C \cap K$ and each function $z_{bl} : [0, L] \mapsto \Psi$ satisfying $\dot{z}_{bl} = \psi_0(x, z_{bl})$, the following holds:

$$\left| \frac{1}{L} \int_0^L [f_0(x, z_{bl}(s)) - f_{av}(x)] ds \right| \leq \sigma_K(L).$$

□

Consider a singularly perturbed hybrid system with states $x \in \mathbb{R}$ and $z \in \mathbb{R}^2$:

$$\left.\begin{array}{l} \dot{x} = \varepsilon f_0(x, z) \\ \dot{z} = \psi_0(z) + \varepsilon\varphi(x, z) \end{array}\right\} \qquad (x, z) \in C \times \Psi \tag{11}$$
$$(x, z)^+ \in G(x, z) \qquad (x, z) \in D \times \Psi$$

where G and D come from (8), $C := \{x : x \geq 0\}$, Ψ is a compact set satisfying $\mathbb{S}^1 \subset \Psi \subset \mathbb{R}^2 \setminus \{0\}$ with $\mathbb{S}^1$ being the unit circle, $f_0(x, z) := (-0.5x + xz_1)$, $\varphi : C \times \Psi \to \Psi$ is locally bounded, and

$$\psi_0(z) := \begin{bmatrix} -\frac{z_1\left(\sqrt{z_1^2+z_2^2}-1\right)^3}{\sqrt{z_1^2+z_2^2}} + z_2(z_1 + z_2 - 1)^2 + z_2\left(1 - \sqrt{z_1^2 + z_2^2}\right)^2 \\ -\frac{z_2\left(\sqrt{z_1^2+z_2^2}-1\right)^3}{\sqrt{z_1^2+z_2^2}} - z_1(z_1 + z_2 - 1)^2 - z_1\left(1 - \sqrt{z_1^2 + z_2^2}\right)^2 \end{bmatrix}. \tag{12}$$

We first show that the boundary layer system for (11) contains equilibria that are neither stable nor attractive. Noting the dynamics $\dot{z} = \psi_0(z)$ in polar coordinates:

$$\begin{array}{l} \dot{\rho} = -(\rho - 1)^3 \\ \dot{\theta} = (\rho\sin(\theta) + \rho\cos(\theta) - 1)^2 + (1 - \rho)^2\ , \end{array} \tag{13}$$

we know that θ is unbounded when the solution of $\dot{z} = \psi_0(z)$ starts off the unit circle $\mathbb{S}^1$, since the first term of righthand of dynamics of θ is positive and second term is not integrable. From Fig. 1, we can get that solutions of $\dot{z} = \psi_0(z)$ that start off the unit circle $\mathbb{S}^1$, tend toward $\mathbb{S}^1$ while rotating in the counterclockwise direction with motion that becomes arbitrarily slow at points arbitrarily close to equilibria $(0, 1)$ or $(1, 0)$. Considering that $\dot{\theta} > 0$ and $\theta(t)$ is unbounded when t grows, we know that the equilibria of the boundary layer system $\mathcal{H}_{bl}$ of system (11) are neither stable nor attractive for any solution of system $\mathcal{H}_{bl}$ that starts in $\Psi \setminus \mathbb{S}^1$.

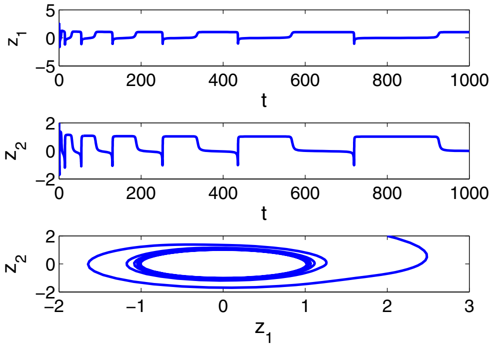

Fig. 1. Trajectories of the solution z of $\dot{z} = \psi_0(z)$.

Letting $K = [-M, M]$ and defining $\Omega := (C \cap K) \times \mathbb{S}^1$, it is clear that Ω is globally asymptotically stable for the boundary layer system with C replaced by $C \cap K$. We next consider if there exists a $\sigma_\Omega \in \mathcal{L}$ such that Assumption 3 holds to invoke Lemma 1. Note that system (13) degenerates into $\dot{\theta} = \phi(\theta)$ when $z(0) \in \mathbb{S}^1$ with $\phi(\theta) := (\sin(\theta) + \cos(\theta) - 1)^2$, and $\theta = 0$ and $\theta = \pi/2$ are the equilibria of $\dot{\theta} = \phi(\theta)$. Considering Def. 6, the average of $f_0(x, z)$ is $-0.5x$ if solutions of system (13) start from the equilibrium $\theta = 0$ and it is $-1.5x$ for the other equilibrium $\theta = \pi/2$. Thus, the single-valued average given in Def. 6, that was employed in [19], can not be used to average $f_0(x, z)$ with respect to ψ_0 of system (11). On the other hand, we next show that the multi-valued average in Def. 1 exists for this example.

Let $\Theta := [0, \pi/2]$ and $\Omega_1 := [-M, M] \times \{\rho = 1, \theta \in \Theta\}$. Note that each solution $z_1(t) = \sin(\theta(t, \theta(0)))$ of system $\mathcal{H}_{bl}$, where $\theta(0)$ is determined by initial condition $z(0)$, picks value in $[0, 1]$ when it starts with $(x, z(0)) \in \Omega_1$. We can get that the function $f_0(x, z)$ of system (11) satisfies $f_0(x, z(s)) \in [-1.5, -0.5]x$ for all solutions (x, z) of system $\mathcal{H}_{bl}$ with $(x, z(0)) \in \Omega_1$. Let $F_{av}(x) := [-1.5, -0.5]x$. Then, for each solution z of system $\mathcal{H}_{bl}$ of (11) starting on the set Ω_1, we can always find $f_{z_{bl}}(s) \in F_{av}(x)$ such that Assumption 3 holds for arbitrary class-$\mathcal{L}$ function σ_Ω.

Noting $\Omega_1 \subset \Omega$, we next consider the solution $z_1 = \sin(\theta(t, \theta(0)))$ of $\mathcal{H}_{bl}$ starting from $\Omega \setminus \Omega_1$. For arbitrary $\delta \in (0, 0.5]$, let

$$\begin{aligned} \vartheta &:= \arcsin(\delta) \in [0, \pi/6] \\ T(\delta) &:= \{t : \theta(t, \vartheta + \pi/2) = 2\pi - \vartheta\}. \end{aligned} \tag{14}$$

For dynamics of $\theta(t)$ that agree with $\dot{\theta} = \phi(\theta)$, noting $\phi : (\pi/2, 2\pi) \to [0, (\sqrt{2}+1)^2]$ is symmetric about $\theta = 5\pi/4$, we can only consider its property on $(\pi/2, 5\pi/4]$. Since $\frac{d\phi(\theta)}{d\theta} > 0$ for $\theta \in (\pi/2, 5\pi/4]$, we have that $\phi(\theta)$ is strictly increasing and then $\dot{\theta} \geq \phi(\vartheta)$ for all $\theta \in [\pi/2 + \vartheta, 5\pi/4]$. With $\phi(\vartheta) = 2(1 - \delta)(1 - \sqrt{1 - \delta^2})$, we have $T(\delta) \leq \frac{3\pi/4 - \arcsin(\delta)}{(1-\delta)(1-\sqrt{1-\delta^2})} := \tilde{T}(\delta)$. Noting that $\tilde{T} : (0, 0.5] \to \mathbb{R}_{>0}$ is continuous, strictly decreasing, bounded away from zero and $\lim_{\delta \to 0} \tilde{T}(\delta) = \infty$, there exists a $\alpha \in \mathcal{K}_\infty$ such that $\alpha(1/\delta) = \tilde{T}(\delta)$ for $\delta \in (0, 0.5]$.

For each solution of $\dot{\theta} = \phi(\theta)$ with $\theta(0) \in (\pi/2, 2\pi)$, let $T_1 := \{t \geq 0 : \theta(t) = \pi/2 + \vartheta\}$ and $T_1 := 0$ if $\theta(t) \geq \pi/2 + \vartheta$ for all $t \geq 0$, let $T_2 \geq T_1 := \{t \geq 0 : \theta(t) = 2\pi - \vartheta\}$ and $T_2 := T_1$ if $\theta(t) \geq 2\pi - \vartheta$ for all $t \geq 0$. With noting $T_2 - T_1 \leq T_\delta$ in (14), we have for each $L > 0$:

$$\begin{aligned} &\frac{1}{L} \int_0^L |\sin(\theta(s))| ds \\ &\quad \leq \frac{1}{L} \int_0^{T_1} 1 ds + \int_{T_1}^{T_2} 1 ds + \int_{T_2}^{L} \delta ds \\ &\quad \leq \frac{T_1}{L} + \frac{\tilde{T}(\delta)}{L} + \delta = \frac{T_1}{L} + \frac{\alpha(1/\delta)}{L} + \delta. \end{aligned} \tag{15}$$

Noting that (15) holds for arbitrary $\delta \in (0, 1/2]$, it holds for

$$\delta = \min \left\{ 0.5, \frac{1}{\alpha^{-1}\left(\sqrt{L}\right)} \right\}. \tag{16}$$

Let $c := \frac{T_1}{L}$ and note that $c \in [0, 1]$ from the definition of L and T_1 in (15). Then, we have for each $M > 0$, $L > 0$, $x \in [-M, M]$ and solution $z_1 : [0, L] \mapsto \Psi$ of system $\mathcal{H}_{bl}$ satisfying $(x, z(0)) \in (\Omega \setminus \Omega_1)$, the function $f_{z_{bl}}(s) = -(0.5 + c)x \in F_{av}(x)$ satisfies:

$$\begin{aligned} &\left| \frac{1}{L} \int_0^L [-(0.5x + x z_1(s)) - f_{z_{bl}}(s)] ds \right| \\ &= \left| \frac{1}{L} \int_0^L [-(0.5x + x z_1(s)) + (0.5 - c)x] ds \right| \\ &\leq |x| \left(\frac{1}{L} \int_0^L |\sin(\theta(s))| ds - c \right), \\ &\leq M \left(\frac{\max\left\{2, \alpha\left(\sqrt{L}\right)\right\}}{L} + \min\left(0.5, \frac{1}{\alpha^{-1}\left(\sqrt{L}\right)}\right)\right), \\ &:= \sigma_\Omega(L). \end{aligned}$$

Noting that σ_Ω is of class-$\mathcal{L}$, we know that Assumption 3 holds for system (11). Invoking Lemma 1 and letting the jump mapping of the averaged system come from (9) in Example 1, we have that the average of system (11) is

$$\begin{aligned} \xi' &\in [-1.5, -0.5]\xi & \xi &\in C \\ \xi^+ &\in -\gamma\xi + [c_3, c_4] & \xi &\in D\,. \end{aligned} \tag{17}$$

We next compare our results with [13], which considers a class of hybrid control systems singularly perturbed by fast but continuous actuators, where a reduced system that omits the actuator dynamics is used in analysis of stability properties of the actual system. To extend the classical singular perturbation theory to the hybrid setting, the equilibrium manifold in Assumption 4 is replaced by a set-valued mapping $H : \mathbb{R}^n \rightrightarrows \mathbb{R}^m$ in [13]. The closed-loop of the hybrid control system considered in [13] is formed as

$$\mathrm{diag}(I_n, \varepsilon I_m)\ y' \in F_1(y) \qquad y \in C \times \Psi$$
$$y^+ \in G_1(y) \qquad y \in D \times \Psi\ , \qquad (18)$$

where $y := (x, z) \in \mathbb{R}^n \times \mathbb{R}^m$, I_n and I_m respectively denote the $n \times n$ and $m \times m$ identity matrices, $F_1 : \mathbb{R}^n \times \mathbb{R}^m \rightrightarrows \mathbb{R}^n \times \mathbb{R}^m$ and $G_1 : \mathbb{R}^n \times \mathbb{R}^m \rightrightarrows \mathbb{R}^n \times \mathbb{R}^m$. In [13], the reduced system $\mathcal{H}_r := \{F_r, G_r, C, D\}$ of the perturbed system (18) is defined as (6) with the set-valued mapping H and

$$F_r(x) := \overline{con}\{v_1 \in \mathbb{R}^n : \qquad (19)$$
$$(v_1, v_2) \in F_1(x, z), z \in H(x), v_2 \in \mathbb{R}^m\},$$
$$G_r(x) := \{v_1 \in \mathbb{R}^n : (v_1, v_2) \in G_1(x, z), (z, v_2) \in \Psi \times \Psi\}.$$

Note that G_r defined above is a projection of G_1 to the subspace of the slow state x, which is same as the definition of G_{av} in (7) for the average system, except that the fast states z are constrained to the compact set Ψ in (19) and it is not required for our main results.

Consider the singularly perturbed hybrid system (11). From the definition of the reduced system in (19) given in [13], noting the fact that the boundary layer system of system (11) converges to $\mathbb{S}^1$ and letting c_3 and c_4 come from Example 1, the reduced system is

$$\xi' \in [-1.5, 0.5]\xi \qquad \xi \in C$$
$$\xi^+ \in -\gamma\xi + [c_3, c_4] \qquad \xi \in D\ . \qquad (20)$$

Note that there are solutions for the reduced system (20) that exponentially grow unbounded.

On the other hand, for the average system (17) where $C = \{\xi : \xi \geq 0\}$, the jump mapping $G_{av} := -\gamma\xi + [c_3, c_4]$ make all solutions starting from the set D go back to the flow set C. We have that the origin is globally exponentially stable for the average system. Then, we can analyze the stability properties of system (11) with global asymptotic stability of its average system (17) using Theorem 2, but we cannot draw this stability conclusion from the reduced system (20) that was used in [13].

V. CONCLUSIONS

We extended the analysis results in [19] to consider a class of hybrid dynamical systems with the singular perturbation theory and the averaging method. We showed that if there exists a well defined average for the actual perturbed hybrid system, the slow solutions of the actual system on compact time domains are arbitrarily close to the solution of the average system that approximates the slow dynamics of the actual system for arbitrarily small values of the singular perturbation parameter. We also showed that the global asymptotic stability of a compact set for the average system implies that the set is semi-globally practically asymptotically stable for the actual perturbed system. Through an example, we showed that the average definition introduced in the present paper allows for a more general class of hybrid systems. Using the same example, we also showed that our results give sharper conclusions than [13] in some cases. The continuity assumption on the slow vector field can be relaxed to an assumption that small perturbations to the solutions of the boundary layer system lead to small changes in the integral that defines the average vector field. Also, one can consider set-valued boundary layer dynamics. These generalizations are useful for recovering the singular perturbation results in [13]. We have not considered these generalization here for ease of exposition.

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