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Network Coherence in Fractal Graphs

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***Abstract*— We study distributed consensus algorithms in fractal networks where agents are subject to external disturbances. We characterize the coherence of these networks in terms of an H_2 norm of the system that captures how closely agents track the consensus value. We show that, in first-order systems, the coherence measure is closely related to the global mean first passage time of a simple random walk. We can therefore draw directly from the literature on random walks in fractal graphs to derive asymptotic expressions for the coherence in terms of the network size and dimension. We then show how techniques employed in the random walks setting can be extended to analyze the coherence of second-order consensus algorithms in fractal graphs with tree-like structures, and we present asymptotic results for these second-order systems.**

I. INTRODUCTION

Distributed consensus algorithms are important tools in the domains of multi-agent systems and the vehicle platooning problem [1], [2], [3] as a means by which agents can reach and maintain agreement on quantities such as velocity, heading, and inter-vehicle spacing using only local communication. In these settings, in addition to verifying the correctness of distributed consensus algorithms, it is also important to consider how robust these algorithms are to external disturbances.

Several recent works have studied the robustness of distributed consensus algorithms for systems with first-order and second-order dynamics in terms of an H_2 norm. This norm is a quantification of the *network coherence*; it captures how well a network can maintain its formation in the face of stochastic external disturbances. For the first-order case, it has been shown that the H_2 norm can be characterized by the trace of the pseudo-inverse of the Laplacian matrix [4], [5], [6], [7]. This value has important meaning not just in consensus systems, but in electrical networks [8], [9], random walks [10], and molecular connectivity [11]. For systems with second-order dynamics, the H_2 norm is also determined by the spectrum of the Laplacian. However, we are unaware of any analogous concepts in other fields.

Our recent work [5], [7] gave scalings for the H_2 norm of first and second order consensus algorithms in torus and lattice networks in terms of the number of nodes and the network dimension. These results showed that there is a marked difference in coherence between first-order and second-order systems and also between networks of different dimensions. For example, in a one-dimensional ring network with first-order dynamics, the per-node variance of the deviation from consensus scales linearly in the number nodes, while in the two-dimensional torus, the per-node variance scales logarithmically in the number of nodes.

In this work, we explore the robustness of consensus algorithms in networks with dimensions between one and two. Namely, we study the coherence of consensus algorithms in self-similar, fractal graphs. For first-order systems, we are able to draw directly from literature on random walks on fractal networks to show that, in a network with N nodes, the per-node variance scales as N^{1/d_f} where d_f is the *fractal dimension* of the network. For networks with second-order consensus dynamics, we are unaware of any such analysis for general fractal graphs. For fractals with tree-like structures, we show how techniques used for the analysis of random walks [12], [13] can be extended to analyze the coherence of second-order consensus systems, and we present asymptotic results for the per-node variance in terms of the network size and spectral dimension.

The remainder of this paper is organized as follows. In Section II, we present the system models for first-order and second-order consensus dynamics and give a formal definition of network coherence for each setting. We also present several properties from other domains that are mathematically similar to network coherence. In Section III, we describe the fractal graph models, and in Section IV we present analytical results on the scalings of the first-order and second-order coherence measures in fractal graphs. Finally, we conclude in Section V.

II. NETWORK COHERENCE

We consider simple, first-order and second-order consensus algorithms over an undirected, connected network modeled by a graph G with N nodes and M edges. Our objective is to quantify the robustness of these algorithms to stochastic perturbations at the nodes using a quantity that we call *network coherence*. In the first-order setting, each node has a single state, and in the second order setting, each node has two states corresponding to position and velocity. This difference leads to different expressions for coherence and therefore different asymptotic scalings, as we show in the sequel.

A. Coherence in Networks with First-Order Dynamics

In the first-order consensus problem, the state of the system, the value at each node, is a vector $x \in \mathbb{R}^N$. Each node state is subject to stochastic disturbances, and the objective is for nodes to maintain consensus at the average of their states.

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Let L be the Laplacian of the graph, defined by $L := D - A$, where D is diagonal matrix of node degrees and A is the adjacency matrix of G. The dynamics of the system are defined as follows,

$$\dot{x} = -Lx + w, \tag{1}$$

where w is an N-vector of zero-mean, white noise processes.

In the absence of the noise processes, the system converges asymptotically to consensus at the the average of the initial states. With the additive noise term, the nodes do not converge to consensus, but instead, node values fluctuate around the average of the *current* node states. Network coherence captures the variance of these fluctuations.

Definition 2.1: The *first-order network coherence* is defined as the mean, steady-state variance of the deviation from the average of all node values,

$$H^{(1)} := \frac{1}{N}\sum_{i=1}^{N}\lim_{t\to\infty}\mathbf{E}\left\{x_i(t) - \frac{1}{N}\sum_{j=1}^{N}x_j(t)\right\}.$$

Let P be the projection operator $P = I - \frac{1}{N}\mathbf{1}\mathbf{1}^*$, where $\mathbf{1}$ is the N-vector of all ones. We define the output of the system (1) to be

$$y = Px. \tag{2}$$

$H^{(1)}$ relates to the H_2 norm of the system defined by (1) and (2) as follows,

$$H^{(1)} = \frac{1}{N}\mathbf{tr}\left(\int_0^\infty e^{-L^*t}Pe^{-Lt}dt\right).$$

It has been shown that $H^{(1)}$ is s completely determined by the spectrum of L [4], [5], [7], [6]. Let the eigenvalues of L be $0 = \lambda_1 < \lambda_2 \leq \ldots \leq \lambda_N$. Then, the network coherence of the first-order system is given by

$$H^{(1)} = \frac{1}{2N}\sum_{i=2}^{N}\frac{1}{\lambda_i}. \tag{3}$$

B. Coherence in Networks with Second-Order Dynamics

In the vehicle formation problem, there are N vehicles, each with a position and a velocity. The objective is for each vehicle to travel at a constant target velocity while maintaining a fixed, pre-specified distance between itself and each of its neighbors. The state of the second-order system consists of a position vector x and a velocity vector v. The states are measured relative to the target velocity $\overline{v}$ and position $\overline{x}(t)$. The system dynamics are given by

$$\begin{bmatrix}\dot{x}\\ \dot{v}\end{bmatrix} = \begin{bmatrix}0 & I\\ -L & -L\end{bmatrix}\begin{bmatrix}x\\ v\end{bmatrix} + \begin{bmatrix}0\\ I\end{bmatrix}w, \tag{4}$$

where w is a $2N$-vector of zero-mean white noise processes. Note that the stochastic disturbances enter only in the velocity terms.

The network coherence of the second-order system is defined in terms of the the node positions only, and it captures how closely the vehicle formation follows the target trajectory in steady-state.

Definition 2.2: The *second-order network coherence* is the mean, steady-state variance of the deviation of each vehicle's position error from the average of all vehicle position errors.

Let the output for the system (4) be

$$y = \begin{bmatrix}P & 0\end{bmatrix}\begin{bmatrix}x\\ v\end{bmatrix}, \tag{5}$$

As in the consensus case, the variance is given by the H_2 norm of the system defined by (4) and (5). This value is also completely determined by the eigenvalues of the Laplacian matrix [5], [7], and is given by

$$H^{(2)} = \frac{1}{2N}\sum_{i=2}^{N}\frac{1}{\lambda_i^2}.$$

C. Related Concepts

The eigenvalues of the Laplacian are linked to the topology of the network, and therefore, it is not surprising that these eigenvalues play a role in many graph problems. In fact, the sum

$$S := \sum_{i=2}^{N}\frac{1}{\lambda_i}$$

that appears in the expression for first order coherence (3) is an important quantity, not just in the study of consensus algorithms, but in several other fields. We can leverage work in these fields to develop analytic expressions for network coherence for different graph topologies. We first review these related properties, and in the following sections, we present analysis for the asymptotic behavior of network coherence measures in several classes of self-similar networks.

1) Effective Resistance in an Electrical Network: The graph G represents an electrical network where each edge is a unit resistor. The *resistance distance* r_{ij} between two nodes i and j is the potential distance between them when a one ampere current source is connected from node j to node i. The *total effective resistance* of the network, also called the *Kirchoff index* [8], [9], is the sum of the resistance distances over all pairs of nodes in the graph,

$$R := \sum_{\substack{i,j\in V\\ i\neq j}} r_{ij}.$$

It has been shown that the total effective resistance depends on the spectrum of the Laplacian matrix as follows (for proof, see [14]),

$$R = 2N\sum_{i=2}^{N}\frac{1}{\lambda_i}.$$

2) Global Mean First Passage Time of a Random Walk: In a simple random walk on a graph G, the probability of moving from one state to the next is $\frac{1}{\Delta}$ where Δ is the out-degree of the current state. The *first passage time* f_{ij} (also called the *hitting time*) is the average number of steps it takes for a random walk starting at node i to reach node j for the first time. The *global mean first passage time* is the average first passage time over all pairs of nodes

$$F = \frac{1}{N(N-1)} \sum_{\substack{i,j \in V \\ i \neq j}} f_{ij}.$$

It has been shown that, for a connected graph, the the following relationship exists between the mean first passage time between nodes i and j and the resistance distance [10],

$$f_{ij} + f_{ji} = 2M r_{ij},$$

where M is the number of edges in the graph. The global mean first passage time is therefore related to the total effective resistance as follows,

$$F = \frac{M}{N(N-1)} \sum_{\substack{i,j \in V \\ i \neq j}} r_{ij} = \frac{2M}{N-1} \sum_{i=2}^{N} \frac{1}{\lambda_i}.$$

If the graph is a tree, then $M = N-1$, and the global mean first passage time is simply

$$F = 2 \sum_{i=2}^{N} \frac{1}{\lambda_i}.$$

3) Quasi-Wiener Index: The Wiener index was proposed by Harry Wiener in 1947 as a measure of molecular connectivity [15]. The graph G represents a molecule, where nodes are atoms and edges are chemical bonds. The distance between two atoms is the length (number of edges) of the shortest path between them. The Wiener index is the sum of the distances between all pairs of non-hydrogen atoms. If the molecular graph is acyclic, this value is exactly [11],

$$W = \sum_{i=2}^{N} \frac{1}{\lambda_i},$$

where N is the number of non-hydrogen atoms. If the graph contains cycles, the right-hand side of the equation is no longer equal to the sum of lengths of the shortest paths. However, this quantity is still utilized in mathematical chemistry and is called the quasi-Wiener index [16].

III. SELF-SIMILAR GRAPHS

Ideally, one would like to find an analytical expression for the network coherence values, but in general, it is a difficult problem to analyze the spectrum of the Laplacian. However, for graphs with special structure, one can sometimes find a closed form for either these eigenvalues, or the sum of their inverses. For example, for d-dimensional torus and lattice networks, it has been shown that the network coherence of systems with first-order dynamics scales as

$$H^{(1)} \approx \begin{cases} N & d = 1 \\ \log(N) & d = 2 \\ 1 & d \geq 3. \end{cases}$$

This result has appeared in the contexts of global mean first passage time [17] and effective resistance [18]. With respect to consensus algorithms, our recent works [5], [7], also present this result, and in addition, show that this is the best achievable asymptotic behavior for any local controller. This same work gives the asymptotic scalings of network coherence for consensus algorithms with second-order dynamics,

$$H^{(2)} \approx \begin{cases} N^3 & d = 1 \\ N & d = 2 \\ N^{1/3} & d = 3 \\ \log(N) & d = 4 \\ 1 & d \geq 5. \end{cases}$$

A natural question that arises is how to analyze coherence scalings for other types of graphs. In order to study the relationship between coherence and the size of the network, we require that the network can grow in a prescribed manner. A good candidate is the class of self-similar graphs. Informally, a self-similar graph is one which exhibits the same structure at every scale.

Self-similar graphs can be characterized by the number of nodes and the graph dimension. However, there is not a single agreed-upon notion of dimension that encapsulates all properties of the graph. In this work, we consider the following two dimension definitions.

Definition 3.1: Let $N(r)$ be the minimum number of balls of radius r required to cover the graph (where distance is defined by the length of the shortest path). The *fractal dimension* (also called the *Hausdorff dimension*) of the graph G is

$$d_f := -\lim_{\epsilon \to 0} \frac{\log(N(\epsilon))}{\log(\epsilon)},$$

where $\log(\cdot)$ denotes the natural logarithm function.

Definition 3.2: Let $\rho(x)$ be the eigenvalue counting function of L, *i.e.* $\rho(x)$ is the number of eigenvalues of L that have magnitude less than or equal to x. The *spectral dimension* of the graph G is [19]

$$d_s := 2 \lim_{x \to \infty} \frac{\log(\rho(x))}{\log(x)}.$$

Torus and lattice graphs are both self-similar graphs, and their fractal and spectral dimensions are equivalent to the natural dimension definition, *e.g.* a 2-dimensional torus has $d_f = d_s = 2$. In this work, we consider two classes of fractal graphs, tree-like fractals and Viscek fractals These graphs have spectral dimension between 1 and 2, and their fractal and spectral dimensions are related as follows [20],

$$d_s = (2d_f)/(d_f + 1).$$

We describe the construction of these graphs below.

Tree-Like Fractals: We employ the model for tree-like fractals that was presented in [12]. Each family of graphs is parameterized by a positive integer k. The graphs are constructed in an iterative manner, and each iteration yields a new graph generation. The process begins with a graph that consists of two nodes connected by a single edge. This graph corresponds to the generation $g = 0$. Given a graph of generation g, the graph of generation $g+1$ is formed by replacing each edge with a path of length 2, *i.e* for each

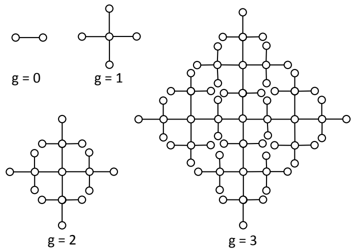

Fig. 1. First four generations of the tree-like fractal for $k = 2$.

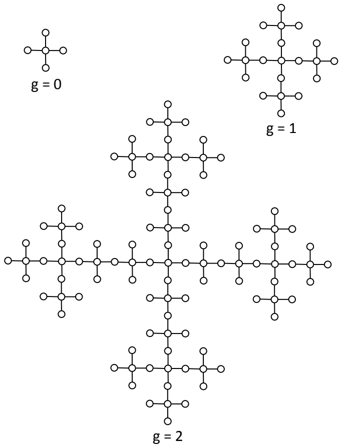

Fig. 2. First three generations of the Vicsek fractal for $f = 4$.

pair of nodes i and j connected by an edge (i, j) in G_g, we replace (i, j) by two edges (i, k) and (k, j) where k is a new node (not existing in graph g). k additional nodes are then attached to every new node. The process is illustrated in Fig. 1 for $k = 2$. This model encompasses several well-know fractal graphs, including the T-graph (k = 2) [21] and the Peano basin fractal (k = 1) [22]. These graphs have fractal dimension $d_f = \log(k + 2)/\log(2)$ and spectral dimension $d_s = 2\log(k + 2)/\log(2(k + 2))$.

Vicsek Fractals: The second class of graphs studied in this work is the generalized Vicsek fractal [23], [24]. This graph is also constructed in an iterative manner. The generation 0 graph is a star graph with $f + 1$ nodes. The graph in generation $g+1$ is generated from generation g by making f copies of the generation g graph and arranging them around the generation g graph. These copies are connected to the center graph by adding edges from the f corners of the center graph, each linking to a corner of a copy. An illustration of this process is shown in Fig. 2 for $f = 4$. The generalized Vicsek fractal has fractal dimension $d_f = \log(f+1)/\log(3)$ and spectral dimension $d_s = 2\log(f + 1)/\log(3f + 3)$.

IV. COHERENCE IN FRACTAL NETWORKS

In this section, we explore the coherence of fractal graphs described in the previous section. These graphs have been investigated in the context of random walks [24], [13], [12], and we can directly apply results in this area to formulate expressions for the coherence in self-similar fractal graphs with first-order dynamics. We note that, for the tree-like and Vicsek fractal graphs, $M \sim N$ and therefore, $H^{(1)} \sim \frac{1}{N}F$.

The first order-coherence for a self-similar (tree-like) fractal graph with spectral dimension d_s and fractal dimension d_f is $H^{(1)} \sim N^{(2/d_s - 1} = N^{1/d_f}$. A list of example graphs with their spectral dimension, fractal dimension, and scaling for $H^{(1)}$ is given in Table I. We note that the T-Graph has fractal dimension 2, the same as a 2-dimensional torus. However, in the torus, first order coherence scales as $\log(N)$ as opposed to $\sqrt{N}$ for the T-Graph. This illustrates that it is the spectral dimension, not the fractal dimension, that determines the asymptotic behavior of network coherence.

As we unaware of any analogs in other domains to the the coherence for second order systems, we cannot look to existing works for solutions to this problem. In remainder of this section, we show how techniques used to analyze random walks in tree-like fractal networks can be extended to analyze the coherence second-order systems. With these these extensions, we derive analytical expressions for the network coherence in terms of the network size and dimension.

Coherence in Tree-Like Fractals: In their recent work, Lin et al. [12] give an analytic expression for the global mean first passage time in tree-like fractals, derived from a recursive expression for the characteristic polynomial of the associated Laplacian matrix. We briefly review their technique and then show how to extend this analysis to determine the coherence of systems with second-order dynamics

Let P_g denote the characteristic polynomial for the Laplacian of the graph of generation g (denoted L_g),

$$P_g(\lambda) = \det(L_g - \lambda I_g).$$

The roots of P_g are the eigenvalues of L_g. As we are only interested in the non-zero eigenvalues of L_g, we instead consider the polynomial

$$P'_g(\lambda) = \frac{1}{\lambda} P_g(\lambda),$$

and note that

$$S_g = \sum_{i=2}^{N} \frac{1}{\lambda_i(L_g)} = \sum_{i=1}^{N-1} \frac{1}{\lambda'_i(L_g)},$$

where $\lambda'_i(L_g)$, $i = 1 \ldots N-1$ are the $N-1$ roots of $P'(g)$. Let $p_g^{(i)}$ denotes the coefficient of the term λ^i, such that

$$P'_g(\lambda) = \sum_{i=0}^{N-1} p_g^{(i)} \lambda^i = p_g^{N-1} \prod_{i=1}^{N-1} (\lambda - \lambda'_i(L_g)). \quad (6)$$

Applying Viète's formulae, the following equality is established,

$$\sum_{i=1}^{N_g} \frac{1}{\lambda_i(g)} = -\frac{p_g^{(1)}}{p_g^{(0)}}. \quad (7)$$

Therefore, to find the sum S_g, one only needs to derive the coefficients $p_g^{(1)}$ and $p_g^{(0)}$.

To determine these coefficients, one first derives a recursion for P'_g. Let Q_g be the characteristic polynomial

[1]See [12]. [2] See [12]. [3] See [13], [24].

TABLE I

EXAMPLES OF SELF-SIMILAR FRACTAL GRAPHS AND THEIR DIMENSIONS AND COHERENCE MEASURES.

Network	**Fractal Dimension**	**Spectral Dimension**	$H_{(1)}$
Peano Basin Fractal[1] (tree-like fractal with $k=1$)	$\frac{\log(3)}{\log(2)}$	$\frac{2\log(3)}{\log(6)}$	$N^{\log(2)/\log(3)}$
T-Graph[2] (tree-like fractal with $k=2$)	2	2/3	$\sqrt{N}$
Generalized Vicesk Fractal V_f [3]	$\frac{\log(f+1)}{\log(3)}$	$\frac{2\log(f+1)}{\log(3f+3)}$	$N^{\log(3)/\log(f+1)}$

of the $(N-1)\times(N-1)$ submatrix of L_g formed by removing a single column and row corresponding to an outermost node. Let R_g be the characteristic polynomial of the $(N-2)\times(N-2)$ submatrix of L_g formed by removing columns and rows corresponding to two outermost nodes. The following equation captures the relationship between the characteristic polynomials of the graphs of generation $g+1$ and g.

$$\begin{aligned}P'_{g+1}(\lambda) =&(k+2)[Q_g(\lambda)]^{k+1}P'_g(\lambda)+(k+1)\left[Q_g(\lambda)\right]^{k+2}\\ Q_{g+1}(\lambda) =& \left[Q_g(\lambda)\right]^{k+2}+(k+1)\lambda R_g(\lambda)\left[Q_g(\lambda)\right]^{k+1}\\ &+(k+1)\lambda R_g(\lambda)[Q_g(\lambda)]^kP'_g(\lambda)\\ R_{g+1}(\lambda) =&2R_g(\lambda)[Q_g(\lambda)]^{k+1}+(m+1)[R_g(\lambda)]^2[Q_g(\lambda)]^m\\ &+k\lambda\left[R_g(\lambda)\right]^2\left[Q_g(\lambda)\right]^{k-1}P'_g(\lambda).\end{aligned}$$

From these equations, Lin et al. [12] derive recursions for the coefficients $p_g^{(0)}$, $q_g^{(0)}$, and $r_g^{(0)}$, and the coefficients $p_g^{(1)}$ and $q_g^{(1)}$. They then solve these recursions arriving at the following expression for the asymptotic order of the global mean first passage time,

$$F_g \sim N_g^{1+\log(2)/\log(k+2)} = N_g^{2/d_s}.$$

where d_s is the spectral dimension of the network. Therefore, the first-order coherence for tree-like fractals is

$$H_g^{(1)} = \frac{1}{N_g}\left(-\frac{p_g^{(1)}}{p_g^{(0)}}\right) \sim N_g^{\frac{\log(2)}{\log(k+2)}} = N_g^{(2/d_s)-1} = N_g^{1/d_f}.$$

Using equation (6) and Viète's formulae, we can also express $H^{(2)}$ in terms of the coefficients of $P'(\lambda)$ as follows,

$$\sum_{i=1}^{N_g}\frac{1}{\left(\lambda_i(g)\right)^2} = \left(\frac{p_g^{(1)}}{p_g^{(0)}}\right)^2+\frac{p_g^{(2)}}{p_g^{(1)}} \qquad (8)$$

In order to solve for $p_g^{(2)}$, we need the recursion equations for $p_g^{(2)}, q_g^{(2)}$, and $r_g^{(1)}$, which are as follows,

$$\begin{aligned}r_{g+1}^{(1)} =&2[q_g^{(0)}]^{k+1}r_g^{(1)}+2(k+1)r_g^{(0)}[q_g^{(0)}]^kq_g^{(1)}\\ &+(k+1)[r_g^{(0)}]^2[q_g^{(0)}]^k+k[r_g^{(0)}]^2[q_g^{(0)}]^{k-1}p_g^{(0)}\\ q_{g+1}^{(2)} =&(k+2)q_g^{(2)}+(k+2)(k+1)q_g^{(1)}+(k+1)r_g^{(1)}\\ &+(k+1)^2r_g^{(0)}q_g^{(1)}+(k+1)p_g^{(0)}r_g^{(1)}\\ &+k(k+1)r_g^{(0)}p_g^{(0)}q_g^{(1)}+r_g^{(0)}p_g^{(1)}\\ p_{g+1}^{(2)} =&(k+2)[q_g^{(0)}]^{k+1}p_g^{(2)}\\ &+(k+1)(k+2)[q_g^{(0)}]^kq_g^{(2)}p_g^{(0)}\\ &+(k+2)(k+1)m[q_g^{(0)}]^{k-1}[q_g^{(1)}]^2p_g^{(0)}\\ &+(k+1)(k+2)[q_g^{(0)}]^{k+1}q_g^{(2)}\\ &+(k+2)(k+1)m[q_g^{(0)}]^k[q_g^{(1)}]^2.\end{aligned}$$

Again, we are only interested in the asymptotic behavior of $H_{(2)}$, and so we only need to derive the highest order term of $p_g^{(2)}$. The details of this derivation are straightforward and are omitted for brevity. The order of $p_g^{(2)}$ is

$$p_g^{(2)} \sim 2^{2g}\left(k+2\right)^{3g}.$$

Combining this result with the equation for $p_g^{(1)}$, we arrive at the following theorem.

Theorem 4.1: For the class of tree-like fractals with second-order noisy consensus dynamics as defined by (4), the mean steady-state variance of the deviation from average is

$$H_g^{(2)} \sim N_g^{1+\log(2)/\log(k+2)} = N_g^{(4/d_s)-1} = N_g^{1+(2/d_f)},$$

where N_g is the number of nodes in the generation g graph.

Coherence in Vicsek Fractals: It has been shown that the following relationship exists between the eigenvalues of the generation $g+1$ graph and the eigenvalues of the generation g graph [24],

$$\lambda_{g+1}^{(i)}\left(\lambda_{g+1}^{(i)}-3\right)\left(\lambda_{g+1}^{(i)}-f-1\right) = \lambda_g^{(i)} \qquad (9)$$

Each eigenvalue from graph of generation g generates three eigenvalues for graph of generation $g+1$.

One can use the above expression to solve for the eigenvalues of the Vicsek fractal explicitly, but, it is not clear how to derive a closed form expression for each $\lambda_g^{(i)}$. In their recent work, Zhang et al. [13], show that, one can obtain a closed expression for S_g without solving for the individual eigenvalues. We briefly review this technique here.

Let $\lambda_{g+1}^{(i,1)}$, $\lambda_{g+1}^{(i,2)}$, and $\lambda_{g+1}^{(i,3)}$ be the three roots of equation (9), Then, applying Viète's formulas, one can obtain the following recursive expression for the sum of the inverses of these roots

$$\frac{1}{\lambda_{g+1}^{(i,1)}}+\frac{1}{\lambda_{g+1}^{(i,2)}}+\frac{1}{\lambda_{g+1}^{(i,3)}} = \frac{3(f+1)}{\lambda_g^{(i)}}. \qquad (10)$$

Summing over all eigenvalues, and taking special care with degenerate eigenvalues, one obtains the following

$$S_g = \frac{(f-2)(f+1)^{g-1}(3^g-1)}{2} + \frac{3^g(f+1)^g-1}{3f+2}.$$

The number of nodes in the graph of generation g is $N_g = (f+1)^g$, and $3^g = N_g^{\log(3)/\log(f+1)}$. Therefore, the global mean first passage time is

$$F_g \sim N_g^{1+\log(3)/log(f+1)} = N_g^{2/d_s}.$$

where d_s is the spectral dimension of the network. The network coherence for Vicsek fractals with first-order consensus dynamics immediately follows,

$$H_g^{(1)} \sim N_g^{\log(3)/\log(f+1)} = N_g^{(2/d_s)-1} = N_g^{1/d_f}.$$

As in the work by Zhang et al. [13], we apply Viète's formulas to equation (9), but this time to obtain a sum over the squares of the inverses of the Laplacian eigenvalues,

$$\frac{1}{\left[\lambda_{g+1}^{(i,1)}\right]^2} + \frac{1}{\left[\lambda_{g+1}^{(i,1)}\right]^2} + \frac{1}{\left[\lambda_{g+1}^{(i,1)}\right]^2} = \frac{(3(f+1))^2}{\left[\lambda_g^{(i)}\right]^2} - \frac{2(f+4)}{\lambda_g^{(i)}}.$$

Using this result to sum over all eigenvalues yields a term of order

$$3^{2g}(f+1)^{2g} = N_g^{2+2\log(3)/\log(f+1)}.$$

From this result, we obtain at the following theorem for $H^{(2)}$.

Theorem 4.2: For generalized Vicsek fractal graphs with second-order noisy consensus dynamics as defined by (4), the mean steady-state variance of the deviation from average is

$$H_g^{(2)} \sim N_g^{1+2\log(3)/\log(f+1)} = N_g^{(4/d_s)-1} = N_g^{1+(2/d_f)},$$

where N_g is the number of nodes in the generation g graph.

V. Conclusion

We have investigated the per node variance of the deviation from consensus as a measure of network coherence in fractal networks with first-order and second-order noisy consensus dynamics. We have shown that, in first-order systems, the coherence measure is closely related to the effective resistance of an electrical network, the global mean first passage time of a simple random walk, and the quasi-Wiener index for chemical networks. Drawing directly from literature on random walks in fractal graphs, we presented asymptotic expressions for the first-order coherence in terms of the network size and dimension. We then extended these results to second-order consensus algorithms in fractal graphs with tree-like structures, and we have shown that the second-order coherence grows as $N^{1+(2/d_f)}$, where d_f is the fractal dimension of the network. A question for future study is whether this result can be generalized to other classes of fractal graphs.

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