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# Robust Adaptive Geometric Tracking Controls on $\mathsf{SO}(3)$ with an Application to the Attitude Dynamics of a Quadrotor UAV

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*Abstract*— **This paper provides new results for a robust adaptive tracking control of the attitude dynamics of a rigid body. Both of the attitude dynamics and the proposed control system are globally expressed on the special orthogonal group, to avoid complexities and ambiguities associated with other attitude representations such as Euler angles or quaternions. By designing an adaptive law for the inertia matrix of a rigid body, the proposed control system can asymptotically follow an attitude command without the knowledge of the inertia matrix, and it is extended to guarantee boundedness of tracking errors in the presence of unstructured disturbances. These are illustrated by numerical examples and experiments for the attitude dynamics of a quadrotor UAV.**

## I. Introduction

The attitude control problem has been extensively studied under various assumptions (see, for example, [1], [2], [3]). One of the distinct features of the attitude dynamics is that its configuration manifold is not linear: it evolves on a nonlinear manifold, referred as the special orthogonal group, $\mathsf{SO}(3)$. This yields important and unique properties that cannot be observed from dynamic systems evolving on a linear space. For example, it has been shown that there exists no continuous feedback control system that asymptotically stabilizes an attitude globally on $\mathsf{SO}(3)$ [4].

Geometric control is concerned with the development of control systems for dynamic systems evolving on nonlinear manifolds that cannot be globally identified with Euclidean spaces [5], [6]. By characterizing geometric properties of nonlinear manifolds intrinsically, geometric control techniques completely avoids singularities and ambiguities that are associated with local coordinates or improper characterizations of a configuration manifold. This approach has been applied to fully actuated rigid body dynamics on Lie groups to achieve almost global asymptotic stability [6], [7], [8], [9].

In this paper, we develop a geometric adaptive controller on $\mathsf{SO}(3)$ to track an attitude and angular velocity command without the knowledge of the inertia matrix of a rigid body. An estimate of the inertia matrix is updated online to provide an asymptotic tracking property. It is also extended to a robust adaptive attitude tracking control system. Stable adaptive control schemes designed without consideration of uncertainties may become unstable in the presence of small disturbances [10]. The presented robust adaptive scheme guarantees the boundedness of the attitude tracking error and the inertia matrix estimation error even if there exist modeling errors or disturbances. Compared with a prior work in [8], the proposed adaptive tracking control system has simpler structures, and the proposed robust adaptive tracking control system can be applied to a more general class of unstructured or non-harmonic uncertainties.

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## II. Attitude Dynamics of a Rigid Body

We consider the rotational attitude dynamics of a fully-actuated rigid body. We define an inertial reference frame and a body fixed frame whose origin is located at the mass center of the rigid body. The configuration of the rigid body is the orientation of the body fixed frame with respect to the inertial frame, and it is represented by a rotation matrix $R \in \mathsf{SO}(3)$, where the special orthogonal group $\mathsf{SO}(3)$ is the group of $3\times 3$ orthogonal matrices with determinant of one, i.e., $\mathsf{SO}(3) = \{R\in\mathbb{R}^{3\times 3}\,|\, R^T R = I,\, \det R = 1\}$.

The equations of motion are given by

$$J\dot\Omega + \Omega\times J\Omega = u + \Delta, \tag{1}$$

$$\dot R = R\hat\Omega, \tag{2}$$

where $J\in\mathbb{R}^{3\times 3}$ is the inertia matrix in the body fixed frame, and $\Omega\in\mathbb{R}^3$ and $u\in\mathbb{R}^3$ are the angular velocity of the rigid body and the control moment, represented with respect to the body fixed frame, respectively. The vector $\Delta\in\mathbb{R}^3$ represents unknown disturbances in the attitude dynamics.

The *hat* map $\wedge : \mathbb{R}^3 \to \mathfrak{so}(3)$ transforms a vector in $\mathbb{R}^3$ to a $3\times 3$ skew-symmetric matrix such that $\hat x y = x\times y$ for any $x, y\in\mathbb{R}^3$. The inverse of the hat map is denoted by the *vee* map $\vee : \mathfrak{so}(3)\to\mathbb{R}^3$. Throughout this paper, the 2-norm of a matrix $A$ is denoted by $\|A\|$, and its Frobenius norm is denoted by $\|A\|_F = \sqrt{\mathrm{tr}[A^T A]}$. We have $\|A\| \leq \|A\|_F \leq \sqrt{r}\|A\|$, where $r$ is the rank of $A$.

## III. Geometric Tracking Control on $\mathsf{SO}(3)$

We develop adaptive control systems to follow a given smooth attitude command $R_d(t)\in\mathsf{SO}(3)$. The kinematics equation for the attitude command can be written as

$$\dot R_d = R_d\hat\Omega_d, \tag{3}$$

where $\Omega_d\in\mathbb{R}^3$ is the desired angular velocity.

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### A. Attitude Error Dynamics

We define an attitude error dynamics that represent the errors in tracking the desired attitude trajectory. First, several properties of the attitude error function studied in [6], [11] are summarized, and we show few additional facts required in this paper. Due to the page limit, the proofs are relegated to [12].

*Proposition 1:* For a given tracking command $(R_d, \Omega_d)$, and the current attitude and angular velocity $(R, \Omega)$, we define an attitude error function $\Psi : \mathsf{SO(3)} \times \mathsf{SO(3)} \to \mathbb{R}$, an attitude error vector $e_R \in \mathbb{R}^3$, and an angular velocity error vector $e_\Omega \in \mathbb{R}^3$ as follows:

$$\Psi(R, R_d) = \frac{1}{2}\mathrm{tr}\big[G(I - R_d^T R)\big]\,, \tag{4}$$

$$e_R(R, R_d) = \frac{1}{2}(GR_d^T R - R^T R_d G)^\vee, \tag{5}$$

$$e_\Omega(R, \Omega, R_d, \Omega) = \Omega - R^T R_d \Omega_d, \tag{6}$$

where the matrix $G \in \mathbb{R}^{3\times3}$ is given by $G = \mathrm{diag}[g_1, g_2, g_3]$ for distinct, positive constants $g_1, g_2, g_3 \in \mathbb{R}$. Then, the following statements hold:

(i) $\Psi$ is locally positive definite about $R = R_d$.

(ii) the left-trivialized derivative of $\Psi$ is given by

$$\mathsf{T}_I^* \mathsf{L}_R\left(\mathbf{D}_R \Psi(R, R_d)\right) = e_R. \tag{7}$$

(iii) the critical points of $\Psi$, where $e_R = 0$, are $\{R_d\} \cup \{R_d \exp(\pi\hat{s})\}$ for $s \in \{e_1, e_2, e_3\}$.

(iv) a lower bound of $\Psi$ is given as follows:

$$b_1 \|e_R(R, R_d)\|^2 \leq \Psi(R, R_d), \tag{8}$$

where the constant $b_1$ is given by $b_1 = \frac{h_1}{h_2 + h_3}$ for

$$h_1 = \min\{g_1 + g_2, g_2 + g_3, g_3 + g_1\},$$
$$h_2 = \max\{(g_1 - g_2)^2, (g_2 - g_3)^2, (g_3 - g_1)^2\},$$
$$h_3 = \max\{(g_1 + g_2)^2, (g_2 + g_3)^2, (g_3 + g_1)^2\}.$$

(v) Let $\psi$ be a positive constant that is strictly less than $h_1$. If $\Psi(R, R_d) < \psi < h_1$, then an upper bound of $\Psi$ is given by

$$\Psi(R, R_d) \leq b_2 \|e_R(R, R_d)\|^2, \tag{9}$$

where the constant $b_2$ is given by $b_2 = \frac{h_1 h_4}{h_5(h_1 - \psi)}$ for

$$h_4 = \max\{g_1 + g_2, g_2 + g_3, g_3 + g_1\}$$
$$h_5 = \min\{(g_1 + g_2)^2, (g_2 + g_3)^2, (g_3 + g_1)^2\}.$$

*Proof:* See [12]. ∎

*Proposition 2:* The error dynamics for $\Psi$, $e_R$, $e_\Omega$ satisfies

$$\frac{d}{dt}(R_d^T R) = R_d^T R \hat{e}_\Omega \tag{10}$$

$$\frac{d}{dt}(\Psi(R, R_d)) = e_R \cdot e_\Omega, \tag{11}$$

$$\dot{e}_R = E(R, R_d) e_\Omega, \tag{12}$$

$$\dot{e}_\Omega = J^{-1}(-\Omega \times J\Omega + u + \Delta) - \alpha_d, \tag{13}$$

where the matrix $E(R, R_d) \in \mathbb{R}^{3\times3}$, and the angular acceleration $\alpha_d \in \mathbb{R}^3$, that is caused by the attitude command, and measured in the body fixed frame, are given by

$$E(R, R_d) = \frac{1}{2}(\mathrm{tr}[R^T R_d G]I - R^T R_d G), \tag{14}$$

$$\alpha_d = -\hat{\Omega} R^T R_d \Omega_d + R^T R_d \dot{\Omega}_d. \tag{15}$$

Furthermore, the matrix $E(R, R_d)$ is bounded by

$$\|E(R, R_d)\| \leq \frac{1}{\sqrt{2}}\mathrm{tr}[G]\,. \tag{16}$$

*Proof:* See [12]. ∎

### B. Adaptive Attitude Tracking

Attitude tracking control systems require the knowledge of an inertia matrix when the given attitude command is not fixed. But, it is difficult to measure the value of an inertia matrix exactly. In general, there is an estimation error:

$$\tilde{J} = J - \bar{J}, \tag{17}$$

where the exact inertia matrix and its estimate are denoted by the matrices $J$ and $\bar{J} \in \mathbb{R}^{3\times3}$, respectively. All of matrices, $J$, $\bar{J}$, $\tilde{J}$ are symmetric.

Here, an adaptive tracking controller for the attitude dynamics of a rigid body is presented to follow a given attitude command without the knowledge of its inertia matrix assuming that there is no disturbance, and that the bounds of the inertia matrix are given.

*Assumption 3:* The minimum eigenvalue $\lambda_m \in \mathbb{R}$, and the maximum eigenvalue $\lambda_M \in \mathbb{R}$ of the true inertia matrix $J$ given at (1) are known.

*Proposition 4:* Assume that there is no disturbance in the attitude dynamics, i.e. $\Delta = 0$ at (1), and Assumption 3 is satisfied. For a given attitude command $R_d(t)$, and positive constants $k_R, k_\Omega, k_J \in \mathbb{R}$, we define a control input $u \in \mathbb{R}^3$, and an update law for $\bar{J}$ as follows:

$$u = -k_R e_R - k_\Omega e_\Omega + \Omega \times \bar{J}\Omega + \bar{J}\alpha_d, \tag{18}$$

$$\dot{\bar{J}} = \frac{k_J}{2}(-\alpha_d e_A^T - e_A \alpha_d^T + \Omega\Omega^T \hat{e}_A - \hat{e}_A \Omega\Omega^T), \tag{19}$$

where $e_A \in \mathbb{R}^3$ is an augmented error vector given by

$$e_A = e_\Omega + c e_R \tag{20}$$

for a positive constant $c$ satisfying

$$c < \min\left\{\sqrt{\frac{2b_1 k_R \lambda_m}{\lambda_M^2}}, \frac{\sqrt{2}k_\Omega}{\lambda_M \mathrm{tr}[G]}, \frac{4k_R k_\Omega}{k_\Omega^2 + \frac{1}{\sqrt{2}}k_R \lambda_M \mathrm{tr}[G]}\right\}. \tag{21}$$

Then, the zero equilibrium of the tracking errors $(e_R, e_\Omega)$ and the estimation error $\tilde{J}$ is stable, and those errors are uniformly bounded. Furthermore, the tracking errors for the attitude and the angular velocity asymptotically converge to zero, i.e. $e_R, e_\Omega \to 0$ as $t \to \infty$.

*Proof:* Consider the following Lyapunov function:

$$\mathcal{V} = \frac{1}{2} e_\Omega \cdot J e_\Omega + k_R \Psi(R, R_d) + cJe_\Omega \cdot e_R + \frac{1}{2k_J}\|\tilde{J}\|_F^2. \tag{22}$$

From (8), we obtain

$$z^T W_{11} z \leq \mathcal{V} \tag{23}$$

where $z = [\|e_R\|; \|e_\Omega\|; \|\tilde{J}\|_F] \in \mathbb{R}^3$, and the matrix $W_1 \in \mathbb{R}^{3\times 3}$ are given by

$$W_{11} = \begin{bmatrix} b_1 k_R & \frac{1}{2}c\lambda_M & 0 \\ \frac{1}{2}c\lambda_M & \frac{1}{2}\lambda_m & 0 \\ 0 & 0 & \frac{1}{2k_J} \end{bmatrix}. \tag{24}$$

Substituting (18) into (13) with $\Delta = 0$, we obtain

$$J\dot{e}_\Omega = -k_R e_R - k_\Omega e_\Omega - \tilde{J}\alpha_d - \Omega \times \tilde{J}\Omega. \tag{25}$$

Using (11), (12), (25), the time-derivative of $\mathcal{V}$ is given by

$$\begin{aligned}\dot{\mathcal{V}} &= -k_\Omega \|e_\Omega\|^2 - ck_R\|e_R\|^2 + cJe_\Omega \cdot Ee_\Omega - ck_\Omega e_\Omega \cdot e_R \\ &\quad - (e_\Omega + ce_R)\cdot(\tilde{J}\alpha_d + \Omega \times \tilde{J}\Omega) + \frac{1}{k_J}\mathrm{tr}\Big[\tilde{J}\dot{\tilde{J}}\Big].\end{aligned}$$

From (20), and using the fact that $x \cdot y = \mathrm{tr}[xy^T] = \mathrm{tr}[yx^T]$ for any $x, y \in \mathbb{R}^3$, and the scalar triple product identity, this can be written as

$$\begin{aligned}\dot{\mathcal{V}} &= -k_\Omega \|e_\Omega\|^2 - ck_R\|e_R\|^2 + cJe_\Omega \cdot Ee_\Omega - ck_\Omega e_\Omega \cdot e_R \\ &\quad + \mathrm{tr}\left[\tilde{J}\left\{-\alpha_d e_A^T - \Omega(e_A \times \Omega)^T + \frac{1}{k_J}\dot{\tilde{J}}\right\}\right].\end{aligned}$$

Since $\dot{\tilde{J}} = -\dot{\bar{J}}$, we can substitute (19) into this. Using the facts that $\mathrm{tr}[\tilde{J}A] = \mathrm{tr}[\tilde{J}A^T]$ for any $A \in \mathbb{R}^{3\times 3}$, and $(e_A \times \Omega)^T = (\hat{e}_A\Omega)^T = -\Omega^T\hat{e}_A$, it reduces to

$$\dot{\mathcal{V}} = -k_\Omega \|e_\Omega\|^2 - ck_R\|e_R\|^2 + cJe_\Omega \cdot Ee_\Omega - ck_\Omega e_\Omega \cdot e_R. \tag{26}$$

From (16), it is bounded by

$$\begin{aligned}\dot{\mathcal{V}} &\leq -(k_\Omega - \frac{c}{\sqrt{2}}\lambda_M \mathrm{tr}[G])\|e_\Omega\|^2 - ck_R\|e_R\|^2 \\ &\quad + ck_\Omega\|e_\Omega\|\|e_R\| = -\zeta^T W_2 \zeta,\end{aligned} \tag{27}$$

where $\zeta = [\|e_R\|; \|e_\Omega\|] \in \mathbb{R}^2$, and the matrix $W_2 \in \mathbb{R}^{2\times 2}$ is given by

$$W_2 = \begin{bmatrix} ck_R & -\frac{ck_\Omega}{2} \\ -\frac{ck_\Omega}{2} & k_\Omega - \frac{c}{\sqrt{2}}\lambda_M\mathrm{tr}[G] \end{bmatrix}. \tag{28}$$

The inequality (21) for the constant $c$ guarantees that the matrices $W_{11}, W_2$ are positive definite.

This implies that the Lyapunov function $\mathcal{V}(t)$ is bounded from below and it is nonincreasing. Therefore, it has a limit, $\lim_{t\to\infty}\mathcal{V}(t) = \mathcal{V}_\infty$, and $e_R, e_\Omega, \bar{J} \in \mathcal{L}_\infty$.[1] From (12), (25), we have $\dot{e}_R, \dot{e}_\Omega \in \mathcal{L}_\infty$. Furthermore $e_R, e_\Omega \in \mathcal{L}_2$ since $\int_0^\infty \zeta(\tau)^T W_2 \zeta(\tau) d\tau \leq \mathcal{V}(0) - \mathcal{V}_\infty < \infty$. According to Barbalat's lemma (or Lemma 3.2.5 in [10]), we have $e_R, e_\Omega \to 0$ as $t \to \infty$. ■

*Remark 5:* This proposition guarantees that the attitude error vector $e_R$ asymptotically converges to zero. But, this does not necessarily imply that $R \to R_d$ as $t \to \infty$. According to Proposition 1, there exist three additional critical points of $\Psi$, namely $\{R_d \exp(\pi\hat{s})\}$ for $s \in \{e_1, e_2, e_3\}$, where $e_R = 0$. This is due to the nonlinear structures of $\mathsf{SO}(3)$, and these cannot be avoided for any continuous attitude control systems [4].

But, we can show that those three additional equilibrium points are unstable, by using linearization. It turned out that these points are saddle equilibria, which have both of stable manifolds and unstable manifolds [13]. The union of the stable manifolds to these undesirable equilibria has a lower dimension than the tangent bundle of the configuration space, and we say that it has an *almost*-global stabilization property.

*Remark 6:* At Assumption 3, the minimum eigenvalue $\lambda_m$ and the maximum eigenvalue $\lambda_M$ of the inertia matrix $J$ are required. But, in Proposition 4, they are only used to find the coefficient $c$ at (21). So, Assumption 3 can be relaxed as requiring an upper bound of $\lambda_m$ and a lower bound of $\lambda_M$, which are relatively simpler to estimate.

### C. Robust Adaptive Attitude Tracking

The adaptive tracking control system developed in the previous section is based on the assumption that there is no disturbance in the attitude dynamics. But, it has been discovered that adaptive control schemes may become unstable in the presence of small disturbances [10]. Robust adaptive control deals with redesigning or modifying adaptive control schemes to make them robust with respect to unmodeled dynamics or bounded disturbances. In this section, we develop a robust adaptive attitude tracking control system assuming that the bound of disturbances are given.

*Assumption 7:* The disturbance term in the attitude dynamics at (1) is bounded by a known constant, i.e. $\|\Delta\| \leq \delta$ for a given positive constant $\delta$.

*Proposition 8:* Suppose that Assumptions 3 and 7 hold. For a given attitude command $R_d(t)$, and positive constants $k_R, k_\Omega, k_J, \sigma, \epsilon \in \mathbb{R}$, we define a control input $u \in \mathbb{R}^3$, and an update law for $\bar{J}$ as follows:

$$u = -k_R e_R - k_\Omega e_\Omega + \Omega \times \bar{J}\Omega + \bar{J}\alpha_d + v, \tag{29}$$

$$v = -\frac{\delta^2 e_A}{\delta\|e_A\| + \epsilon}, \tag{30}$$

$$\dot{\bar{J}} = \frac{k_J}{2}(-\alpha_d e_A^T - e_A\alpha_d^T + \Omega\Omega^T\hat{e}_A - \hat{e}_A\Omega\Omega^T - 2\sigma\bar{J}), \tag{31}$$

where $e_A \in \mathbb{R}^3$ is an augmented error vector given at (20) for a positive constant $c$ satisfying (21). Then, if $\sigma$ and $\epsilon$ are sufficiently small, the zero equilibrium of the tracking errors $(e_R, e_\Omega)$ and the estimation error $\tilde{J}$ are uniformly bounded.

*Proof:* Consider the Lyapunov function $\mathcal{V}$ at (22). For a positive constant $\psi < h_1$, define $D \subset \mathsf{SO}(3)$ as

$$D = \{R \in \mathsf{SO}(3) \,|\, \Psi < \psi < h_1\}$$

From Proposition 1, $\mathcal{V}$ is bounded in $D$ by

$$z^T W_{11} z \leq \mathcal{V} \leq z^T W_{12} z, \tag{32}$$

[1] A function $f : \mathbb{R} \to \mathbb{R}$ belongs to the $\mathcal{L}_p$ space for $p \in [1, \infty)$, if the following $p$-norm of the function exits, $\|f\|_p = \left\{\int_0^\infty |f(\tau)|^p \, d\tau\right\}^{1/p}$.

where $z=[\|e_R\|;\,\|e_\Omega\|;\,\|\tilde{J}\|_F]\in\mathbb{R}^2$, the matrix $W_{11}\in\mathbb{R}^{2\times 2}$ is given by (24), and the matrix $W_{12}$ is given by

$$W_{12}=\begin{bmatrix} b_2k_R & \frac{1}{2}c_2\lambda_M & 0\\ \frac{1}{2}c_2\lambda_M & \frac{1}{2}\lambda_M & 0\\ 0 & 0 & \frac{1}{2k_J}\end{bmatrix}.$$

The time-derivative of $\mathcal{V}$ along the presented control inputs is written as

$$\dot{\mathcal{V}}=-k_\Omega\|e_\Omega\|^2-ck_R\|e_R\|^2+cJe_\Omega\cdot Ee_\Omega-ck_\Omega e_\Omega\cdot e_R \\ +e_A\cdot(\Delta+v)+\sigma\mathrm{tr}\Big[\tilde{J}\bar{J}\Big]. \quad (33)$$

Compared with (26), this has three additional terms caused by $\Delta, v$ and $\sigma$. From Assumption 7 and (30), the second last term of (33) is bounded by

$$e_A\cdot(\Delta+v)\leq\delta\|e_A\|-\frac{\delta^2\|e_A\|^2}{\delta\|e_A\|+\epsilon}=\frac{\delta\|e_A\|}{\delta\|e_A\|+\epsilon}\epsilon\leq\epsilon. \quad (34)$$

The last term of (33) is bounded by

$$\mathrm{tr}\Big[\tilde{J}\bar{J}\Big]=\mathrm{tr}\Big[\tilde{J}(J-\tilde{J})\Big]=\sum_{1\leq i,j\leq 3}(-\tilde{J}_{ij}^2+J_{ij}\tilde{J}_{ij}) \\ \leq\sum_{1\leq i,j\leq 3}(-\frac{1}{2}\tilde{J}_{ij}^2+\frac{1}{2}J_{ij}^2)=-\frac{1}{2}\|\tilde{J}\|_F^2+\frac{1}{2}\|J\|_F^2.$$

Using the relation between a Frobenius norm and a matrix 2-norm, we have $\|J\|_F\leq\sqrt{3}\|J\|=\sqrt{3}\lambda_M$. Therefore,

$$\mathrm{tr}\Big[\tilde{J}\bar{J}\Big]\leq-\frac{1}{2}\|\tilde{J}\|_F^2+\frac{3}{2}\lambda_M^2. \quad (35)$$

Substituting (34), (35) into (33), we obtain

$$\dot{\mathcal{V}}\leq-z^TW_3z+\frac{3}{2}\sigma\lambda_M^2+\epsilon \quad (36)$$

where the matrix $W_3\in\mathbb{R}^{3\times 3}$ is given by

$$W_3=\begin{bmatrix} ck_R & -\frac{ck_\Omega}{2} & 0\\ -\frac{ck_\Omega}{2} & k_\Omega-\frac{c}{\sqrt{2}}\lambda_M\mathrm{tr}[G] & 0\\ 0 & 0 & \frac{1}{2}\sigma\end{bmatrix}. \quad (37)$$

The inequality (21) for the constant $c$ guarantees that the matrices $W_{11}, W_{12}, W_3$ become positive definite. Then, we have

$$\dot{\mathcal{V}}\leq-\frac{\lambda_{\min}(W_2)}{\lambda_{\max}(W_{12})}\mathcal{V}+\frac{3}{2}\sigma\lambda_M^2+\epsilon, \quad (38)$$

where $\lambda_{\min}(\cdot)$ and $\lambda_{\max}(\cdot)$ represent the minimum eigenvalue and the maximum eigenvalue of a matrix, respectively. This implies that $\dot{\mathcal{V}}<0$ when $\mathcal{V}>\frac{\lambda_{\max}(W_{12})}{\lambda_{\min}(W_3)}(\frac{3}{2}\sigma\lambda_M^2+\epsilon)\triangleq d_1$.

Let a sublevel set of $\mathcal{V}$ be $L_\gamma=\{(R,\Omega,\bar{J})\in\mathsf{SO(3)}\times\mathbb{R}^3\times\mathbb{R}^{3\times 3}\,|\,\mathcal{V}\leq\gamma\}$ for a constant $\gamma>0$. If the following inequality for $\gamma$ is satisfied

$$\gamma<\frac{\psi}{b_2}\lambda_{\min}(W_{11})\triangleq d_2,$$

we can guarantee that $L_\gamma\subset D\times\mathbb{R}^3\times\mathbb{R}^{3\times 3}$, since it implies that $\|z\|^2<\frac{\psi}{b_2}$, which leads $\Psi\leq b_2\|e_R\|^2\leq b_2\|z\|^2<\psi$.

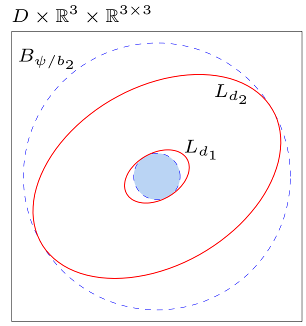



Fig. 1. Boundedness of the error: Outside of the shaded region, represented by $\{\lambda_{\min}(W_3)\|z\|^2\geq(\frac{3}{2}\sigma\lambda_M^2+\epsilon)\}$, we have $\dot{\mathcal{V}}\leq 0$ from (36). Inside of the larger ball, $B_{\psi/b_2}=\{\|z\|^2\leq\psi/b_2\}\subset D\times\mathbb{R}^3\times\mathbb{R}^{3\times 3}$, equations (32) and (38) hold. The inequality (39) guarantees that the smallest sublevel set $L_{d_1}$ of $\mathcal{V}$, covering the shaded area, lies inside of the largest sublevel set $L_{d_2}$ of $\mathcal{V}$ in $B_{\psi/b_2}$, i.e. $L_{d_1}\subset L_{d_2}$. Therefore, along any solution starting in $L_{d_2}$, $\mathcal{V}$ decreases until the solution enters $L_{d_1}$, thereby yielding uniform boundedness.

Then, from (38), a sublevel set $L_\gamma$ is a positively invariant set, when $d_1<\gamma<d_2$, and it becomes smaller until $\gamma=d_1$. In order to guarantee the existence of such $L_\gamma$, the following inequality should be satisfied

$$d_1=\frac{\lambda_{\max}(W_{12})}{\lambda_{\min}(W_3)}(\frac{3}{2}\sigma\lambda_M^2+\epsilon)<\frac{\psi}{b_2}\lambda_{\min}(W_{11})=d_2, \quad (39)$$

which can be achieved by choosing sufficiently small $\sigma$ and $\epsilon$. Then, according to Theorem 5.1 in [14], for any initial condition satisfying $\mathcal{V}(0)<d_2$, its solution exponentially converges to the following set:

$$L_{d_1}\subset\left\{\|z\|^2\leq\frac{\lambda_{\max}(W_{12})}{\lambda_{\min}(W_{11})\lambda_{\min}(W_2)}\left(\frac{3}{2}\sigma\lambda_M^2+\epsilon\right)\right\}.$$

■

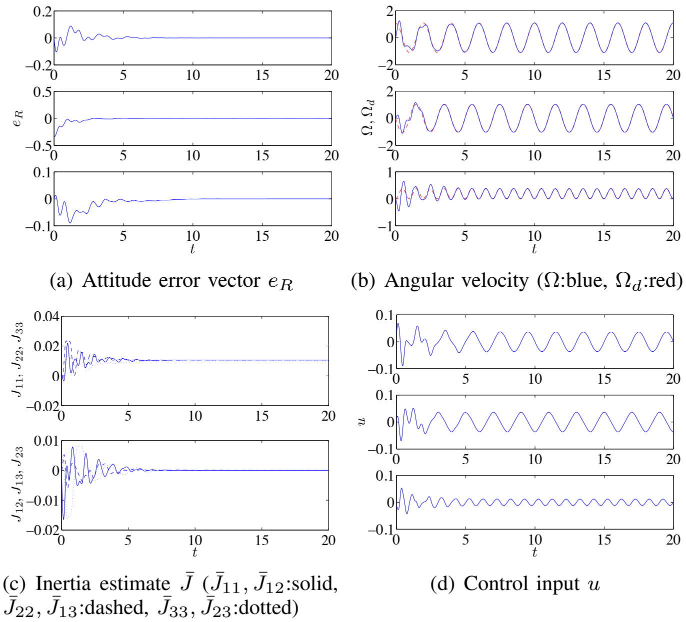
*Remark 9:* The robust adaptive control system in Proposition 8 is referred to as fixed $\sigma$-modification [10], where robustness is achieved at the expense of replacing the asymptotic tracking property of Proposition 4 by boundedness. This property can be improved by the following approaches: (i) the leakage term $-2\sigma\bar{J}$ at (31) can be replaced by $-2\sigma(\bar{J}-J^\star)$, where $J^*$ denotes the best possible prior estimate of the inertia matrix. This shifts the tendency of $\bar{J}$ from zero to $J^\star$, thereby reducing the ultimate bound, (ii) a switching $\sigma$-modification or $\epsilon_1$-modification can be used to improve the convergence properties in the expense of discontinuities, (iii) the constant $\epsilon$ at (30) can be replaced by $\epsilon\exp(-\beta t)$ for any $\beta>0$ to reduce the ultimate bound. The corresponding stability analyses are similar to the presented case, and they are deferred to a future study.

## IV. Numerical Examples

Parameters of a rigid body model and control systems are chosen as follows[2]:

$$J=\begin{bmatrix} 1.059\times 10^{-2} & -5.156\times 10^{-6} & 2.361\times 10^{-5}\\ -5.156\times 10^{-6} & 1.059\times 10^{-2} & -1.026\times 10^{-5}\\ 2.361\times 10^{-5} & -1.026\times 10^{-5} & 1.005\times 10^{-2}\end{bmatrix},$$

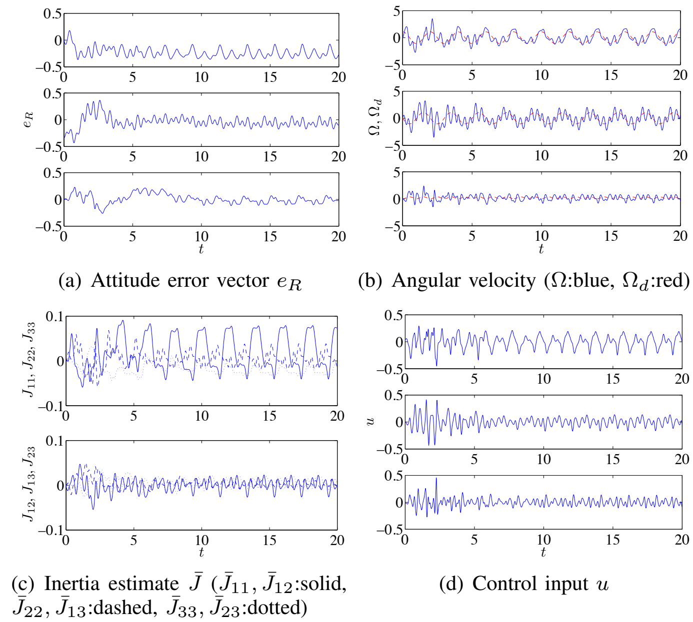
[2]All of variables are written in kilograms, meters, seconds, and radians.



(a) Attitude error vector $e_R$ (b) Angular velocity ($\Omega$:blue, $\Omega_d$:red)

(c) Inertia estimate $\bar{J}$ ($\bar{J}_{11}, \bar{J}_{12}$:solid, $\bar{J}_{22}, \bar{J}_{13}$:dashed, $\bar{J}_{33}, \bar{J}_{23}$:dotted) (d) Control input $u$

Fig. 2. Adaptive attitude tracking without disturbances



(a) Attitude error vector $e_R$ (b) Angular velocity ($\Omega$:blue, $\Omega_d$:red)

(c) Inertia estimate $\bar{J}$ ($\bar{J}_{11}, \bar{J}_{12}$:solid, $\bar{J}_{22}, \bar{J}_{13}$:dashed, $\bar{J}_{33}, \bar{J}_{23}$:dotted) (d) Control input $u$

Fig. 3. Adaptive attitude tracking with disturbances

$$k_R = 0.0424, \quad k_\Omega = 0.0296, \quad k_J = 0.1,$$
$$c = 1.0, \quad \sigma = 0.01, \quad \epsilon = 0.002, \quad \delta = 0.2.$$

Initial conditions are given by

$$\bar{J}(0) = 0.001I, \quad R(0) = I, \quad \Omega(0) = 0.$$

The desired attitude command is described by using 3-2-1 Euler angles [15], i.e. $R_d(t) = R_d(\phi(t), \theta(t), \psi(t))$, and these angles are chosen as

$$\phi(t) = \frac{\pi}{9}\sin(\pi t), \quad \theta(t) = \frac{\pi}{9}\cos(\pi t), \quad \psi(t) = 0.$$

We consider three cases:

(i) Adaptive attitude tracking control system presented at Proposition 4 without disturbances.

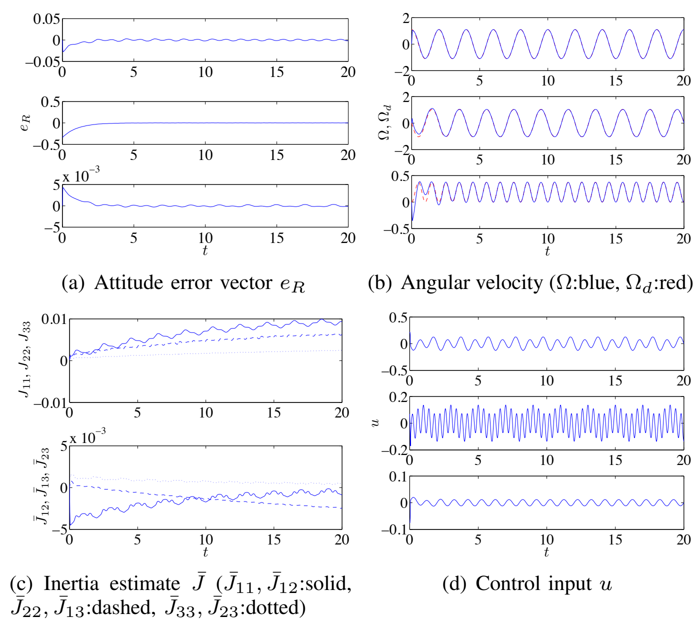
(ii) Adaptive attitude tracking control system presented at Proposition 4 with the following disturbances:

$$\Delta = 0.1 \begin{bmatrix} \sin(2\pi t) & \cos(5\pi t) & R_{11}(t) \end{bmatrix}.$$

(iii) Robust adaptive attitude tracking control system presented at Proposition 8 with the above disturbance model.

It has been shown that general-purpose numerical integrators fail to preserve the structure of the special orthogonal group $\mathsf{SO}(3)$, and they may yields unreliable computational results for complex maneuvers of rigid bodies [16]. In this paper, we use a geometric numerical integrators, referred to as a Lie group variational integrator, to preserve the underlying geometric structures of the attitude dynamics accurately [17].

Simulation results are illustrated at Figures 2-4. When there is no disturbance, the adaptive attitude tracking control system presented at Proposition 4 follows the given attitude command accurately at Fig. 2. But, these convergence properties are degraded in the presence of disturbances. At Fig. 3, the tracking errors do not converge to zero asymptotically, and the estimate of the inertia matrix and control inputs fluctuate. These are significantly improved by the robust adaptive tracking controller discussed at Proposition 8. At Fig. 4, the tracking errors for the attitude and the angular velocity are close to zero, and the estimate of the inertia matrix is bounded. These show that the proposed robust adaptive approach is effective in following an attitude command in the presence of disturbances.



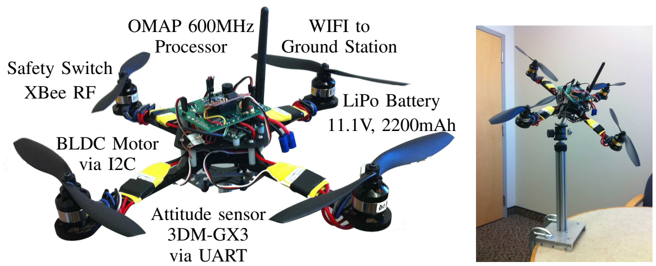
(a) Attitude error vector $e_R$ (b) Angular velocity ($\Omega$:blue, $\Omega_d$:red)

(c) Inertia estimate $\bar{J}$ ($\bar{J}_{11}, \bar{J}_{12}$:solid, $\bar{J}_{22}, \bar{J}_{13}$:dashed, $\bar{J}_{33}, \bar{J}_{23}$:dotted) (d) Control input $u$

Fig. 4. Robust adaptive attitude tracking with disturbances

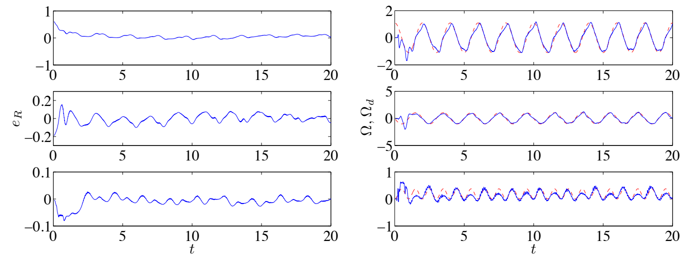
## V. Experiment on a Quadrotor UAV

A quadrotor unmanned aerial vehicle (UAV) is composed of two pairs of counter-rotating rotors and propellers. Due

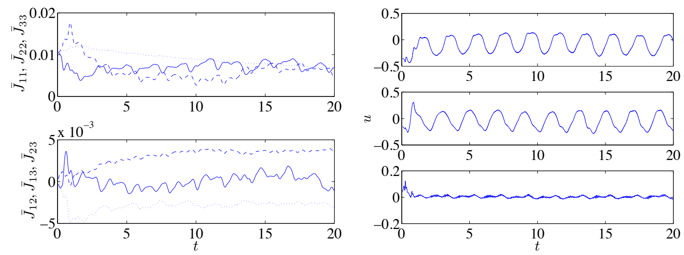



(a) Hardware configuration (b) Attitude control testbed

Fig. 5. Attitude control experiment for a quadrotor UAV



(a) Attitude error vector $e_R$ (b) Angular velocity ($\Omega$:blue, $\Omega_d$:red)



(c) Inertia estimate $\bar{J}$ ($\bar{J}_{11}, \bar{J}_{12}$:solid, $\bar{J}_{22}, \bar{J}_{13}$:dashed, $\bar{J}_{33}, \bar{J}_{23}$:dotted (kgm$^2$)) (d) Control input $u$

Fig. 6. Robust adaptive attitude tracking experiment

to its simple mechanical structure, it has been envisaged for various applications such as surveillance or mobile sensor networks as well as for educational purposes.

We have developed a hardware system for a quadrotor UAV. It is composed of the following parts:

- Gumstix Overo computer-in-module (OMAP 600MHz processor), running a (non-realtime) Linux operating system. It communicates to a ground station via WIFI.
- Microstrain 3DM-GX3 attitude sensor, connected to Gumstix via UART.
- Phifun motor speed controller, connected to Gumstix via I2C.
- Roxxy 2827-35 Brushless DC motors.
- MaxStream XBee RF module, which is used for an extra safety switch.

To test the attitude dynamics only, it is attached to a spherical joint. As the center of rotation is below the center of gravity, there exists a destabilizing gravitational moment, and the resulting attitude dynamics is similar to an inverted rigid body pendulum.

We apply the robust adaptive attitude control system at Proposition 8 to this quadrotor UAV. The control input at (29) is augmented with an additional term to eliminate the gravitational moment. The disturbances are mainly due to the error in canceling the gravitational moment, the friction in the spherical joint, as well as sensor noises and thrust measurement errors.

The attitude tracking command and control input parameters are identical to the numerical examples discussed in the previous section, except the following variables:

$$k_J = 0.01, \quad \sigma = 0.01, \quad \epsilon = 0.35.$$

The corresponding experimental results are illustrated at Fig. 6. Overall, it exhibits a good attitude command tracking performance, while the second component of the attitude error vector $e_R$, and the third component of the angular velocity tracking error are relatively large. The estimates of the inertia matrix are bounded.

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