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Optimized Distributed Control and Network Topology Design for Interconnected Systems

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Abstract— **This paper considers optimized network topology design and distributed control for linear discrete-time systems consisting of subsystems interconnected through states, inputs, and a cost function. By using a distributed control law, which makes use of the communicated states of other subsystems, closed-loop performance is increased at the expense of communication costs. This raises the question of how to find a topology and associated distributed control law with optimal trade-off between communication costs and closed-loop performance. As an answer to this question, we propose an approach to simultaneous optimization of network topology and control law with respect to a cost function which combines a quadratic performance criterion with costs associated to the presence of communication links. The problem is formulated as mixed-integer semi-definite problem (MISDP) where the discrete optimization of the network topology subject to communication constraints and embedded subproblems for structured controller synthesis lead to an upper bound for the combined cost. An example is used to illustrate the method.**

I. INTRODUCTION

Control of interconnected dynamical systems is a challenging problem with many applications, for example in power grids and process control. We consider a class of systems in which subsystems are coupled over states, inputs, and a performance criterion. The control of systems coupled via states has already been tackled to a considerable extent within the framework of decentralized control [1]. Within this framework, the overall system is decomposed by exploiting structural interconnection constraints in order to obtain a system structure which allows to design decentralized control laws. Results have been published on how to alter the decomposition if a system does not admit a stabilizing decentralized controller [2]. However, even if a stabilizing decentralized controller exists, performance might be significantly degraded compared to a centralized approach.

The use of digital communication networks allows for potentially higher performance by exchanging information on the states of other subsystems. This results in distributed control laws [3] which typically offer better performance and can be used even if no stabilizing decentralized control law exists. On the other hand, the network induces phenomena such as time-varying delays and packet dropouts. Much of the work in the area of networked control systems (NCS) is focused either on the effect these phenomena have on closed-loop stability of centralized controllers connected to the plant through a communication network, or on synthesis of controllers that are robust with regard to the network uncertainties [4]. Most of the work on distributed control assumes a certain type of communication topology (such as communication only between neighbors). In [5], the distributed control of identical and dynamically decoupled systems is considered, and in [6] dual decomposition is used for distributed optimization of local controllers. However, digital communication networks often provide flexibility with regard to the topology of the network, resulting in additional freedom in the controller design. At the same time communication between subsystems also leads to additional costs, for example energy consumption and hardware costs. This raises the question which network topology and corresponding distributed control law is optimal with regard to a cost function considering closed-loop performance as well as communication costs.

Results on optimal topology design for fixed local controllers can be found in [7], where the special case of average-consensus is considered, and in [8], where the focus is on the $\mathcal{H}_2$-gain of locally controlled systems coupled through outputs. Some results for topology design of linear continuous-time systems that are interconnected through states can be found in [9], where the underlying assumption is that a decentralized stabilizing controller can be found by a linear matrix inequality (LMI) approach. While the latter work starts from a rather general formulation, an explicit procedure to obtain a controller improving the decay rate of the system by additional feedback via communication is specified only for the assumptions that the subsystems are scalar, that the communication is bidirectional, that all communication links are equally expensive, and that the system matrix A is symmetric. Except for an upper bound on the communication costs no communication constraints are considered. In addition, in many situations the decay rate does not appear as a good performance measure because input costs can not be considered. Furthermore, using a decentralized controller as basis for the distributed scheme may lead to reduced performance.

In contrast to previous work, this paper considers linear discrete-time systems which are interconnected through states as well as inputs and a performance criterion. We propose a mixed integer semi-definite programming (MISDP) approach to simultaneously optimize the communication topology and the distributed controllers with respect to quadratic infinite horizon cost of the closed-loop system as well as communication costs and constraints. The proposed method does not require the existence of a stabilizing decentralized controller. The combined approach to controller synthesis and network topology design leads to a combination

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of a discrete optimization problem with regard to the communication topology and a continuous optimization problem for the distributed controllers. The communication topology introduces structural constraints on the controller, resulting in a non-convex controller synthesis problem [10]. Suboptimal solutions to this problem can be obtained by means of bilinear matrix inequalities (BMI) or iterative procedures based on LMIs [11]. Both of these approaches are computationally expensive. Since the structured controller synthesis problems are subproblems that are repeatedly solved in the optimization of the network topology, these methods are not suitable for the problem considered here (see the discussion in Sec. II-C). In [12] an iterative LMI approach is used to design static $\mathcal{H}_\infty$ controllers with a minimized number of communication links, however constraints on the communication topology are not considered. In [13] a convex parametrization for optimizing frequency domain controllers subject to structural constraints is proposed. However, for unstable plants a stabilizing controller that satisfies the structural constraints has to be known a priori. In the context of this work, the structural constraints arise from the network topology which is not fixed. Thus, a controller that satisfies all possible structural constraints would have to be a decentralized controller.

To avoid these issues we propose sufficient convex LMI conditions for the structured controller synthesis problem based on the well known ideas in [14]. In combination with a Big-M reformulation [15] of the constraints resulting from the network topology, a MISDP is obtained which can be solved by branch and bound techniques. In the next section the system class, the performance criteria, and the resulting optimization problem are defined. Sec. III presents the proposed MISDP approach, and in Sec. IV the method is illustrated by an example.

II. PROBLEM FORMULATION

A. System Dynamics and Performance Criterion

The considered class of systems is one where the global system consists of a set of N linear discrete-time subsystems that are coupled through their states, the inputs, and a performance criterion. The dynamics of the global system is defined by the following difference equation:

$$x_{k+1}^{(g)} = Ax_k^{(g)} + Bu_k^{(g)}, \tag{1}$$

where $x_k^{(g)} \in \mathbb{R}^{n_g}$ and $u_k^{(g)} \in \mathbb{R}^{m_g}$ represent the global states and inputs obtained by stacking the respective vectors of the local subsystems together according to:

$$x_k^{(g)} = \left[x_k^{(1)^T}, \dots, x_k^{(N)^T}\right]^T, u_k^{(g)} = \left[u_k^{(1)^T}, \dots, u_k^{(N)^T}\right]^T.$$

Herein, $x_k^{(i)} \in \mathbb{R}^{n_i}, u_k^{(i)} \in \mathbb{R}^{m_i} \quad \forall i \in \{1, \dots, N\}$ are the local states and inputs.

The dimensions of the global system are $n_g = \sum_{i=1}^N n_i$ and $m_g = \sum_{i=1}^N m_i$. The matrices A and B can be partitioned according to the states and inputs of the subsystems:

$$A = \begin{bmatrix} A_{1,1} & \cdots & A_{1,N} \\ \vdots & \ddots & \vdots \\ A_{N,1} & \cdots & A_{N,N} \end{bmatrix}, B = \begin{bmatrix} B_{1,1} & \cdots & B_{1,N} \\ \vdots & \ddots & \vdots \\ B_{N,1} & \cdots & B_{N,N} \end{bmatrix}, \tag{2}$$

where $A_{i,j} \in \mathbb{R}^{n_i \times n_j}, B_{i,j} \in \mathbb{R}^{n_i \times m_j} \quad \forall i, j \in \{1, \dots, N\}$. The objective is to fix the control inputs by the following static control law with gain matrix $K \in \mathbb{R}^{m_g \times n_g}$:

$$u_k^{(g)} = \underbrace{\begin{bmatrix} K_{1,1} & \cdots & K_{1,N} \\ \vdots & \ddots & \vdots \\ K_{N,1} & \cdots & K_{N,N} \end{bmatrix}}_{:=K} x_k^{(g)}. \tag{3}$$

As performance measure for the closed-loop system, consider an optimal infinite horizon quadratic cost function for the discrete-time dynamics (1), which is with (3):

$$V^*\left(x_k^{(g)}\right) = \min_K \sum_{i=0}^{\infty} x_{k+i}^{(g)^T} (Q + K^T RK) x_{k+i}^{(g)}. \tag{4}$$

Here, $Q = Q^T > 0 \in \mathbb{R}^{n_g \times n_g}$ and $R = R^T > 0 \in \mathbb{R}^{m_g \times m_g}$ determine symmetric positive definite weighting matrices for the states and control inputs. Notice that (4) formulates coupling through the cost function. The optimal infinite horizon cost function is a quadratic function [16]:

$$V^*\left(x_k^{(g)}\right) = x_k^{(g)^T} P^* x_k^{(g)}. \tag{5}$$

where $P^* = P^{*T} > 0 \in \mathbb{R}^{n_g \times n_g}$.

Proposition 1 *The pair P and K is optimal with respect to* (4)*, if and only if it minimizes the following optimization problem.*

$$\min_{P,K} \quad \textit{trace}(P), \qquad \textit{s.t.} \tag{6}$$

$$P = P^T > 0 \tag{7}$$

$$\left(P - (A+BK)^T P (A+BK) - Q - K^T RK\right) \geq 0 \tag{8}$$

Proof: Considering the system dynamics (1) and controller (3), it can be verified that (8) is equivalent to:

$$x_k^{(g)^T} P x_k^{(g)} \geq x_k^{(g)^T} (Q + K^T RK) x_k^{(g)} + x_{k+1}^{(g)^T} P x_{k+1}^{(g)}. \tag{9}$$

Induction for (9) leads to:

$$x_k^{(g)^T} P x_k^{(g)} - x_{k+l+1}^{(g)^T} P x_{k+l+1}^{(g)} \geq \sum_{i=0}^{l} x_{k+i}^{(g)^T} (Q + K^T RK)\, x_{k+i}^{(g)}. \tag{10}$$

Denote the feasible set of (6) by H. It follows that $\lim_{l \to \infty} x_{k+l}^{(g)^T} P x_{k+l}^{(g)} = 0 \quad \forall P, K \in H$. For $l \to \infty$ in (10), it holds that $P \geq P^* \quad \forall P, K \in H$. By Bellman's optimality principle the solution to (4) provides K and the smallest P such that (9) holds. It follows that $P^*, K^* \in H$ and P^* minimizes (6). Because of $Q > 0$ and $R > 0$, the minimum of both (4) and (6) is unique, hence the optimal solutions to (6) and (4) are identical. ■

B. Communication Topology and Cost

The structure of the control system consisting of the communication network, the interconnected subsystems, and the distributed controllers is shown in Fig. 1. Motivated by the flexibility provided by digital communication networks, we consider the case that the network topology can be chosen within constraints. In general, only a centralized control law is optimal with regard to the criteria presented in Sec. II-A. However this might be expensive, or impossible, to implement. Therefore, communication constraints and costs for realizing a communication link are considered in the optimization of the distributed control system. In the following, we present a model of the network topology and associated communication costs which, in combination with the performance criterion of Sec. II-A, are used in optimizing the overall system.

Assumption 1 *In this paper, we assume that any delay induced by the communication network is negligible compared to the dynamics of the system under control.* [1]

The topology of the communication network is described by the directed graph $\mathcal{G} = (V, E)$ with nodes $V = \{1, \dots, N\}$ and edges $E \subseteq V \times V$. The set of systems communicating with the $i - th$ subsystem is given by $\mathcal{N}_i = \{j \in V : (i, j) \in E\}$. In the remaining parts of this paper, the graph $\mathcal{G}$ is represented by the matrix $D \in \mathcal{D}$, where $\mathcal{D}$ is the set of admissible network topologies. The matrix D has the following structure:

$$D = \begin{bmatrix} \delta_{1,1} & \cdots & \delta_{1,N} \\ \vdots & \ddots & \vdots \\ \delta_{N,1} & \cdots & \delta_{N,N} \end{bmatrix}, \tag{11}$$

where the boolean entries $\delta_{i,j} \in \{1, 0\}$ of D are given by

$$\delta_{i,j} = \begin{cases} 1 & j \in \mathcal{N}_i \quad \text{or} \quad i = j \\ 0 & \text{otherwise} \end{cases}. \tag{12}$$

Thus, the boolean variables $\delta_{i,j}$ indicate whether or not information is communicated from subsystem j to the controller of subsystem i. This results in the following condition for the controller:

$$(\delta_{i,j} = 0) \implies (K_{i,j} = 0). \tag{13}$$

Finally for considering the costs of communication, we propose the following cost function with weights $c_{i,j}$ for a communication link between the j-th and i-th subsystem, such that:

$$J_{com} = \sum_{i=1}^{N} \sum_{j=1/i}^{N} c_{i,j} \delta_{i,j} \tag{14}$$

The costs $c_{i,j}$ associated with a communication link could, for example, be chosen based on the following aspects:

1) hardware costs,
2) energy consumption for communicating information,
3) distance between subsystems,
4) or the number of hops in a multi-hop wireless network.

Remark 1 *The integer model of the communication topology allows for a more comprehensive definition of communication costs containing logical statements such as AND, OR, and IF [15] – an example is provided in Sec. IV.*

C. Combined Optimization of Closed-loop Performance and Communication Cost

Based on the performance criterion (4) and the communication cost (14), a mixed integer problem (MIP) is formulated. Combining (6) and (14) results in the following optimization problem which minimizes the sum of the quadratic infinite horizon cost (4) and the communication cost (14):

$$\min_{P,K,D} \quad J = \text{trace}(P) + \sum_{i=1}^{N} \sum_{j=1/i}^{N} c_{i,j} \delta_{i,j} \tag{15}$$

s.t.

$$P - (A + BK)^T P (A + BK) - Q - K^T R K \geq 0 \tag{16}$$
$$P = P^T > 0 \tag{17}$$
$$D \in \mathcal{D} \tag{18}$$
$$(\delta_{i,j} = 0) \implies (K_{i,j} = 0) \tag{19}$$

This problem involves a BMI constraint for P and K as well as logical implication constraints on K which render the problem non-convex. The latter constraints can be formulated in a MIP framework. Solutions to a MIP problem are obtained by iterative procedures; a branch and bound algorithm, e.g., solves relaxed subproblems to obtain lower bounds on the cost, fixes integer variables and, if an integer feasible solution is found, solves the original problem to obtain an upper bound on the cost.

However, BMI solvers are relatively inefficient and provide only locally optimal solutions [2]. Hence, both the integer constraints as well as the BMI constraints have to be relaxed to obtain a lower bound, but often only relatively conservative upper and lower bounds are obtained.

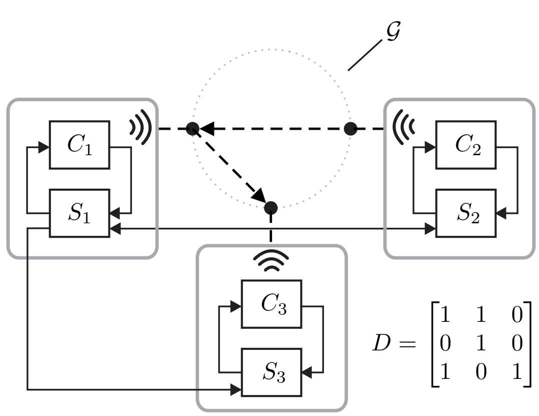

Fig. 1. Example of a graph $\mathcal{G}$ of a communication network (dashed), interconnected local subsystems S_i and local controllers C_i.

[1] An extension of the proposed method to the case of possibly unknown, bounded delays is currently under investigation.

[2] Globally optimal solutions can be obtained by repeatedly solving BMIs and LMIs in a branch and bound algorithm, resulting in far worse efficiency.

If the integer variables are fixed, the problem (15) becomes a structured controller synthesis problem, for which iterative LMI methods have been proposed. However, these methods in general either require a feasible initial solution or do not guarantee convergence to a feasible solution, and the solutions may not be locally optimal [11].

Thus, in the next section, we will focus on deriving sufficient, convex conditions that ensure that (16) and (19) hold.

III. MAIN RESULT

A. LMI Reformulation

The following theorem gives sufficient LMI conditions for the BMI constraint (16), allowing the reformulation of (15) into a MISDP. We use an extended LMI parametrization, similar to the approach known from [14].

Theorem 1 *The non-convex constraint* (16) *holds with* $P = Y^{-1}$ *and* $K = LG^{-1}$ *if there exists* $G \in \mathbb{R}^{n_g \times n_g}$, $Y = Y^T > 0 \in \mathbb{R}^{n_g \times n_g}$ *and* $L \in \mathbb{R}^{m_g \times n_g}$*, such that the following LMI holds:*

$$\begin{bmatrix} G+G^T-Y & (AG+BL)^T & G^T & L^T \\ AG+BL & Y & 0 & 0 \\ G & 0 & Q^{-1} & 0 \\ L & 0 & 0 & R^{-1} \end{bmatrix} > 0, \quad (20)$$

where 0 *denotes zero matrices of appropriate dimensions.*

Proof: Multiplying (16) from the left and the right by $Y = P^{-1}$ as well as making the inequality strict leads to:

$$Y - (AY+BKY)^T Y^{-1} (AY+BKY) \\ -YQY + (KY)^T RKY > 0. \quad (21)$$

Applying the Schur complement [17] to (21) yields:

$$\begin{bmatrix} Y & (AY+BKY)^T & Y^T & (KY)^T \\ AY+BKY & Y & 0 & 0 \\ Y & 0 & Q^{-1} & 0 \\ KY & 0 & 0 & R^{-1} \end{bmatrix} > 0. \quad (22)$$

Consider the following change of coordinates:

$$T = \begin{bmatrix} PG & 0 & 0 & 0 \\ 0 & I_{n_g} & 0 & 0 \\ 0 & 0 & I_{n_g} & 0 \\ 0 & 0 & 0 & I_{m_g} \end{bmatrix}, \quad (23)$$

where I_q denotes the identity matrix with dimension q. From the upper left block of the LMI (20) it follows that $G+G^T - Y > 0$, which requires $G+G^T > Y > 0$ and $G > 0$. Hence, T is nonsingular. Multiplying (22) by T^T from the left and by T from the right results in:

$$\begin{bmatrix} G^T PG & (AG+BKG)^T & G^T & (KG)^T \\ AG+BKG & Y & 0 & 0 \\ G & 0 & Q^{-1} & 0 \\ KG & 0 & 0 & R^{-1} \end{bmatrix} > 0. \quad (24)$$

Furthermore, $Y > 0$ implies $P > 0$. As proposed in [18], this implies $(Y-G)^T P (Y-G) \geq 0$, thus it can be verified that $Y - G - G^T + G^T PG \geq 0$, and it follows that:

$$G^T PG \geq G + G^T - Y. \quad (25)$$

Substituting (25) and $L = KG$ into (24) completes the proof. ■

The cost function of the optimization problem (15) depends on P while the LMI (20) depends on Y. However, an arbitrarily close upper bound on P can be obtained as follows. Denote an upper bound of Y^{-1} by $\hat{P}$. Applying the Schur complement for non-strict inequalities, $\hat{P} \geq Y^{-1}$ is equivalent to:

$$\begin{bmatrix} \hat{P} & I_{n_g} \\ I_{n_g} & Y \end{bmatrix} \geq 0. \quad (26)$$

B. Structured Controller Synthesis

Finally, we need to establish conditions which guarantee that the structure of the controller $K = LG^{-1}$ satisfies (13). Hence, constraints on L and G have to be found which ensure that the system of linear equations $L = KG$ has a solution K with the structure given by $\mathcal{G}$. Partitioning L and G such that they are compatible with the structure of K leads to:

$$\begin{bmatrix} L_{1,1} & \cdots & L_{1,N} \\ \vdots & \ddots & \vdots \\ L_{N,1} & \cdots & L_{N,N} \end{bmatrix} = \begin{bmatrix} K_{1,1} & \cdots & K_{1,N} \\ \vdots & \ddots & \vdots \\ K_{N,1} & \cdots & K_{N,N} \end{bmatrix} \begin{bmatrix} G_{1,1} & \cdots & G_{1,N} \\ \vdots & \ddots & \vdots \\ G_{N,1} & \cdots & G_{N,N} \end{bmatrix} \quad (27)$$

We parametrize the feedback such that $(K_{i,j} = 0) \implies (L_{i,j} = 0)$, i.e. $\neg\delta_{i,j} \implies (L_{i,j} = 0)$. Clearly this is not a necessary condition for the constraint (13), however we use this approach to obtain sufficient convex conditions at the expense of reducing the size of the feasible set compared to the non-convex problem (15). From (27) it can be verified that:

$$L_{i,j} = K_{i,i}G_{i,j} + K_{i,j}G_{j,j} + \sum_{z=1\backslash\{i,j\}}^{N} K_{i,z}G_{z,j} \quad (28)$$

If there is no communication between the j-th and i-th subsystem (i.e. $\neg\delta_{i,j}$), this results in

$$\neg\delta_{i,j} \implies (K_{i,i}G_{i,j} + \sum_{z=1/\{i,j\}}^{N} K_{i,z}G_{z,j} = 0), \quad (29)$$

where $i, j \in \{1, \ldots, N\}$ and $z \in \{1, \ldots, N\} \setminus \{i, j\}$. A sufficient condition to satisfy (29) is that all the summands are 0. This is ensured if $\neg\delta_{i,j}$ implies $(G_{i,j} = 0)$ and $(K_{i,z} = 0) \vee (G_{z,j} = 0) \quad \forall i, j, z$. The latter is equivalent to $\neg\delta_{i,j} \implies \neg\delta_{i,z} \vee (G_{z,j} = 0) \quad \forall i, j, z$ and $\neg\delta_{i,j} \wedge \delta_{i,z} \implies (G_{z,j} = 0) \quad \forall i, j, z$. This results in the following conditions:

$$\neg\delta_{i,j} \implies (L_{i,j} = 0) \quad \forall i, j \quad (30)$$
$$\neg\delta_{i,j} \implies (G_{i,j} = 0) \quad \forall i, j \quad (31)$$
$$\neg\delta_{i,j} \wedge \delta_{i,z} \implies (G_{z,j} = 0) \quad \forall i, j, z. \quad (32)$$

In order to derive a MISDP, the conditions (30) are reformulated using the so called Big-M method [15], resulting in the constraints:

$$-M\delta_{i,j} \leq L_{i,j} \leq M\delta_{i,j} \quad \forall i, j \quad (33)$$
$$-M\delta_{i,j} \leq G_{i,j} \leq M\delta_{i,j} \quad \forall i, j$$
$$-M(\delta_{i,j} - \delta_{i,z} + 1) \leq G_{z,j} \leq M(\delta_{i,j} - \delta_{i,z} + 1) \quad \forall i, j, z$$

where M is a sufficiently large number. In order to avoid poor relaxations of the MISDP, which result in poor solver performance, M should be chosen as small as possible. Here, M has to be larger than the entries of G and L, which are often small (e.g. smaller than 1 in the example in Sec. IV).

Theorem 2 *Suppose the LMI* (20) *holds for L and G subject to* (33). *Then the controller $K = LG^{-1}$ satisfies the structural constraint* (13) *imposed by the communication graph $\mathcal{G}$.*

Proof: The constraints (33) guarantee that $L = KG$ has a solution K that satisfies (13). If the LMI (20) is feasible, G is nonsingular and the system of linear equations $L = KG$ has exactly one solution K. Hence K can be obtained from $K = LG^{-1}$ and satisfies (13). ■

If the communication graph $\mathcal{G}$ is full, L and G are unconstrained and (20) solves the LQR problem (4).

C. MISDP Formulation

Combining the results from theorem 1 and 2 leads to the following MISDP, which minimizes the sum of the upper bound $\hat{P}$ of the quadratic infinite horizon cost and the communication cost.

$$\min_{P,K,D} \quad J = \text{trace}\left(\hat{P}\right) + \sum_{i=1}^{N} \sum_{j=1/i}^{N} c_{i,j}\delta_{i,j} \tag{34}$$

s.t.

$$\begin{bmatrix} G + G^T - Y & (AG + BL)^T & G^T & L^T \\ AG + BL & Y & 0 & 0 \\ G & 0 & Q^{-1} & 0 \\ L & 0 & 0 & R^{-1} \end{bmatrix} > 0$$

$$\begin{bmatrix} \hat{P} & I_{n_g} \\ I_{n_g} & Y \end{bmatrix} \geq 0$$

$$-M\delta_{i,j} \leq L_{i,j} \leq M\delta_{i,j} \quad \forall i,j$$

$$-M\delta_{i,j} \leq G_{i,j} \leq M\delta_{i,j} \quad \forall i,j$$

$$-M(\delta_{i,j} - \delta_{i,z} + 1) \leq G_{z,j} \leq M(\delta_{i,j} - \delta_{i,z} + 1) \quad \forall i,j,z$$

IV. SIMULATION RESULTS

To illustrate the proposed method, consider the following system:

$$\begin{bmatrix} x_{k+1}^{(1)} \\ x_{k+1}^{(2)} \\ x_{k+1}^{(3)} \end{bmatrix} = \left[\begin{array}{cc|cc|cc} 1 & 0.1 & 0 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 & 0 & 0 \\ \hline 0.1 & 0.1 & 1 & 0.1 & 0 & 0.1 \\ 0 & 0 & 0 & 1 & 0 & 0 \\ \hline 0 & 0 & 0 & 0.1 & 1 & 0.1 \\ 0 & 0 & 0 & 0 & 0 & 1 \end{array}\right] x_k^{(g)} + \left[\begin{array}{c|c|c} 0 & 0 & 0 \\ 0.1 & 0 & 0 \\ \hline 0 & 0 & 0 \\ 0 & 0.1 & 0 \\ \hline 0 & 0 & 0 \\ 0 & 0 & 0.1 \end{array}\right] u_k^{(g)}. \tag{35}$$

Here, $x_k^{(i)} \in \mathbb{R}^2$, $u_k^{(i)} \in \mathbb{R}^1$ $\forall i = \{1, \ldots, 3\}$ are the local states and inputs, and the second subsystem is interconnected with the first and third subsystem. This example is motivated by the first example given in [2], where it is demonstrated that (i) a similar system can not be asymptotically stabilized by decentralized feedback, and (ii) feedback from the second to the first subsystem (i.e. $\delta_{1,2} = 1$) is sufficient to asymptotically stabilize the system. By using Euler discretization, the results for the continuous-time example in [2] hold for (35).

The weighting matrix for the infinite horizon cost function are chosen as follows and introduce further coupling between the first and third subsystem:

$$Q = \begin{bmatrix} 1 & 0.1 & 0 & 0 & 0.2 & 0 \\ 0.1 & 1 & 0 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0.2 & 0 & 0 \\ 0 & 0 & 0.2 & 1 & 0 & 0 \\ 0.2 & 0 & 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 & 0 & 1 \end{bmatrix}, \quad R = \begin{bmatrix} 0.1 & 0 & 0 \\ 0 & 0.2 & 0 \\ 0 & 0 & 0.3 \end{bmatrix}. \tag{36}$$

First we consider the case that all communication links are equally expensive (i.e. $c_{i,j} = c \;\; \forall i, j \backslash i$ for a chosen c). The resulting topologies and costs are shown in Table I. For comparison, a centralized LQR controller is included in the last row. It can be seen that the proposed method results in the LQR controller if the communication graph is full and that communication links are removed when their costs increase.

The MISDP was solved using the Matlab toolbox YALMIP [19], using branch and bound and the solver SeDuMi for the semi-definite subproblems. Computation times for this example range between 8s and 16s on a Core2 Duo 2.2 GHz with 1 GB of RAM. For the case $c = 15$, the following distributed controller is obtained:

$$K = \begin{bmatrix} -4.03 & -5.81 & -3.62 & 0.97 & 0 & 0 \\ 0 & 0 & -0.83 & -5.12 & 0 & 0 \\ 0 & 0 & 0 & 0 & -2.21 & -3.74 \end{bmatrix} \tag{37}$$

Simulation results for this controller and a centralized LQR controller are shown in Fig. 2. Compared to the centralized controller, the distributed controller results in slower convergence and larger control amplitudes. However the communication costs of the centralized controller are $J_{com} = 90$, compared to $J_{com} = 15$ for the distributed controller, leading to overall costs of $J = 154.97$ (centralized) and $J = 113.84$ (distributed). Despite using only one communication link, the performance of the distributed controller is very similar to that of the centralized controller. In addition, the influence of the cost function (36) can be seen; e.g., the input signals with larger weights are smaller.

Finally, we consider a more complicated scenario, in which

TABLE I

RESULTS FOR SYSTEM (35) WITH DIFFERENT COMMUNICATION COSTS.

c	# of links	D	trace($\hat{P}$)	J_{com}	J
2	6	$\begin{bmatrix} 1 & 1 & 1 \\ 1 & 1 & 1 \\ 1 & 1 & 1 \end{bmatrix}$	64.97	12	76.97
9	4	$\begin{bmatrix} 1 & 1 & 1 \\ 0 & 1 & 1 \\ 0 & 1 & 1 \end{bmatrix}$	70.43	36	106.43
9.5	2	$\begin{bmatrix} 1 & 1 & 0 \\ 1 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}$	89.26	19	108.26
15	1	$\begin{bmatrix} 1 & 1 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}$	98.84	15	113.84
15 (LQR)	6	$\begin{bmatrix} 1 & 1 & 1 \\ 1 & 1 & 1 \\ 1 & 1 & 1 \end{bmatrix}$	64.97	90	154.97

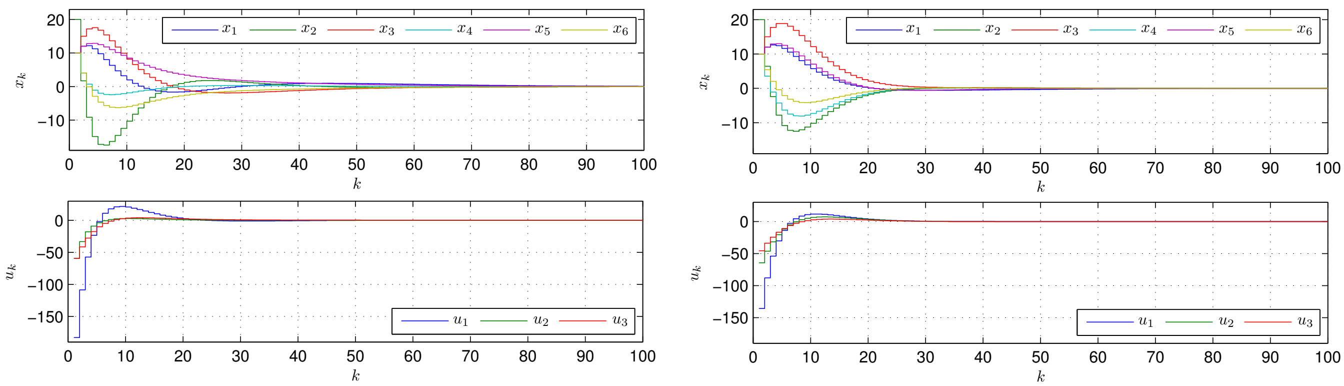

Fig. 2. Simulation results for a distributed controller (left) and a centralized LQR controller (right) with high communication costs.

each subsystem can only communicate with one other subsystem and transmitting information is expensive for the second subsystem. This is modeled by the constraints $\sum_{j=1}^{3} \delta_{i,j} \leq 2$ and $\sum_{i=1}^{3} \delta_{i,j} \leq 2$, and costs $c_{1,2} = c_{3,2} = 40$, $c_{1,3} = c_{3,1} = 8$, $c_{2,1} = c_{2,3} = 4$. Furthermore, the cost of bidirectional communication between two subsystems is lower than the cost of the two individual links between the systems. This is modeled by introducing slack variables $S_{i,j}$, such that $J_{com} = S_{1,2} + S_{1,3} + S_{2,3}$ and

$$S_{i,j} \geq c_{i,j}\delta_{i,j}, \quad S_{i,j} \geq c_{j,i}\delta_{j,i} \tag{38}$$
$$S_{i,j} \geq \tilde{c}_{i,j}(\delta_{i,j} + \delta_{j,i} - 1) \quad \forall (i,j) \in \{(1,2),(1,3),(2,3)\}.$$

Here, $\tilde{c}_{i,j}$ is the cost for implementing bidirectional communication between the i-th and j-th subsystem. These costs are $\tilde{c}_{1,2} = 41$, $\tilde{c}_{1,3} = 41$, $\tilde{c}_{2,3} = 10$. The optimization results in trace$(\hat{P}) = 108.34$, $J_{com} = 10$ and in a network topology which avoids the expensive outgoing communication from subsystem two and which has two active communication links ($\delta_{3,1} = \delta_{1,3} = 1$).

V. CONCLUSIONS AND FUTURE WORKS

In this paper, an approach to optimal distributed control considering closed-loop performance and communication costs was presented. Based on a quadratic performance criterion, sufficient convex LMI conditions for structured controller synthesis are obtained. In combination with a model of the costs of communication topologies, a MISDP is obtained which allows joint optimization of the network topology and distributed controllers. An extension of the proposed method to delayed feedback is the focus of ongoing work. Robustness with regard to link failures is seen as interesting topic for future research. More efficient methods to solve or reformulate MISDPs should be explored to achieve better scalability of the proposed method with respect to the system size. For instance, in [20] it is proposed to convert the MISDP to a mixed integer nonlinear program (MINLP) with smooth convex NLP subproblems.

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